The e-positivity of chromatic symmetric functions

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CHROMATIC POLYNOMIAL: BIRKHOFF 1912

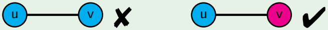
Given G with vertices $V(G)$ a proper colouring κ of G in k colours is

$$\kappa : V(G) \to \{1, 2, 3, \ldots, k\}$$

so if $u, v \in V(G)$ are joined by an edge then

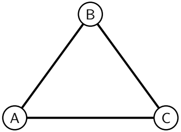
$$\kappa(u) \neq \kappa(v).$$

EXAMPLE



Chromatic polynomial: Birkhoff 1912

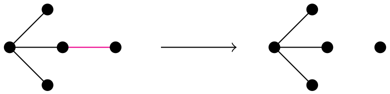
Given G the chromatic polynomial $\chi_G(k)$ is the number of proper colourings with k colours.



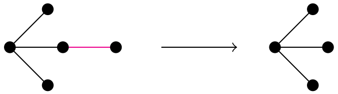
has $\chi_G(k) = k(k-1)(k-2)$.

Deletion-contraction

Delete ϵ: remove edge ϵ to get $G - \epsilon$.



Contract ϵ: shrink edge ϵ + identify vertices to get G/ϵ.



Theorem (Deletion-contraction)

$$\chi_G(k) - \chi_{G-\epsilon}(k) + \chi_{G/\epsilon}(k) = 0$$

Chromatic symmetric function: Stanley 1995

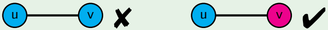
Given G with vertices $V(G)$ a proper colouring κ of G is

$$\kappa : V(G) \to \{1, 2, 3, \ldots\}$$

so if $u, v \in V(G)$ are joined by an edge then

$$\kappa(u) \neq \kappa(v).$$

Example

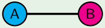


Chromatic symmetric function: Stanley 1995

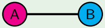
Given a proper colouring κ of vertices $v_1, v_2, \ldots, v_N$ associate a monomial in commuting variables $x_1, x_2, x_3, \ldots$

$$x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_N)}.$$

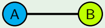
Example



gives $x_1 x_2$.



gives $x_2 x_1 = x_1 x_2$.



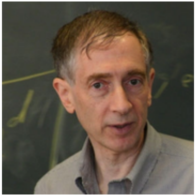
gives $x_1 x_3$.

Chromatic symmetric function: Stanley 1995

Given G with vertices $v_1, v_2, \ldots, v_N$ the chromatic symmetric function is

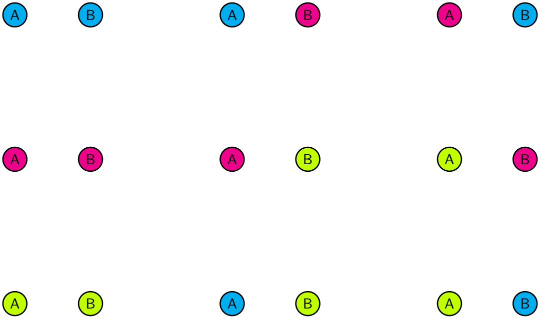
$$X_G = \sum_{\kappa} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_N)}$$

where the sum over all proper colourings κ.



Chromatic symmetric function: Stanley 1995

has $X_G(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_2x_3 + 2x_1x_3$.



Multi-deletion

Deletion-contraction fails, as contraction gives degree change.

Theorem (Triple-deletion: Orellana-Scott 2014)

Let G be such that $\epsilon_1, \epsilon_2, \epsilon_3$ form a triangle. Then

$$X_G - X_{G-\{\epsilon_1\}} - X_{G-\{\epsilon_2\}} + X_{G-\{\epsilon_1,\epsilon_2\}} = 0.$$

Theorem (k-deletion: Dahlberg-vW 2018)

Let G be such that $\epsilon_1, \epsilon_2, \ldots, \epsilon_k$ form a k-cycle for $k \geq 3$. Then

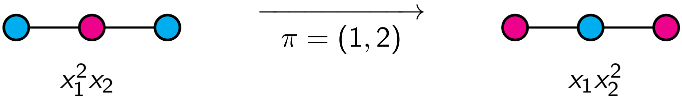
$$\sum_{S \subseteq [k-1]} (-1)^{|S|} X_{G - \cup_{i \in S}\{\epsilon_i\}} = 0.$$

Symmetric functions

A symmetric function is a formal power series f in commuting variables $x_1, x_2, \ldots$ such that for all permutations π

$$f(x_1, x_2, \ldots) = f(x_{\pi(1)}, x_{\pi(2)}, \ldots).$$

X_G is a symmetric function.



