Concepts of Dimension for Convex Geometries

William T. Trotter
wtt.math@gmail.com
Georgia Tech Mathematics
College of Sciences

MSU Combinatorics and Graph Theory Seminar October 4, 2023

Co-Author

Kolja Knauer
Universitat de Barcelona

Separating Parameters - A Modern Research Theme

- Let f and g be monotonic parameters defined on a class of structures, such as graphs, posets, lattices, etc.

Separating Parameters - A Modern Research Theme

- Let f and g be monotonic parameters defined on a class of structures, such as graphs, posets, lattices, etc.
- When is f bounded in terms of g?

Separating Parameters - A Modern Research Theme

- Let f and g be monotonic parameters defined on a class of structures, such as graphs, posets, lattices, etc.
- When is f bounded in terms of g?
- We say that f can be separated from g when there is an infinite sequence of structures on which g is bounded and f is not.

Separating Parameters - A Modern Research Theme

- Let f and g be monotonic parameters defined on a class of structures, such as graphs, posets, lattices, etc.
- When is f bounded in terms of g?
- We say that f can be separated from g when there is an infinite sequence of structures on which g is bounded and f is not.
- Perhaps the single most widely studied example are classes of graphs with f being chromatic number and g being clique number.

Dushnik-Miller Dimension—Abreviated to Dimension

- Let $\sigma = (L_1, \ldots, L_n)$ be a sequence of linear orders on the ground set of a poset P. For $x, y \in P$, let $q(x, y, \sigma)$ be the 0–1 sequence with coordinate i set to 1 if and only if $x \leq y$ in L_i.

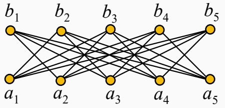
Dushnik-Miller Dimension—Abreviated to Dimension

- Let $\sigma = (L_1, \ldots, L_n)$ be a sequence of linear orders on the ground set of a poset P. For $x, y \in P$, let $q(x, y, \sigma)$ be the 0–1 sequence with coordinate i set to 1 if and only if $x \leq y$ in L_i.
- We say σ is a Dushnik-Miller realizer of P if $x \leq y$ in P if and only if $q(x, y, \sigma)$ is the constant 0–1 sequence $(1, \ldots, 1)$ of length n.

Dushnik-Miller Dimension—Abreviated to Dimension

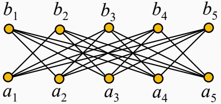
- Let $\sigma = (L_1, \ldots, L_n)$ be a sequence of linear orders on the ground set of a poset P. For $x, y \in P$, let $q(x, y, \sigma)$ be the 0–1 sequence with coordinate i set to 1 if and only if $x \leq y$ in L_i.
- We say σ is a Dushnik-Miller realizer of P if $x \leq y$ in P if and only if $q(x, y, \sigma)$ is the constant 0–1 sequence $(1, \ldots, 1)$ of length n.
- The dimension of P, denoted $\dim(P)$, is the least positive integer n for which P has a Dushnik-Miller realizer $\sigma = (L_1, \ldots, L_n)$ of length n.

Dimension and Standard Example Number



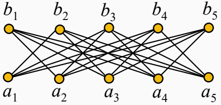
- For $n \geq 2$, the standard example S_n is the height 2 poset with minimal elements $a_1, \ldots, a_n$, maximal elements $b_1, \ldots, b_n$, and $a_i < b_j$ in S_n if and only $i \neq j$.

Dimension and Standard Example Number



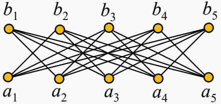
- For $n \geq 2$, the standard example S_n is the height 2 poset with minimal elements $a_1, \ldots, a_n$, maximal elements $b_1, \ldots, b_n$, and $a_i < b_j$ in S_n if and only $i \neq j$.
- The dimension of S_n is n.

