Generalized Parking Function Polytopes

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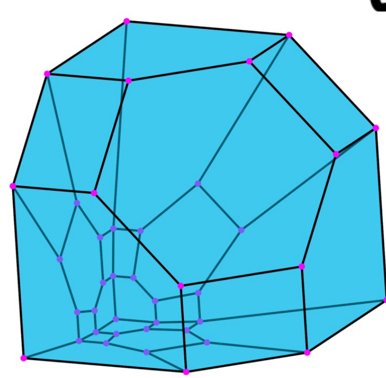
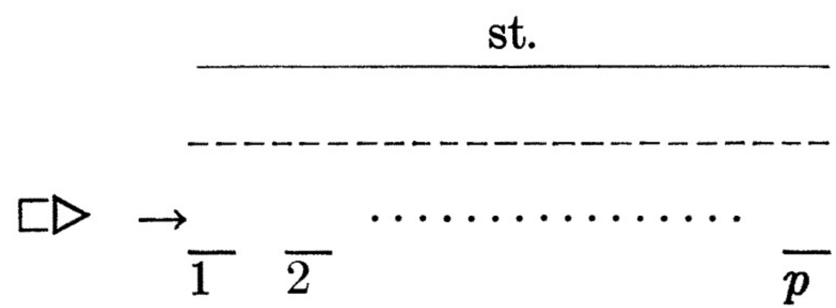


Classical Parking Functions

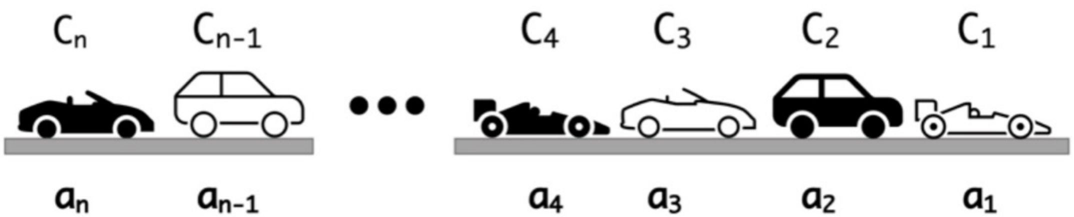
Konheim and Weiss (1966):

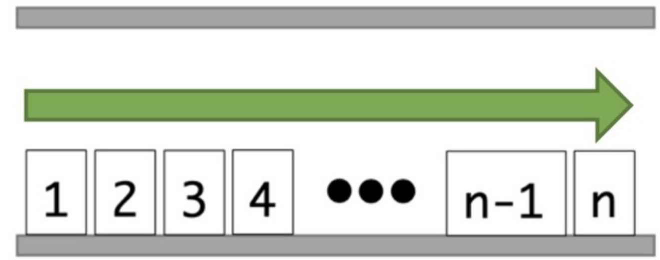
6. A parking problem—the case of the capricious wives. Let st. be a street with p parking places. A car



occupied by a man and his dozing wife enters st. at the left and moves towards the right. The wife awakens at a capricious moment and orders her husband to park immediately! He dutifully parks at his present location, if it is empty, and if not, continues to the right and parks at the next available space. If no space is available he leaves st.

Classical Parking Functions





Car C_i has parking preference a_i. If a_i is occupied, then C_i takes the next available parking spot. We call $(a_1, \ldots, a_n)$ a parking function (of length n) if all cars can successfully park.

$n = 2 : 11, 12, 21$ $\cancel{22}$

$n = 3 : 111, 112, 121, 211, 113, 131, 311, 122,$
$212, 221, 123, 132, 213, 231, 312, 321$

Classical Parking Functions

Proposition: Let α be a list $(a_1, \ldots, a_n)$ of positive integers. Take $b_1 \leq b_2 \leq \cdots \leq b_n$ to be the nondecreasing rearrangement of α. Then α is a parking function if and only if $b_i \leq i$.

Corollary: Every permutation of the entries of a parking function is also a parking function.

Theorem (Pyke 1959, Konheim & Weiss 1966): Let $pf(n)$ be the number of parking functions of length n. Then

$$pf(n) = (n+1)^{n-1}.$$

Classical Parking Function Polytope

Let PF_n denote the classical parking function polytope; i.e., the convex hull of all parking functions of length n in $\mathbb{R}^n$.