Let

$$\Lambda = \bigoplus_{N \geq 0} \Lambda^N \subset \mathbb{Q}[[x_1, x_2, \ldots]]$$

be the algebra of symmetric functions with Λ^N spanned by . . .

Classical basis: power sum

A partition $\lambda = \lambda_1 \geq \cdots \geq \lambda_\ell > 0$ of N is a list of positive integers whose sum is N: $3221 \vdash 8$.

The i-th power sum symmetric function is

$$p_i = x_1^i + x_2^i + x_3^i + \cdots$$

and for $\lambda = \lambda_1 \cdots \lambda_\ell$

$$p_\lambda = p_{\lambda_1} \cdots p_{\lambda_\ell}.$$

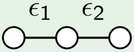
Example

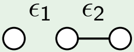
$p_{21} = p_2 p_1 = (x_1^2 + x_2^2 + x_3^2 + \cdots)(x_1 + x_2 + x_3 + \cdots)$

Classical basis: power sum

Given $S \subseteq E(G)$, $\lambda(S)$ is the partition determined by the connected components of G restricted to S.

Example

$G =$ 

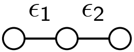
G restricted to $S = \{\epsilon_2\}$ is 

and $\lambda(S) = 21$.

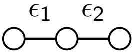
Theorem (Stanley 1995)

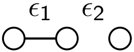
$$X_G = \sum_{S \subseteq E(G)} (-1)^{|S|} p_{\lambda(S)}$$

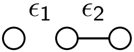
Classical basis: power sum

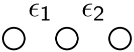
$G =$ 

G restricted to

- $S = \{\epsilon_1, \epsilon_2\}$ is 

and $\lambda(S) = 3$
- $S = \{\epsilon_1\}$ is 

and $\lambda(S) = 21$
- $S = \{\epsilon_2\}$ is 

and $\lambda(S) = 21$
- $S = \emptyset$ is 

and $\lambda(S) = 111$.

$$X_G = p_3 - 2p_{21} + p_{111}$$

Classical basis: elementary

The i-th elementary symmetric function is

$$e_i = \sum_{j_1 < \cdots < j_i} x_{j_1} \cdots x_{j_i}$$

and for $\lambda = \lambda_1 \cdots \lambda_\ell$

$$e_\lambda = e_{\lambda_1} \cdots e_{\lambda_\ell}.$$

Example

$$e_{21} = e_2 e_1 = (x_1x_2 + x_1x_3 + x_2x_3 + \cdots)(x_1 + x_2 + x_3 + \cdots)$$

$G =$  $\qquad X_G = 3e_3 + e_{21}$

Classical basis: elementary

Theorem (Stanley 1995)

If

$$X_G = \sum_{\lambda} c_\lambda e_\lambda$$

then

$$\sum_{\lambda \text{ with } k \text{ parts}} c_\lambda = \text{ number of acyclic orientations with } k \text{ sinks.}$$

Example

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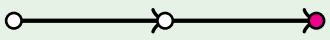
Example

$G =$  $\qquad X_G = 3e_3 + e_{21}$

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Example

$G =$ 

$$X_G = 3e_3 + e_{21}$$

Classical basis: elementary

Theorem (Stanley 1995)

If

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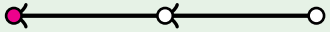
Example

$G =$ 

$$X_G = 3e_3 + e_{21}$$

Classical basis: elementary

Theorem (Stanley 1995)

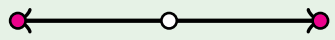
If

$$X_G = \sum_{\lambda} c_\lambda e_\lambda$$

then

$$\sum_{\lambda \text{ with } k \text{ parts}} c_\lambda = \text{ number of acyclic orientations with } k \text{ sinks.}$$

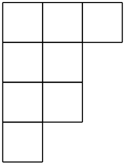
Example

$G =$  $\quad X_G = 3e_3 + e_{21}$

Partitions and diagrams

A partition $\lambda = \lambda_1 \geq \cdots \geq \lambda_\ell > 0$ of N is a list of positive integers whose sum is N: $3221 \vdash 8$.

The diagram $\lambda = \lambda_1 \geq \cdots \geq \lambda_\ell > 0$ is the array of boxes with λ_i boxes in row i from the top.