Dimension and Standard Example Number



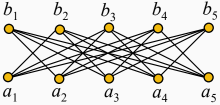
- For $n \geq 2$, the standard example S_n is the height 2 poset with minimal elements $a_1, \ldots, a_n$, maximal elements $b_1, \ldots, b_n$, and $a_i < b_j$ in S_n if and only $i \neq j$.
- The dimension of S_n is n.
- The standard example number of a poset P, denoted $\mathrm{se}(P)$, is 1 if P does not contain S_2;

Dimension and Standard Example Number



- For $n \geq 2$, the standard example S_n is the height 2 poset with minimal elements $a_1, \ldots, a_n$, maximal elements $b_1, \ldots, b_n$, and $a_i < b_j$ in S_n if and only $i \neq j$.
- The dimension of S_n is n.
- The standard example number of a poset P, denoted $\mathrm{se}(P)$, is 1 if P does not contain S_2; otherwise, $\mathrm{se}(P)$ is the largest $n \geq 2$ such that P contains the standard example S_n.

Dimension and Standard Example Number



- For $n \geq 2$, the standard example S_n is the height 2 poset with minimal elements $a_1, \ldots, a_n$, maximal elements $b_1, \ldots, b_n$, and $a_i < b_j$ in S_n if and only $i \neq j$.
- The dimension of S_n is n.
- The standard example number of a poset P, denoted $\mathrm{se}(P)$, is 1 if P does not contain S_2; otherwise, $\mathrm{se}(P)$ is the largest $n \geq 2$ such that P contains the standard example S_n.
- $\dim(P) \geq \mathrm{se}(P)$ for all posets P.

Separating Dimension and Standard Example Number

- Dushnik-Miller 1941 For every $d \geq 3$, there is a poset P with $\mathrm{se}(P) = 3$ and $\dim(P) > d$.

Separating Dimension and Standard Example Number

- Dushnik-Miller 1941 For every $d \geq 3$, there is a poset P with $\mathrm{se}(P) = 3$ and $\dim(P) > d$.
- Bogart, Rabinovitch and WTT 1975 For every $d \geq 3$, there is a poset P with $\mathrm{se}(P) = 1$ and $\dim(P) > d$.

Separating Dimension and Standard Example Number

- Dushnik-Miller 1941 For every $d \geq 3$, there is a poset P with $\mathrm{se}(P) = 3$ and $\dim(P) > d$.
- Bogart, Rabinovitch and WTT 1975 For every $d \geq 3$, there is a poset P with $\mathrm{se}(P) = 1$ and $\dim(P) > d$.
- Füredi, Hajnal, Rödl and WTT 1991 If P is a poset with $\mathrm{se}(P) = 1$ and $\dim(P) > d$, then the height of P is $\Omega(\log \log d)$.

Separating Dimension and Standard Example Number

- Dushnik-Miller 1941 For every $d \geq 3$, there is a poset P with $\mathrm{se}(P) = 3$ and $\dim(P) > d$.
- Bogart, Rabinovitch and WTT 1975 For every $d \geq 3$, there is a poset P with $\mathrm{se}(P) = 1$ and $\dim(P) > d$.
- Füredi, Hajnal, Rödl and WTT 1991 If P is a poset with $\mathrm{se}(P) = 1$ and $\dim(P) > d$, then the height of P is $\Omega(\log \log d)$.
- Felsner and WTT 2000 For every $d \geq 3$, there is a height 2 poset P with $\mathrm{se}(P) = 2$ and $\dim(P) > d$.

Separating Dimension and Standard Example Number

- Dushnik-Miller 1941 For every $d \geq 3$, there is a poset P with $\mathrm{se}(P) = 3$ and $\dim(P) > d$.
- Bogart, Rabinovitch and WTT 1975 For every $d \geq 3$, there is a poset P with $\mathrm{se}(P) = 1$ and $\dim(P) > d$.
- Füredi, Hajnal, Rödl and WTT 1991 If P is a poset with $\mathrm{se}(P) = 1$ and $\dim(P) > d$, then the height of P is $\Omega(\log\log d)$.
- Felsner and WTT 2000 For every $d \geq 3$, there is a height 2 poset P with $\mathrm{se}(P) = 2$ and $\dim(P) > d$.
- Regardless, it is of interest to investigate classes of posets for which dimension is bounded in terms of standard example number.