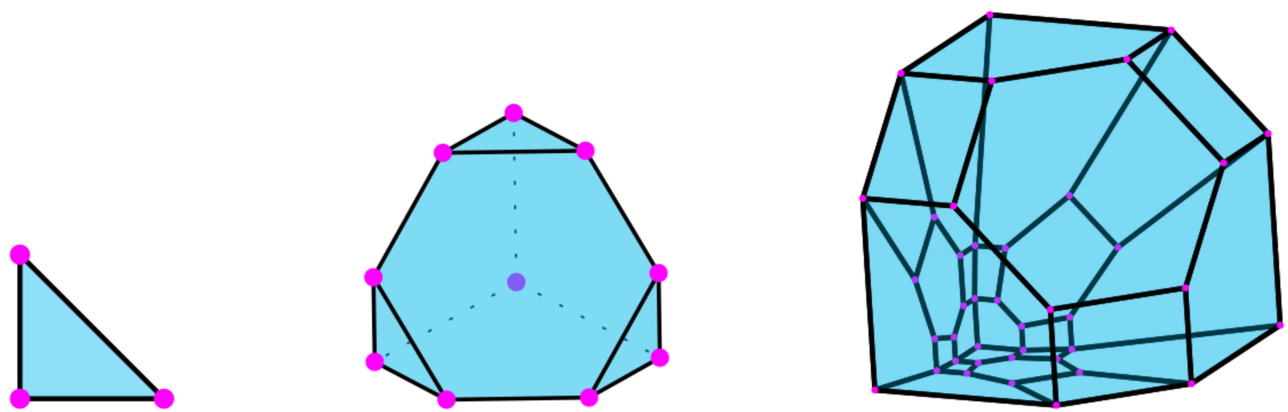


Figure: PF_2, PF_3, and PF_4.

Classical Parking Function Polytope

In 2020, Richard Stanley asked for:

- the number of vertices of PF_n,
- the number of faces of PF_n,
- the number of lattice points $PF_n \cap \mathbb{Z}^n$,
- the volume of PF_n.

These were resolved by Amanbayeva & Wang (2022).

Collaborators

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Generalizing the Parking Function Polytope

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n_{>0}$.

Define an **$\mathbf{x}$-parking function** to be a sequence $(a_1, \ldots, a_n)$ of positive integers whose nondecreasing rearrangement $b_1 \le b_2 \le \cdots \le b_n$ satisfies $b_i \le x_1 + x_2 + \cdots + x_i$.

Theorem (Yan 2001):

For $\mathbf{x} = (a, b, b, \ldots, b) \in \mathbb{Z}^n_{>0}$, the number of $\mathbf{x}$-parking functions is given by

$$a(a+nb)^{n-1}.$$

Generalizing the Parking Function Polytope

Define the **x**-parking function polytope $\mathcal{X}_n(a,b)$ as the convex hull of all **x**-parking functions of length n in $\mathbb{R}^n$ for $\mathbf{x} = (a,b,\dots,b) \in \mathbb{Z}^n_{>0}$.

