New generalizations of the Foulkes Conjecture on q–binomial coefficients

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January 10, 2024

I. Lattice paths and q-binomial coefficients

II. The Foulkes Conjecture for q-binomial coefficients

III. New generalizations: Inequalities and asymptotics

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- How many from the origin to (a, b)?
  $\binom{a+b}{a}$ ("$a + b$ choose a").
  - This is the number of size-a subsets of a size-$(a + b)$ set.
  - Also the number of binary strings with a 0's and b 1's.
  - $\binom{a+b}{a} = \frac{(a+b)!}{a!\,b!}$.

I. Lattice paths and q-binomial coefficients

- A lattice path can also be interpreted as the diagram of an integer partition: a way of writing a positive integer as a sum of positive integers.
- Lattice paths from $(0, 0)$ to (a, b) correspond to partitions whose diagrams fit inside an $a \times b$ rectangle.
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$$1 + q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6 = \binom{5}{2}_q .$$

- $\binom{a+b}{a}_q$ denotes the polynomial in q where the coefficient of q^m gives the number of paths from $(0, 0)$ to (a, b) with area m. These polynomials are called q-binomial coefficients or Gaussian coefficients.

I. Lattice paths and q-binomial coefficients

The q-binomial coefficients have q-analogs of many properties of the binomial coefficients. In each one, setting $q = 1$ recovers the binomial coefficient property.

- Define the "q-integer" $[k]_q = 1 + q + \cdots + q^{k-1}$, and define the "q-factorial" $[n]_q! = [1]_q [2]_q \cdots [n]_q$. Then

$$\binom{a+b}{a}_q = \frac{[a+b]_q!}{[a]_q!\,[b]_q!}.$$

- $\binom{n}{k}_q = \binom{n-1}{k}_q + q^{n-k}\binom{n-1}{k-1}_q.$
- $\prod_{i=0}^{n-1}(1 + q^i x) = \sum_{k=0}^{n} q^{k(k-1)/2}\binom{n}{k}_q x^k.$
- If we set q to be the order of a finite field F, then $\binom{n}{k}_q$ is the number of k-dimensional subspaces of an n-dimensional vector space over F.

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II. The Foulkes Conjecture for q-binomial coefficients

When does $\binom{c+d}{c}_q - \binom{a+b}{a}_q$ have non-negative coefficients?

- $\binom{a+b}{a}_q$ has degree ab, so it makes sense to compare $\binom{a+b}{a}_q$ and $\binom{c+d}{c}_q$ if $ab = cd$.

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Foulkes Conjecture for q–binomial coefficients (F. Bergeron 2016): If $ab = cd$ and $a \leqslant b, c, d$, then $\binom{c+d}{c}_q - \binom{a+b}{a}_q$ has non-negative coefficients.

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- The combinatorial interpretation is that the number of size-n partitions inside a given rectangle is at least the number for a skinnier rectangle of the same area.
- Zanello 2018 proved it for $a = 3$ using Zeilberger's KOH theorem.

I. The Foulkes Conjecture and prior generalizations

That is a specialization of a conjecture on symmetric functions:

Generalized Foulkes Conjecture (for symmetric functions)
(Rebecca Vessenes 2004): If $ab = cd$ and $a \leqslant b, c, d$, then
$h_c \circ h_d - h_a \circ h_b$ is Schur positive.

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- The equivalent statements from representation theory are:
  (1) $1\uparrow_{S_a \wr S_b}^{S_{ab}}$ is a subrepresentation of $1\uparrow_{S_c \wr S_d}^{S_{cd}}$
  (2) The $GL(V)$-module $S^a(S^b(V))$ is a submodule of $S^c(S^d(V))$
- Specializes to q-binomial version since
  $$h_a \circ h_b \circ (1+q) = \binom{a+b}{a}_q.$$
- François Bergeron has several conjectures that refine the one by Vessenes.
- Original: if $a < b$ then $h_b \circ h_a - h_a \circ h_b$ is Schur positive, conjectured by H. O. Foulkes 1950.

I. The Foulkes Conjecture and prior generalizations

Foulkes Conjecture for q–binomial coefficients (F. Bergeron 2016): If $ab = cd$ and $a \leqslant b, c, d$, then $\binom{c+d}{c}_q - \binom{a+b}{a}_q$ has non-negative coefficients.

A related result is the q-unimodality of $\binom{n}{k}_q$ for fixed n and varying k:

Theorem (Lynne Butler 1987): If $k \leqslant n/2$, then $\binom{n}{k}_q - \binom{n}{k-1}_q$ has non-negative coefficients.

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For what values of a, b, c, d does $\binom{c+d}{c}_q - \binom{a+b}{a}_q$ have non-negative coefficients?

Without loss of generality, $a \leqslant b$ and $c \leqslant d$.