- Baker, Fishburn and Roberts 1970 If P is a poset with a 0 and a 1, and the order diagram of P is planar, then $\dim(P) \leq 2$.

Dimension and Posets with Planar Cover Graphs I

- Baker, Fishburn and Roberts 1970 If P is a poset with a 0 and a 1, and the order diagram of P is planar, then $\dim(P) \leq 2$.
- WTT 1978 For every $d \geq 1$, there is a poset P with a 0 and a 1 such that the cover graph of P is planar, and $\dim(P) > d$.

Dimension and Posets with Planar Cover Graphs I

- Baker, Fishburn and Roberts 1970 If P is a poset with a 0 and a 1, and the order diagram of P is planar, then $\dim(P) \leq 2$.
- WTT 1978 For every $d \geq 1$, there is a poset P with a 0 and a 1 such that the cover graph of P is planar, and $\dim(P) > d$.
- Kelly 1981 For every $d \geq 1$, there is a poset P with a planar order diagram such that $\dim(P) > d$.

Dimension and Posets with Planar Cover Graphs I

- Baker, Fishburn and Roberts 1970 If P is a poset with a 0 and a 1, and the order diagram of P is planar, then $\dim(P) \leq 2$.
- WTT 1978 For every $d \geq 1$, there is a poset P with a 0 and a 1 such that the cover graph of P is planar, and $\dim(P) > d$.
- Kelly 1981 For every $d \geq 1$, there is a poset P with a planar order diagram such that $\dim(P) > d$.
- Conjecture In the class of posets with planar cover graphs, dimension is bounded in terms of standard example number.

Resolution of the Conjecture

- Theorem (Blake, Hodor, Micek, Seweryn and WTT, 2023+) If P is a poset with a planar cover graph, then $\dim(P) = O((\mathrm{se}(P)^6)$.

Resolution of the Conjecture

- Theorem (Blake, Hodor, Micek, Seweryn and WTT, 2023+) If P is a poset with a planar cover graph, then $\dim(P) = O((\text{se}(P)^6)$.
- Theorem (Blake, Hodor, Micek, Seweryn and WTT, 2023+) If P is a poset with a planar order diagram, then $\dim(P) = O(\text{se}(P))$.

Convex Geometries

- Let X be a finite set. A family P of subsets of X is called a convex geometry with ground set X if

Convex Geometries

- Let X be a finite set. A family P of subsets of X is called a convex geometry with ground set X if
  1. $\emptyset, X \in P$;

Convex Geometries

- Let X be a finite set. A family P of subsets of X is called a convex geometry with ground set X if
    1. $\emptyset, X \in P$;
    2. $A \cap B \in P$ whenever $A, B \in P$; and

Convex Geometries

- Let X be a finite set. A family P of subsets of X is called a convex geometry with ground set X if
  1. $\emptyset, X \in P$;
  2. $A \cap B \in P$ whenever $A, B \in P$; and
  3. $A \in P$ and $A \neq X$, there is an element $\alpha \in X - A$ such that $A \cup \{\alpha\} \in P$.