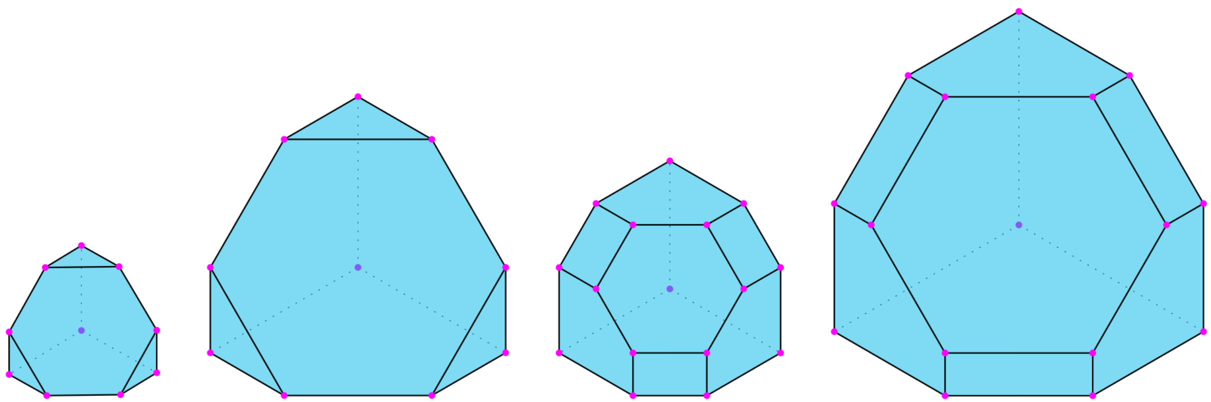


FIGURE The **x**-parking function polytopes, from left to right: $\mathcal{X}_3(1,1)$, $\mathcal{X}_3(1,2)$, $\mathcal{X}_3(2,1)$, and $\mathcal{X}_3(2,2)$. Observe that $\mathcal{X}_3(1,2)$ is a dilate of $\mathcal{X}_3(1,1)$. Note that when $a > 1$, there are new facets that do not appear when $a = 1$.

⚠ We focus on $\mathbf{x} = (a,b,\dots,b)$ following work of Yan.

Face Structure of $\mathcal{X}_n(a,b)$

Proposition (Hanada, Lentfer, ARVM 2023):

The vertices of $\mathcal{X}_n(a,b)$ are all permutations of
$$(\underbrace{1,\dots,1}_{k},\underbrace{a+kb,\ a+(k+1)b,\ \dots,\ a+(n-2)b,\ a+(n-1)b}_{n-k}),$$
for all $0 \le k \le n$.

Furthermore, the number of vertices is
$$\begin{cases} n!\left(1/1! + \cdots + 1/n!\right) & \text{if } a=1, \\ n!\left(1/0! + 1/1! + \cdots + 1/n!\right) & \text{if } a>1. \end{cases}$$

Proposition (Hanada, Lentfer, ARVM 2023):

The $\mathbf{x}$-parking function polytope $\mathcal{X}_n(a,b)$ is given by the minimal inequality description:

For all $1 \le i \le n$,
$$1 \le x_i \le (n-1)b + a,$$
for all $1 \le i < j \le n$,
$$x_i + x_j \le ((n-2)b + a) + ((n-1)b + a),$$
for all $1 \le i < j < k \le n$,
$$x_i + x_j + x_k \le ((n-3)b + a) + ((n-2)b + a) + ((n-1)b + a),$$
$$\vdots$$
for all $1 \le i_1 < i_2 < \cdots < i_{n-2} \le n$,
$$x_{i_1} + x_{i_2} + \cdots + x_{i_{n-2}} \le (2b + a) + \cdots + ((n-2)b + a) + ((n-1)b + a),$$
if $a > 1$, for all $1 \le i_1 < i_2 < \cdots < i_{n-1} \le n$,
$$x_{i_1} + x_{i_2} + \cdots + x_{i_{n-1}} \le (b + a) + \cdots + ((n-2)b + a) + ((n-1)b + a),$$
and (regardless of a),
$$x_1 + x_2 + \cdots + x_n \le a + \cdots + ((n-2)b + a) + ((n-1)b + a).$$

Face Structure of $\mathcal{X}_n(a,b)$

Corollary:
The number of facets of $\mathcal{X}_n(a,b)$ is
$$\begin{cases} 2^n - 1 & \text{if } a = 1, \\ 2^n - 1 + n & \text{if } a > 1. \end{cases}$$

Proposition (Hanada, Lentfer, ARVM ~~2022~~ 2023):
For each vertex v of $\mathcal{X}_n(a,b)$, there are exactly n edges of $\mathcal{X}_n(a,b)$ with v as one of the vertices. That is, $\mathcal{X}_n(a,b)$ is a simple polytope.

Proposition (Hanada, Lentfer, ARVM ~~2022~~ 2023):
Let f_k be the number of k-dimensional faces of $\mathcal{X}_n(a,b)$ for $k \in (0, \dots, n)$.

Then if $a = 1$,
$$f_k = \sum_{m=0,\ m\neq 1}^{n-k} \binom{n}{m} \cdot (n-k-m)! \cdot S(n-m+1, n-k-m+1),$$
and if $a > 1$,
$$f_k = \sum_{m=0}^{n-k} \binom{n}{m} \cdot (n-k-m)! \cdot S(n-m+1, n-k-m+1),$$
where $S(n,k)$ are the Stirling numbers of the second kind.