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- Easy necessary conditions: $ab \leqslant cd$ and $a \leqslant b, c, d$.
- Easy sufficient condition: $a \leqslant c$ and $b \leqslant d$ (the $a \times b$ rectangle is inside the $c \times d$ rectangle).
- Hard sufficient condition: $a + b = c + d = n$ and $a \leqslant b, c, d$. This is because the sequence $\binom{n}{k}_q$ is q-unimodal in k (Lynne Butler 1987).

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Conjecture (T. 2024+): The "easy necessary conditions" are also sufficient: if $ab \leqslant cd$ and $a \leqslant b, c, d$, then $\binom{c+d}{c}_q - \binom{a+b}{a}_q$ has non-negative coefficients.

III. New generalizations: Inequalities and asymptotics

Conjecture (T. 2024+): If $ab \leqslant cd$ and $a \leqslant b, c, d$, then

$$\binom{c+d}{c}_q - \binom{a+b}{a}_q$$ has non-negative coefficients.

- The $ab \leqslant cd$ version implies the $ab = cd$ version. Does the $ab = cd$ version imply the $ab \leqslant cd$ version?

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- The $ab \leqslant cd$ version implies the $ab = cd$ version. Does the $ab = cd$ version imply the $ab \leqslant cd$ version?
- Fake proof of the $ab \leqslant cd$ version assuming the $ab = cd$ version: setting $b' = cd/a$, we get $b \leqslant b'$ and so

$$\binom{a+b}{a}_q \leqslant_q \binom{a+b'}{a}_q \leqslant_q \binom{c+d}{c}_q.$$

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$$\binom{a+b}{a}_q \leqslant_q \binom{a+b'}{a}_q \leqslant_q \binom{c+d}{c}_q.$$

  - The flaw: b' is not always an integer! Indeed, sometimes there are not integers a', b' such that $a \leqslant a'$ and $b \leqslant b'$ and $a'b' = cd$.

III. New generalizations: Inequalities and asymptotics

Some parameters covered by my conjecture that are easy consequences of the $ab = cd$ version and Butler's q-unimodality:

a	b	c	d	ab	cd
2	4	3	3	8	9
3	5	4	4	15	16
3	6	4	5	18	20
3	8	5	5	24	25

Some parameters covered by my conjecture that are not easy consequences of those things:

a	b	c	d	ab	cd
2	7	3	5	14	15
3	9	4	7	27	28
4	11	5	9	44	45

II. New generalizations: Inequalities and asymptotics

What about $h_c \circ h_d - h_a \circ h_b$ when $ab \leqslant cd$?

- $h_c \circ h_d$ and $h_a \circ h_b$ are homogeneous with different degrees when $ab \neq cd$, so it does not make sense to compare them directly.
- However, we can "dehomogenize" them by setting $x_1 = 1$.

Conjecture (T. 2024+): If $ab \leqslant cd$ and $a \leqslant b, c, d$, then $h_c \circ h_d \circ (1 + h_1) - h_a \circ h_b \circ (1 + h_1)$ is Schur positive.

III. New generalizations: Inequalities and asymptotics

What if our paths were made of length-$(1/t)$ steps instead of the usual length-1 steps? This would fix the fake proof by no longer requiring the parameters to be integers.

- Equivalently: instead of looking at area-n paths to (a, b), we can look at area-t^2n paths to (ta, tb).

Let $N(a, b, n)$ be the coefficient of q^n in $\binom{a+b}{a}_q$, i.e. the number of area-n lattice paths from $(0, 0)$ to (a, b).

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Conjecture (almost theorem) (T. 2024+): If $ab \leqslant cd$ and $a < b, c, d$, then $N(ta, tb, t^2n) = o(N(tc, td, t^2n))$ as $t \to \infty$.

- The proof uses asymptotic formulas for $N(a, b, n)$ from Melczer, Panova, & Pemantle 2020 — their results are under the conditions $a/b \to A$ and $n/a^2 \to B$ for constants A and B.

III. New generalizations: Inequalities and asymptotics

Related known results:

- Lajos Takács 1986: Let $\theta(a, b)$ be the size of a partition chosen uniformly at random from those inside the $a \times b$ rectangle. Then, as $a, b \to \infty$, the distribution of $\theta(a, b)$ is asymptotic to a normal distribution with mean $\frac{ab}{2}$ and variance $\frac{ab(a+b+1)}{12}$.
  - A local limit theorem is also proved when $\left|n - \frac{1}{2}ab\right| = O\left(\sqrt{ab(a+b)}\right)$.
- Pak & Panova 2017: Let $N(a, b, n)$ be the number of size-n partitions inside an $a \times b$ rectangle. If $n \leqslant ab/2$, then

$$N(a, b, n) - N(a, b, n-1) > A \cdot s^{-9/4} 2^{\sqrt{s}},$$

  where $s = \min\{2n, a^2\}$.
  - This is a stronger version of the fact (Sylvester 1878) that $N(a, b, n) - N(a, b, n-1) \geqslant 0$.