Convex Geometries

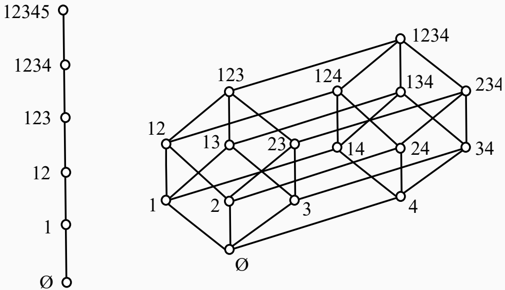
- Let X be a finite set. A family P of subsets of X is called a convex geometry with ground set X if
  1. $\emptyset, X \in P$;
  2. $A \cap B \in P$ whenever $A, B \in P$; and
  3. $A \in P$ and $A \neq X$, there is an element $\alpha \in X - A$ such that $A \cup \{\alpha\} \in P$.
- A convex geometry is a lattice, with $A \wedge B = A \cap B$, and $A \vee B = \cap\{C \in P : A \cup B \subseteq C\}$.

Convex Geometries

- Let X be a finite set. A family P of subsets of X is called a convex geometry with ground set X if
  1. $\emptyset, X \in P$;
  2. $A \cap B \in P$ whenever $A, B \in P$; and
  3. $A \in P$ and $A \neq X$, there is an element $\alpha \in X - A$ such that $A \cup \{\alpha\} \in P$.
- A convex geometry is a lattice, with $A \wedge B = A \cap B$, and $A \vee B = \cap\{C \in P : A \cup B \subseteq C\}$.
- Note that P is closed under intersections, but may not be closed under unions.

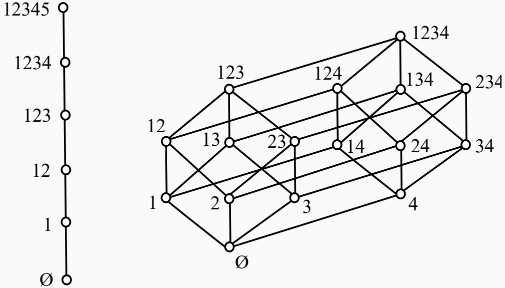
Convex Geometries

- Let X be a finite set. A family P of subsets of X is called a convex geometry with ground set X if
  1. $\emptyset, X \in P$;
  2. $A \cap B \in P$ whenever $A, B \in P$; and
  3. $A \in P$ and $A \neq X$, there is an element $\alpha \in X - A$ such that $A \cup \{\alpha\} \in P$.
- A convex geometry is a lattice, with $A \wedge B = A \cap B$, and $A \vee B = \cap\{C \in P : A \cup B \subseteq C\}$.
- Note that P is closed under intersections, but may not be closed under unions.
- Convex geometries have also been called *anti-matroids*.



- On the left, we show a linear geometry.

Two Important Special Cases of a Convex Geometry



- On the left, we show a linear geometry.
- On the right, we show a Boolean algebra. These are also called subset lattices.

Distributive Lattices I

- Distributive lattices are the convex geometries that are closed under unions.

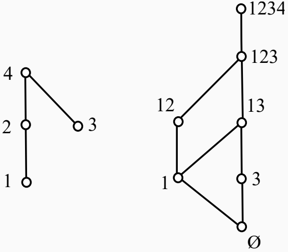
Distributive Lattices I

- Distributive lattices are the convex geometries that are closed under unions.
- Equivalently, distributive lattices are the convex geometries that result when X is a finite poset, and P consists of all down sets of X.

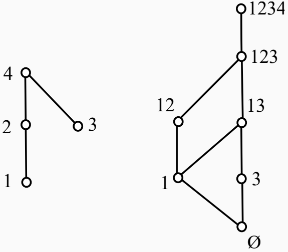
Distributive Lattices I

- Distributive lattices are the convex geometries that are closed under unions.
- Equivalently, distributive lattices are the convex geometries that result when X is a finite poset, and P consists of all down sets of X.
- Linear geometries and Boolean algebras are distributive lattices.

Distributive Lattices II

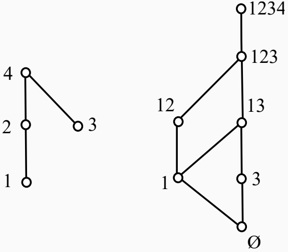


- On the left, we show a poset X with ground set $\{1, 2, 3, 4\}$.