Proposition (Hanada, Lentfer, ARVM ~~2022~~ 2023):
For fixed n and for $a = 1$, the $\mathcal{X}_n(1,b)$ are combinatorially equivalent for all $b \geq 1$. Additionally, for fixed n, the $\mathcal{X}_n(a,b)$ are combinatorially equivalent for all $a > 1$ and $b \geq 1$.

Volume of $\mathcal{X}_n(a,b)$

Proposition (Hanada, Lentfer, ARVM ~~2022+~~ 2023):

(1) The $\mathbf{x}$-parking function polytope $\mathcal{X}_n(1,b)$ is a b-dilate of $\mathcal{X}_n(1,1)$.

(2) for fixed $a \geq 1$, $\mathcal{X}_n(a+(b-1)(a-1),b)$ is a b-dilate of $\mathcal{X}_n(a,1)$.

Theorem (Hanada, Lentfer, ARVM ~~2022+~~ 2023):

Fix two positive integers a, b. Define a sequence $\{V_n^{a,b}\}_{n\geq 0}$ by $V_0^{a,b}=1$ and $V_n^{a,b} = \mathrm{Vol}(\mathcal{X}_n(a,b))$ for all positive integers n. Then $V_1^{a,b} = a-1$ and for $n \geq 2$, $V_n^{a,b}$ is given recursively by

$$V_n^{a,b} = \frac{1}{n}\sum_{k=0}^{n-1}\binom{n}{k}\frac{(b(n-k))^{n-k-1}(nb+kb-b+2a-2)}{2}V_k^{a,b}.$$

Generalization of Amanbayeva & Wang's recursive volume formula for PFn.

Volume of $\mathcal{X}_n(a,b)$

closed formula!

Theorem (Hanada, Lentfer, ARVM ~~2022+~~ 2023):

For any positive integers a, b, n, the normalized volume $\mathrm{NVol}(\mathcal{X}_n(a,b))$ is given by

$$\mathrm{NVol}(\mathcal{X}_n(a,b)) = -n!\left(\frac{b}{2}\right)^n \sum_{i=0}^{n} \binom{n}{i} (2i-3)!!\left(2n-1+\frac{2a-2}{b}\right)^{n-i}.$$

Connections to other polytopes

A **weakly increasing x-parking function** associated to a positive integer vector $(a, b, \ldots, b)$ is a weakly increasing sequence $(a_1, \ldots, a_n)$ of positive integers who satisfy

$$a_i \leq a + (i-1)b.$$

Denote the **weakly increasing x-parking function polytope** by $\mathcal{X}_n^w(a,b)$.

For any $\mathbf{x} \in \mathbb{R}^n$, the **Pitman-Stanley polytope** $PS_n(\mathbf{x})$ is defined to be

$$\left\{\mathbf{y} \in \mathbb{R}^n : y_i \geq 0 \text{ and } \sum_{j=1}^{i} y_j \leq \sum_{j=1}^{i} x_j \;\; \forall 1 \leq i \leq n\right\}.$$

Proposition (Hanada, Lentfer, ARVM ~~2022+~~ 2023)**:**

The weakly increasing x-parking function $\mathcal{X}_n^w(a,b)$ is integrally equivalent to $PS_n(a-1, b, \ldots, b)$.

Corollary:

Let $t \in \mathbb{Z}_{\geq 0}$. The number of lattice points in the t-dilate of $\mathcal{X}_n^w(a,b)$ is given by

$$|t\mathcal{X}_n^w(a,b) \cap \mathbb{Z}^n| = \frac{1}{n!}(t(a-1)+1)(t(a-1+nb)+2)(t(a-1+nb)+3)\cdots(t(a-1+nb)+n).$$

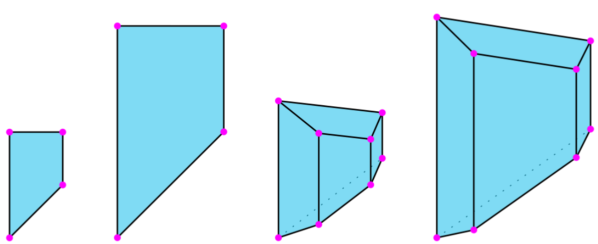


FIGURE The weakly increasing x-parking function polytopes, from left to right: $\mathcal{X}_3^w(1,1)$, $\mathcal{X}_3^w(1,2)$, $\mathcal{X}_3^w(2,1)$, $\mathcal{X}_3^w(2,2)$. Note that when $a = 1$, they are two-dimensional, and when $a > 1$, they are three-dimensional.