3221

Semi-standard Young tableaux

A semi-standard Young tableau (SSYT) T of shape λ is a filling with $1, 2, 3, \ldots$ so rows weakly increase and columns increase.

1	1	1
2	4	
4	5	
6		

Given an SSYT T we have

$$x^T = x_1^{\#1s} x_2^{\#2s} x_3^{\#3s} \cdots .$$

$$x_1^3 x_2 x_4^2 x_5 x_6$$

Classical basis: Schur

The Schur function is

$$s_\lambda = \sum_{T \text{ SSYT of shape } \lambda} x^T.$$

Example

$$s_{21} = x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + 2x_1x_2x_3 + \cdots$$

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}$$

$G =$  $\qquad X_G = s_{21} + 4s_{111}$

(Wang-Wang 2020) Intricate formula for X_G.

Are these chromatic?

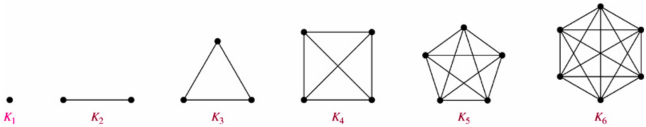
Question: Are classical symmetric functions ever examples of chromatic symmetric functions of a connected graph?

Answer:

Theorem (Cho-vW 2018)

Only the elementary symmetric functions, namely

$$e_N = \frac{1}{N!} X_{K_N}.$$

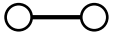


New bases

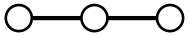
Pick favourite connected graph on 1 vertex:

$G_1 =$ 

Pick favourite connected graph on 2 vertices:

$G_2 =$ 

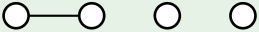
Pick favourite connected graph on 3 vertices:

$G_3 =$ 

And so on ...

Let G_λ be the disjoint union $G_{\lambda_1} \cup \cdots \cup G_{\lambda_\ell}$.

Example

$G_{211} =$ 

New bases

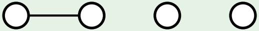
Theorem (Cho-vW 2016)

$$\Lambda = \mathbb{Q}[X_{G_1}, X_{G_2}, \ldots] \qquad \Lambda^N = \mathrm{span}_{\mathbb{Q}}\{X_{G_\lambda} \mid \lambda \vdash N\}$$

where

$$X_{G_\lambda} = X_{G_{\lambda_1}} \cdots X_{G_{\lambda_\ell}}.$$

Example

$G_{211} =$ 

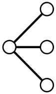
$$X_{G_{211}} = X_{G_2} X_{G_1} X_{G_1}$$
$$= 2e_2 e_1 e_1 = 2e_{211}$$

e-positivity and Schur-positivity

G is e-positive if X_G is a positive linear combination of e_λ.

G is Schur-positive if X_G is a positive linear combination of s_λ.

 has $X_G = e_{21} + 3e_3$ ✔ $X_G = 4s_{111} + s_{21}$ ✔

 has $X_G = e_{211} - 2e_{22} + 5e_{31} + 4e_4$ ✘ $X_G = 8s_{1111} + 5s_{211} - s_{22} + s_{31}$ ✘

K_{13} : Smallest graph that is not e-positive. Smallest graph that is not Schur-positive.

e-positivity and Schur-positivity

For $\lambda = \lambda_1 \cdots \lambda_\ell$

$$e_\lambda = \sum_\mu K_{\mu\lambda} s_{\mu^t}$$

where $K_{\mu\lambda} = \#$ SSYTs of shape μ filled with λ_1 1s, ..., λ_ℓ ℓs, and μ^t is the transpose of μ along the downward diagonal.

Hence $K_{\mu\lambda} \geq 0$ and

e-positivity implies Schur-positivity.

Example

$$e_{21} = s_{21} + s_{111}$$

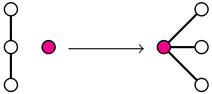
1	1
2	

1	1	2

e-positivity and Schur-positivity

Conjecture (Stanley-Stembridge 1993)

If G is an incomparability graph of a $(3+1)$-free poset then X_G is e-positive.

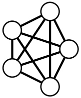


Theorem (Gasharov 1996)

If G is an incomparability graph of a $(3+1)$-free poset then X_G is Schur-positive.

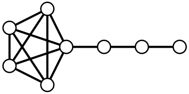
Known e-positive graphs

- Complete graphs K_m.