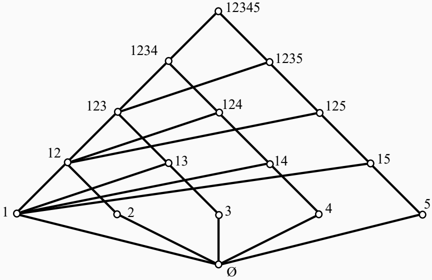
- On the left, we show a poset X with ground set $\{1, 2, 3, 4\}$.
- On the right, we show the family of all down sets of X, ordered by inclusion.

Distributive Lattices II



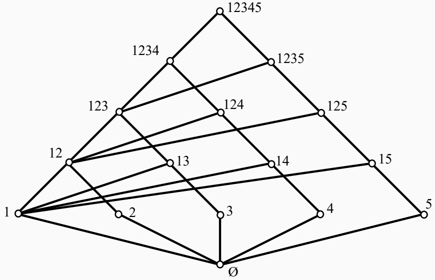
- On the left, we show a poset X with ground set $\{1, 2, 3, 4\}$.
- On the right, we show the family of all down sets of X, ordered by inclusion.
- This distributive lattice is neither a linear geometry nor a Boolean algebra.

An Example of a Convex Geometry I



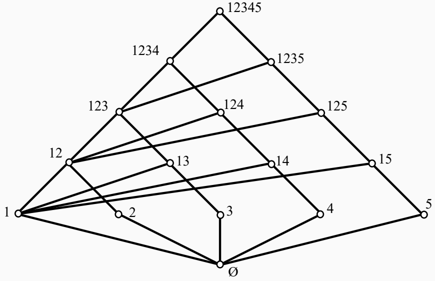
- The ground set X is $\{1, 2, 3, 4, 5\}$.

An Example of a Convex Geometry I



- The ground set X is $\{1, 2, 3, 4, 5\}$.
- The sets $\{1, 3\}$ and $\{5\}$ show that P is *not* a distributive lattice, i.e., P is not closed under unions.

An Example of a Convex Geometry I



- The ground set X is $\{1, 2, 3, 4, 5\}$.
- The sets $\{1, 3\}$ and $\{5\}$ show that P is *not* a distributive lattice, i.e., P is not closed under unions.
- Note that the maximum up-degree is 5, and the maximum down-degree is 2.

The Wedge Operation on Convex Geometries

- If $n \geq 1$ and P_i is a convex geometry with ground set X for each $i \in [n]$, then the family of all sets of the form $A_1 \cap \cdots \cap A_n$, where $A_i \in P_i$ for each $i \in [n]$, is a convex geometry with ground set X.

The Wedge Operation on Convex Geometries

- If $n \geq 1$ and P_i is a convex geometry with ground set X for each $i \in [n]$, then the family of all sets of the form $A_1 \cap \cdots \cap A_n$, where $A_i \in P_i$ for each $i \in [n]$, is a convex geometry with ground set X.
- We denote this geometry as $P_1 \wedge \cdots \wedge P_n$.

The Wedge Operation on Convex Geometries

- If $n \geq 1$ and P_i is a convex geometry with ground set X for each $i \in [n]$, then the family of all sets of the form $A_1 \cap \cdots \cap A_n$, where $A_i \in P_i$ for each $i \in [n]$, is a convex geometry with ground set X.
- We denote this geometry as $P_1 \wedge \cdots \wedge P_n$.
- Note that $P_1 \wedge \cdots \wedge P_n$ is the smallest convex geometry with ground set X containing all sets in $P_1 \cup \cdots \cup P_n$.

Convex Dimension I

- When P is a convex geometry with ground set X, the convex dimension of P, denoted $\mathrm{cdim}(P)$, is the least integer n such that there are linear geometries $P_1, \ldots, P_n$, with ground set X, such that $P = P_1 \wedge \cdots \wedge P_n$.