Connections to other polytopes

The *partial permutahedron* $\mathcal{P}(n,p)$ is the polytope with all permutations of the vectors

$$(0,\ldots,0,p-k+1,\ldots,p-1,p),$$

for all $0 \le k \le \min(n,p)$, as vertices.

Proposition (Hanada, Lentfer, ARVM ~~2022~~ 2023):

$$\{\mathcal{P}(n,p)\} \longleftrightarrow \{\mathcal{X}_n(a,b)\}$$

for $p \ge n-1$, $b=1$, and $n>1$.

- Specifically, if $n \ge 2$ and $p \ge n-1$, $\mathcal{P}(n,p) \cong \mathcal{X}_n(a,b)$ if and only if $b=1$ and $a=p-n+2$.

- If $n=1$ and $p \ge n-1$, then $\mathcal{P}(1,p) \cong \mathcal{X}_1(a)$ if and only if $a=p-1$.

Revisiting PF_n

$$\mathrm{PF}_n = \mathcal{X}_n(1,1) \cong P(n, n-1)$$

Proposition (Hanada, Lentfer, ARVM ~~2022~~ 2023):

The classical parking function polytope PF_n is integrally equivalent to the partial permutahedron $P(n, n-1)$.

Revisiting PF_n

An additional result on the face structure of PF_n:

Proposition (Hanada, Lentfer, ARVM ~~2022+~~ 2023):

(1) The regular permutahedron Π_n appears as a facet of PF_n exactly once.

(2) The $(n-1)$-dimensional parking function polytope PF_{n-1} appears as a facet of PF_n exactly n times.

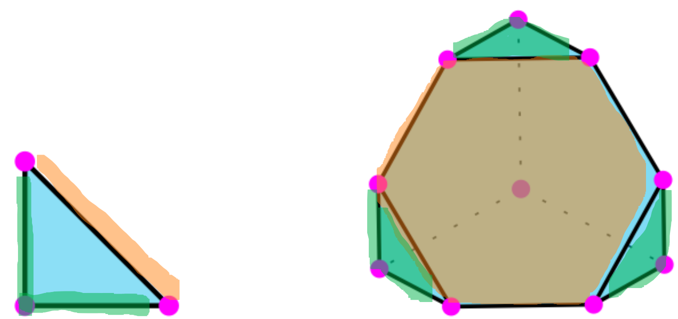


Figure: PF_2, PF_3,

Volume of PF_n

Theorem (Hanada, Lentfer, ARVM ~~2022+~~ 2023):

The following are equivalent volume formulas for PF_n:

(i) *with* $\mathrm{NVol}(\mathsf{PF}_0) = 1$ *and* $\mathrm{NVol}(\mathsf{PF}_1) = 0$*, for* $n \geq 2$ *we have recursively,*

$$\mathrm{NVol}(\mathsf{PF}_n) = (n-1)! \sum_{k=0}^{n-1} \binom{n}{k} \frac{(n-k)^{n-k-1}(n+k-1)}{2} \frac{\mathrm{NVol}(\mathsf{PF}_k)}{k!}.$$

Amanbayeva & Wang, 2022

(ii) *with* $\mathrm{NVol}(\mathsf{PF}_0) = 1$ *and* $\mathrm{NVol}(\mathsf{PF}_1) = 0$*, for* $n \geq 2$ *we have recursively,*

$$\mathrm{NVol}(\mathsf{PF}_n) = (n-1)! \sum_{k=1}^{n} k^{k-2} \frac{\mathrm{NVol}(\mathsf{PF}_{n-k})}{(n-k)!} \left(k(n-1) - \binom{k}{2} \right) \binom{n}{k}.$$

Behrend et al., 2022+

(iii)

$$\mathrm{NVol}(\mathsf{PF}_n) = -\frac{n!}{2^n} \sum_{i=0}^{n} \binom{n}{i} (2i-3)!!(2n-1)^{n-i}.$$