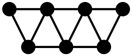


- Paths P_n (Stanley 1995).

- Lollipop graphs $L_{m,n}$ (Gebhard-Sagan 2001).

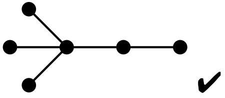


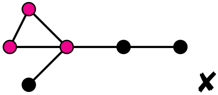
- Triangular ladders (Dahlberg 2018).



- Complement of G is bipartite (Stanley-Stembridge 1993).

e-positivity of trees: Dahlberg, She, vW 2020





N	1	2	3	4	5	6	7	8	9	10	11	12	13
trees	1	1	1	2	3	6	11	23	47	106	235	551	1301
e-pos	1	1	1	1	2	1	3	1	2	2	5	1	4

e-positivity of trees

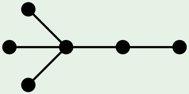
Theorem (Dahlberg-She-vW 2020)

Any tree with N vertices and a vertex of degree

$$d \geq \log_2 N + 1$$

is not e-positive.

Example



is not e-positive.

Conjecture (Dahlberg-She-vW 2020)

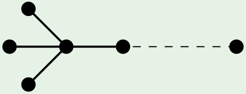
Any tree with N vertices and a vertex of degree

$$d \geq 4$$

is not e-positive.

(Zheng 2020) True for $d \geq 6$.

Example



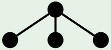
is not e-positive.

e-positivity test of Wolfgang III 1997

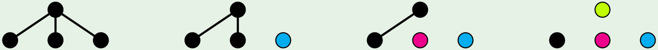
A graph has a connected partition of type $\lambda = \lambda_1 \cdots \lambda_\ell$ if we can find disjoint subsets of vertices $V_1, \ldots, V_\ell \in V(G)$ so

- $V_1 \cup \cdots \cup V_\ell = V(G)$
- restricting edges to each V_i gives connected components with λ_i vertices.

Example



has connected partitions of type $4, 31, 211$ and 1111



but is missing a connected partition of type 22.

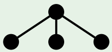
e-positivity test of Wolfgang III 1997

Theorem (Wolfgang III 1997)

If a connected graph G with N vertices is e-positive, then G has a connected partition of type λ for every partition $\lambda \vdash N$.

Test: If G does not have a connected partition of some type then G is not e-positive.

Example



does not have a connected partition of type 22. Hence it is not e-positive.

Schur-positivity of trees

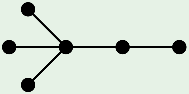
Theorem (Dahlberg-She-vW 2020)

Any tree with N vertices and a vertex of degree

$$d > \left\lceil \frac{N}{2} \right\rceil$$

is not Schur-positive.

Example



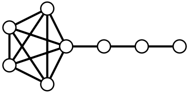
is not Schur-positive.

Why e-positivity?

- Stanley-Stembridge conjecture.
- e-positivity implies Schur-positivity.
- If Schur-positive, then it arises as the Frobenius image of some representation of a symmetric group.
- If Schur-positive, then it arises as the character of a polynomial representation of a general linear group.

Infinitely many positive bases

The lollipop graph $L_{m,n}$ is complete graph K_m connected to degree 1 vertex in path P_n.

$L_{5,3} =$ 

Different lollipop graphs have different chromatic functions.

Theorem (Dahlberg-vW 2018)

Every distinct set $\{\mathcal{L}_1, \mathcal{L}_2, \ldots\}$ where $\mathcal{L}_i = L_{m_i,n_i}$, $m_i + n_i = i$ gives distinct set of generators $\{X_{\mathcal{L}_1}, X_{\mathcal{L}_2}, \ldots\}$ such that

$$\Lambda = \mathbb{Q}[X_{\mathcal{L}_1}, X_{\mathcal{L}_2}, \ldots].$$

Questions

- Which chromatic bases are e-positive? Since a chromatic basis is e-positive iff each generator is e-positive ...

- When is X_G e-positive?

- ... or not. Stanley 1995:

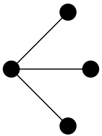
  We don't know of a graph which is not contractible to K_{13} (even regarding multiple edges of a contraction as a single edge) which is not e-positive.

  We do.