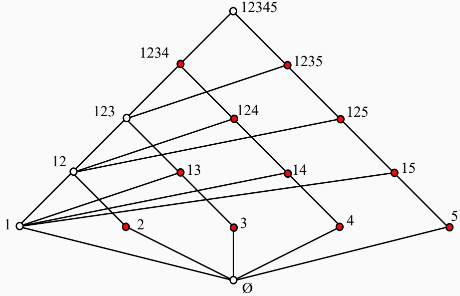
Convex Dimension I

- When P is a convex geometry with ground set X, the convex dimension of P, denoted $\mathrm{cdim}(P)$, is the least integer n such that there are linear geometries $P_1, \ldots, P_n$, with ground set X, such that $P = P_1 \wedge \cdots \wedge P_n$.
- Edelman and Jamison 1985 The convex dimension of P is the width of the subposet of P consisting of the elements that have up-degree 1.

Convex Dimension I

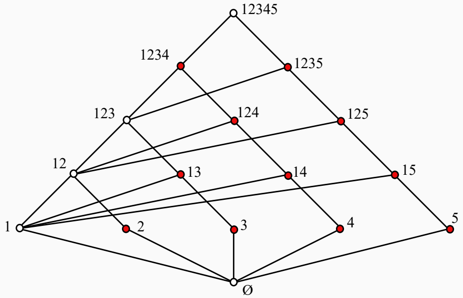
- When P is a convex geometry with ground set X, the convex dimension of P, denoted $\mathrm{cdim}(P)$, is the least integer n such that there are linear geometries $P_1, \ldots, P_n$, with ground set X, such that $P = P_1 \wedge \cdots \wedge P_n$.
- Edelman and Jamison 1985 The convex dimension of P is the width of the subposet of P consisting of the elements that have up-degree 1.
- Elements that have up-degree 1 are also called meet-irreducible elements.

An Example of a Convex Geometry II



- The width of the subposet of meet-irreducible elements is 4.

An Example of a Convex Geometry II



- The width of the subposet of meet-irreducible elements is 4.
- The convex dimension is 4.

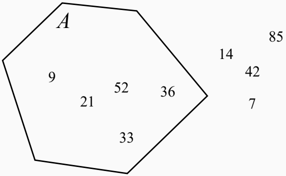
Dimension and Convex Dimension

- Dilworth 1950 The dimension of a distributive lattice is the same as its convex dimension, i.e., the width of the subposet of meet-irreducible elements.

Dimension and Convex Dimension

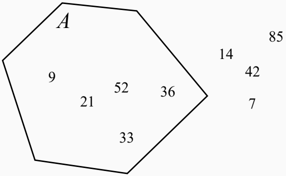
- Dilworth 1950 The dimension of a distributive lattice is the same as its convex dimension, i.e., the width of the subposet of meet-irreducible elements.
- Edelman and Jamison 1985 If P is a convex geometry, then $\dim(P) \leq \operatorname{cdim}(P)$.

VC-Dimension



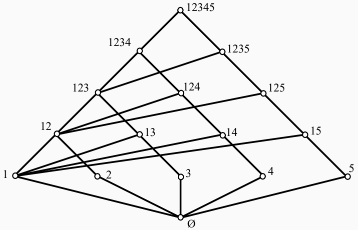
- When $\mathcal{F}$ is a family of subsets of a finite set X, the VC-dimension of $\mathcal{F}$ is the largest integer n for which there are n-elements $a_1, \ldots, a_n$ of X such that for each subset $S \subseteq [n]$, there is a set $A = A_S$ in $\mathcal{F}$ with $a_i \in A$ if and only if $i \in S$.