Hanada, Lentfer, ARVM ~~2022~~ 2023

(iv) *for* $n \geq 2$*,*

$$\mathrm{NVol}(\mathsf{PF}_n) = \frac{n!}{2^n} \sum_{i=0}^{n} (2i-1)(2i-1)!! \binom{n}{i} (2n-1)^{n-i-1}$$

$$= n! \frac{n-1}{2^{n-1}} \sum_{i=0}^{n-2} (2i+1)!! \binom{n-2}{i} (2n-1)^{n-i-2}.$$

Shevelev, 1997

(v) $\mathrm{NVol}(\mathsf{PF}_n)$ *equals the number of* $n \times n$ $(0,1)$*-matrices with two 1's in each row that have positive permanent.*

Volume of PF_n

Theorem (Hanada, Lentfer, ARVM ~~2024+~~ 2023):

The following are equivalent volume formulas for PF_n:

(i) with $\mathrm{NVol}(\mathsf{PF}_0) = 1$ and $\mathrm{NVol}(\mathsf{PF}_1) = 0$, for $n \geq 2$ we have recursively,

$$\mathrm{NVol}(\mathsf{PF}_n) = (n-1)! \sum_{k=0}^{n-1} \binom{n}{k} \frac{(n-k)^{n-k-1}(n+k-1)}{2} \frac{\mathrm{NVol}(\mathsf{PF}_k)}{k!}.$$

(ii) with $\mathrm{NVol}(\mathsf{PF}_0) = 1$ and $\mathrm{NVol}(\mathsf{PF}_1) = 0$, for $n \geq 2$ we have recursively,

$$\mathrm{NVol}(\mathsf{PF}_n) = (n-1)! \sum_{k=1}^{n} k^{k-2} \frac{\mathrm{NVol}(\mathsf{PF}_{n-k})}{(n-k)!} \left(k(n-1) - \binom{k}{2} \right) \binom{n}{k}.$$

(iii) $$\mathrm{NVol}(\mathsf{PF}_n) = -\frac{n!}{2^n} \sum_{i=0}^{n} \binom{n}{i} (2i-3)!!(2n-1)^{n-i}.$$

(iv) for $n \geq 2$, $$\mathrm{NVol}(\mathsf{PF}_n) = \frac{n!}{2^n} \sum_{i=0}^{n} (2i-1)(2i-1)!! \binom{n}{i} (2n-1)^{n-i-1}$$
$$= n! \frac{n-1}{2^{n-1}} \sum_{i=0}^{n-2} (2i+1)!! \binom{n-2}{i} (2n-1)^{n-i-2}.$$

(v) $\mathrm{NVol}(\mathsf{PF}_n)$ equals the number of $n \times n$ $(0,1)$-matrices with two 1's in each row that have positive permanent.

Proof Idea:

① The equivalence of (i) and (ii) follow from the integral equivalence between PF_n and $P(n, n-1)$.

② We derive (iii) by using the exponential generating function for the recursive volume formula (i) and then using Ramanujan's Master Theorem.

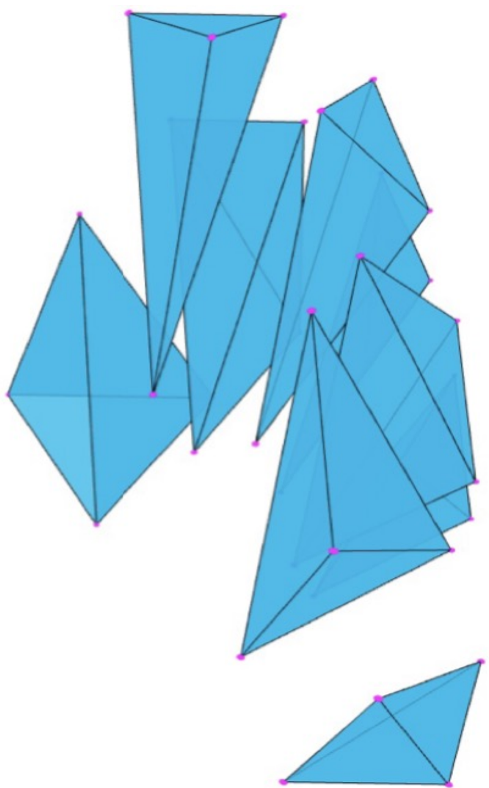
We then use the Lambert W function, Charlier polynomials, and generalized Laguerre polynomials to simplify the result.