DVF

Dahlberg-Foley-vW 2020 (JEMS)

Arch-nemesis: the claw aka K_{13}

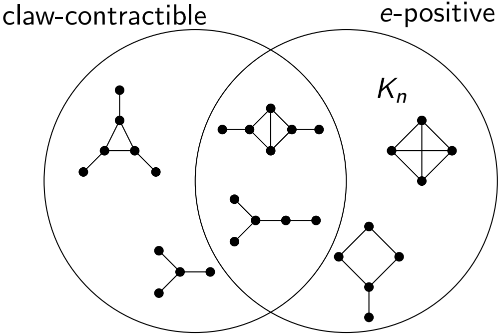


Contracts to the claw: shrinking edges + identifying vertices + removing multiple edges = claw.

A picture speaks 1000 words

Stanley 1995:

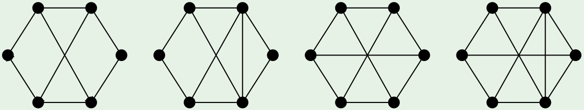
We don't know of a graph which is not contractible to K_{13} (even regarding multiple edges of a contraction as a single edge) which is not e-positive.



Claw-contractible-free: Brouwer-Veldman 1987

G is claw-contractible-free if and only if deleting all sets of 3 non-adjacent vertices gives disconnection.

Example



Claw-contractible-free: Brouwer-Veldman 1987

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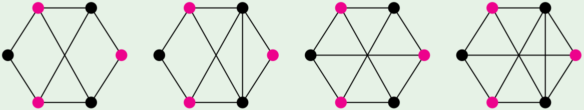
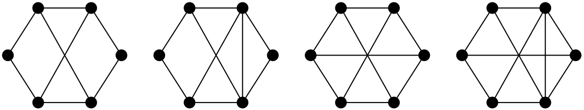
Example



...WITH CHROMATIC SYMMETRIC FUNCTION

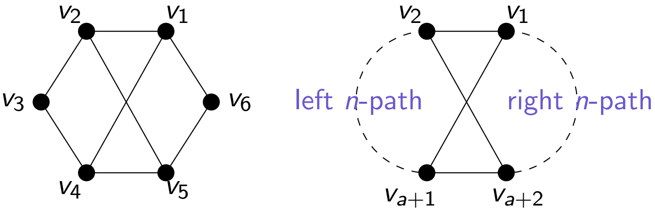


$$2e_{222} - 6e_{33} + 26e_{42} + 28e_{51} + 102e_{6}$$
$$2e_{321} - 6e_{33} + 24e_{42} + 40e_{51} + 120e_{6}$$
$$2e_{222} - 12e_{33} + 30e_{42} + 24e_{51} + 186e_{6}$$
$$2e_{321} - 6e_{33} + 20e_{42} + 32e_{51} + 228e_{6}$$

Smallest counterexamples to Stanley's statement.

Infinite family: saltire graphs

The saltire graph $SA_{n,n}$ for $n \geq 3$ is given by



with $SA_{3,3}$ on the left.

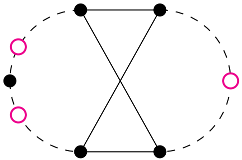
Infinite family: saltire graphs

Theorem (Dahlberg-Foley-vW 2020)

$SA_{n,n}$ for $n \geq 3$ is claw-contractible-free and

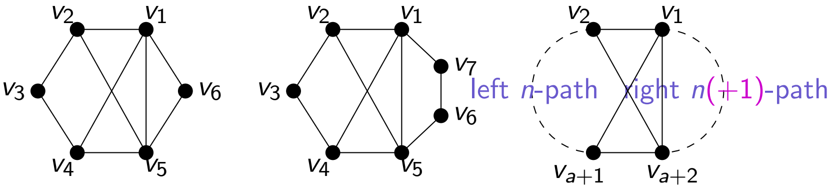
$$[e_{nn}]X_{SA_{n,n}} = -n(n-1)(n-2).$$

CCF:



For any n: augmented saltire graphs

The augmented saltire graphs $AS_{n,n}$, $AS_{n,n+1}$ for $n \geq 3$.



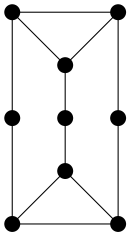
Theorem (Dahlberg-Foley-vW 2020)

$AS_{n,n}$ and $AS_{n,n+1}$ for $n \geq 3$ are claw-contractible-free and

$$[e_{nn}]X_{AS_{n,n}} = [e_{(n+1)n}]X_{AS_{n,n+1}} = -n(n-1)(n-2).$$

Claw-free: Beineke 1970

Claw-free: does not contain the claw as an induced subgraph of the graph.



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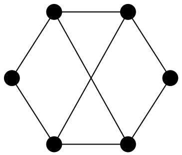

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Claw-free: Beineke 1970

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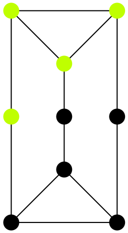

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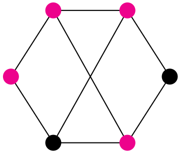

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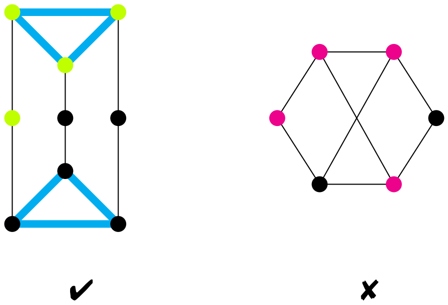
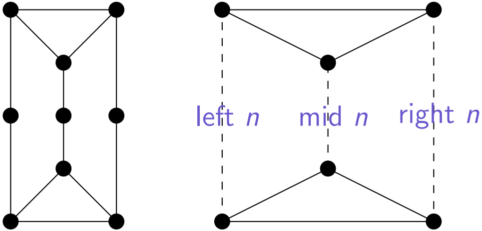
Claw-free: Beineke 1970

G is claw-free if there exists an edge partition giving complete graphs, every vertex in at most two.



And claw-free: triangular tower graphs

The triangular tower graph $TT_{n,n,n}$ for $n \geq 3$ is given by



with $TT_{3,3,3}$ on the left.

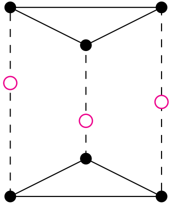
And claw-free: triangular tower graphs

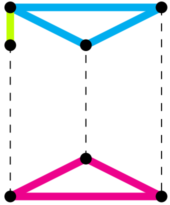
Theorem (Dahlberg-Foley-vW 2020)

$TT_{n,n,n}$ for $n \geq 3$ is claw-contractible-free, claw-free and

$$[e_{nnn}]X_{TT_{n,n,n}} = -n(n-1)^2(n-2).$$

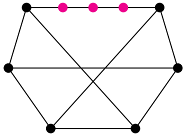
CCF+CF:





Conjectures

1. Bloated $K_{3,3}$:



with $3n$ vertices has

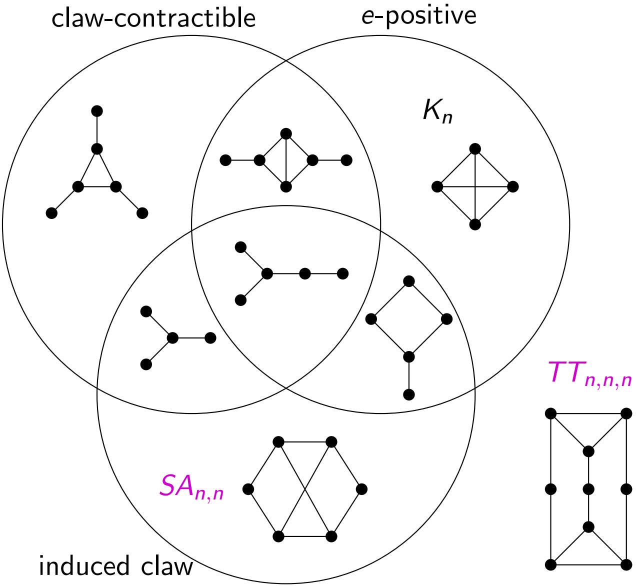
$$-(3 \times 2^n)e_{3^n}.$$

2. No G exists that is connected, claw-contractible-free, claw-free and not e-positive with 10, 11 vertices.

Scarcity

- $N = 6$: 4 of 112 connected graphs ccf and not e-positive.
- $N = 7$: 7 of 853 connected graphs ccf and not e-positive.
- $N = 8$: 27 of 11117 connected graphs ccf and not e-positive.
- Of 293 terms in $TT_{7,7,7}$ only $-$ve at e_{777}.
- Of 564 terms in $TT_{8,8,8}$ only $-$ves at e_{888} and $-1944e_{444444}$.
- Of 1042 terms in $TT_{9,9,9}$ only $-$ves at e_{999}, $-768e_{333333333}$.

A picture speaks 1000 words



In general, *e*-positivity has nothing to do with the claw.

Thank you very much!