③ formulas (iii) & (iv) are shown to be equivalent using Zeilberger's creative telescoping algorithm.

④ formulas (iv) & (v) are shown by Sheveler to be equivalent.

Open Problems

Corollary:
~~Conjecture:~~ The parking function polytope PF_n admits a regular unimodular triangulation.



Problem 1: Find a bijection between the simplices of a unimodular triangulation of PF_n and $(0,1)$-matrices with two 1's in each row with positive permanent.

Problem 2:
A.) Determine a formula for the Ehrhart polynomial (or h^*-polynomial) of PF_n.
B.) Find a combinatorial or geometric interpretation for their coefficients?
C.) Is PF_n Ehrhart positive?

So what about $\mathbf{x}$-parking function polytopes for general $\mathbf{x}$?

Collaborators

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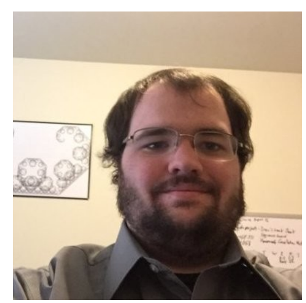

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Face Structure of $\mathfrak{X}_n$

Proposition (MB-SB-7C-SD-AD-DD-HL-7M-AM-GN-ARVM 2023+): The $\boldsymbol{x}$-parking function polytope $\mathfrak{X}_n$ is given by the minimal inequality description:

$$1 \leq x_i, \qquad \textit{for} \quad 1 \leq i \leq n$$

and

$$x_{i_1} + x_{i_2} + \cdots + x_{i_k} \leq k(b_1 + \cdots + b_{n-k+1}) + (k-1)b_{n-k+2} + \cdots + 2b_{n-1} + b_n,$$

for each possible chain $1 \leq i_1 < \cdots < i_k \leq n$ *and each value* $k \in \{1, 2, ..., n\} \setminus \{n-1\}$, *if* $b_1 = 1$, *and* $k \in \{1, 2, ..., n\}$, *if* $b_1 > 1$.

More concisely, $\mathfrak{X}_n$ *is the set of points* $(x_1, \ldots, x_n) \in \mathbb{R}^n$ *such that* $x_i \geq 1$ *for* $1 \leq i \leq n$ *and*

$$\sum_{i \in I} x_i \leq \sum_{j=1}^{n} \min\{\, |I|, j \,\}\, b_{n-j+1}$$

for all $I \subset [n]$ *if* $b_1 \geq 2$, *and for all* $I \subset [n] \setminus \{\, n-1 \,\}$ *if* $b_1 = 1$.

Equivalently, given $(b_1, \ldots, b_n) \in \mathbb{Z}^n_{>0}$, *write* $b(t)$ *for the polynomial* $\sum_{k=0}^{n} b_k t^k$. *Then the inequalities defining the corresponding generalized parking polytope are* $x_i \geq 1$ *for* $1 \leq i \leq n$, *together with*

$$\sum_{i \in I} x_i \leq [t^n] \left(\frac{1 - t^{|I|}}{(1-t)^2} \cdot b(t) \right)$$

for all $I \subseteq [n]$.

Proposition (MB-SB-7C-SD-AD-DD-HL-7M-AM-GN-ARVM 2023+): The vertices of $\mathfrak{X}_n$ are all possible permutations of

$$(\underbrace{1, 1, \ldots, 1}_{k}, \underbrace{b_1 + \cdots + b_{k+1}, b_1 + \cdots + b_{k+2}, \ldots, b_1 + \cdots + b_{k+n}}_{n-k})$$

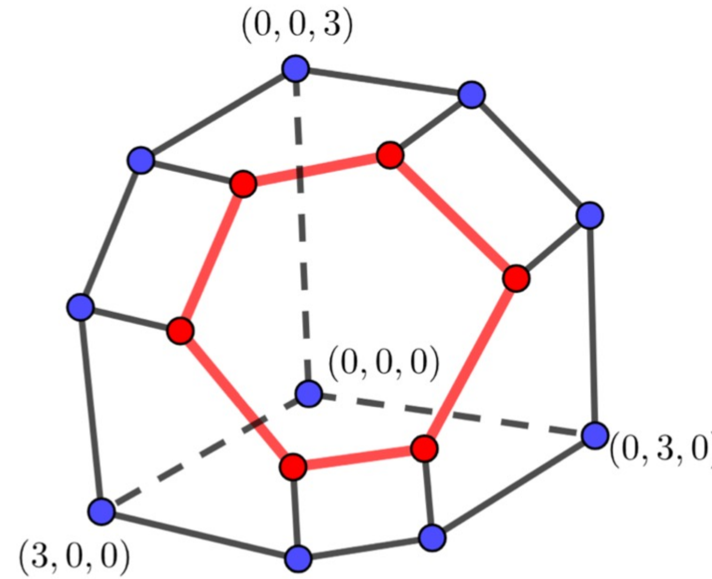
for all $0 \leq k \leq n$.

Furthermore, the number of vertices is

$$\begin{cases} n!\left(1/1! + \cdots + 1/n!\right) & \text{if } b_1 = 1, \\ n!\left(1/0! + 1/1! + \cdots + 1/n!\right) & \text{if } b_1 > 1 ! \end{cases}$$

The Stellohedron St_n

The stellohedron St_n, considered by Postnikov, Reiner and Williams (2008), is a polytope that can be constructed by starting with the simplex $\Delta_n = (0, e_1, e_2, \dots, e_n)$ and truncating the faces not containing 0 starting from those of dimension $0, 1, \dots, n-1$.



The h-polynomial of St_n is the binomial Eulerian polynomial

$$\tilde{A}_n(z) = 1 + z \sum_{k=1}^{n} \binom{n}{k} A_k(z),$$

where $A_k(z) = \sum_{\sigma \in S_n} z^{\operatorname{des}(\sigma)}$ is the Eulerian polynomial.

Face Structure of $\mathcal{X}_n$

Theorem (MB-SB-TC-SD-AD-DD-HL-TM-AM-GN-ARVM 2023+):

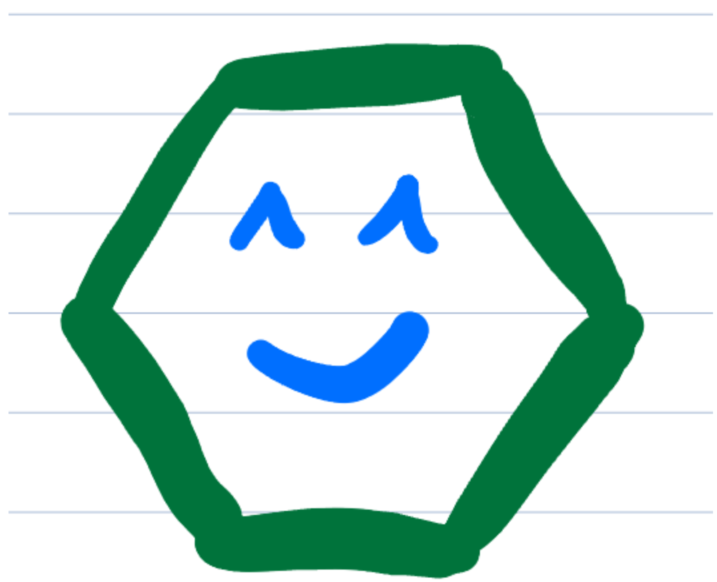
The h-polynomial of $\mathcal{X}_n$ is:

$$h(\mathcal{X}_n; z) = \begin{cases} h(PF_n; z) = \widetilde{A}_n(z) - nzA_{n-1}(z) & , \text{if } b_1 = 1 \\ h(St_n; z) = \widetilde{A}_n(z) & , \text{if } b_1 > 1. \end{cases}$$

X_n is a generalized permutahedron

A generalized permutahedron is a polytope that can be obtained from the standard permutahedron by changing the edge lengths while preserving the edge directions.

Fin... ¡Gracias!



Mitsuki Hanada, John Lentfer, and **Andrés R. Vindas-Meléndez,** *Generalized parking function polytopes,* to appear in *Annals of Combinatorics*, arXiv: 2212.06885.