VC-Dimension



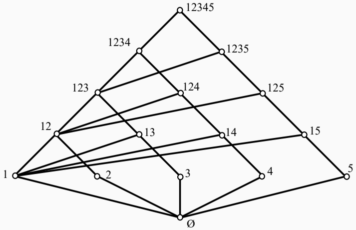
- When $\mathcal{F}$ is a family of subsets of a finite set X, the VC-dimension of $\mathcal{F}$ is the largest integer n for which there are n-elements $a_1, \ldots, a_n$ of X such that for each subset $S \subseteq [n]$, there is a set $A = A_S$ in $\mathcal{F}$ with $a_i \in A$ if and only if $i \in S$.
- If $X = [85]$, and $\mathcal{F}$ has VC-dimension 9, as evidenced by the set $\{7, 9, 14, 21, 33, 36, 42, 52, 85\}$, then 2^9 sets are required. We illustrate one of them.

An Example of a Convex Geometry II



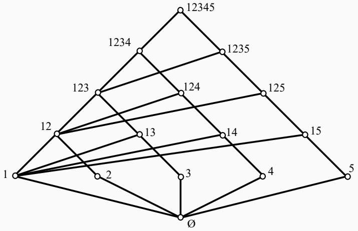
- The elements 2 and 4 together with the sets $\emptyset$, $\{1, 2\}$, $\{1, 3, 4\}$ and $\{1, 2, 3, 4\}$ show that the VC-dimension of P is at least 2.

An Example of a Convex Geometry II



- The elements 2 and 4 together with the sets $\emptyset$, $\{1, 2\}$, $\{1, 3, 4\}$ and $\{1, 2, 3, 4\}$ show that the VC-dimension of P is at least 2.
- By inspection, the VC-dimension of P is 2.

An Example of a Convex Geometry II



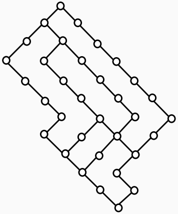
- The elements 2 and 4 together with the sets $\emptyset$, $\{1, 2\}$, $\{1, 3, 4\}$ and $\{1, 2, 3, 4\}$ show that the VC-dimension of P is at least 2.
- By inspection, the VC-dimension of P is 2.
- Challenge Show that the dimension of P is 3.

VC-Dimension and Down-Degree

- Edelman and Jamison 1985 When P is a convex-geometry, the VC-dimension of P is the maximum down-degree in P.

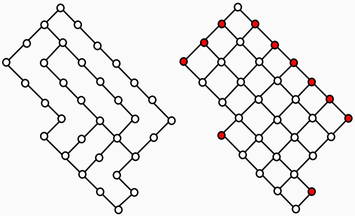
VC-Dimension and Down-Degree

- Edelman and Jamison 1985 When P is a convex-geometry, the VC-dimension of P is the maximum down-degree in P.
- Edelman and Jamison 1985 When L is a finite lattice, there is a convex geometry P such that L is isomorphic to P (as a poset) if and only if for every element y of L with y distinct from the zero of L, the interval of L between y and the meet of all elements covered by y is a Boolean algebra.



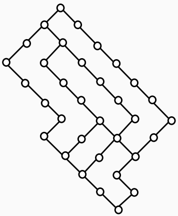
- In the figure, we show two lattices.

Recognizing Convex Geometries



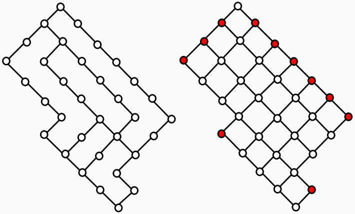
- In the figure, we show two lattices.
- Both lattices have planar order diagrams, and both have dimension 2.

Recognizing Convex Geometries



- In the figure, we show two lattices.
- Both lattices have planar order diagrams, and both have dimension 2.
- Only the lattice on the right is a convex geometry.

Recognizing Convex Geometries



- In the figure, we show two lattices.
- Both lattices have planar order diagrams, and both have dimension 2.
- Only the lattice on the right is a convex geometry.
- It has convex dimension 2.

Separating Parameters for Convex Geometries I

- There are three parameters for convex geometries that are (essentially) the same.

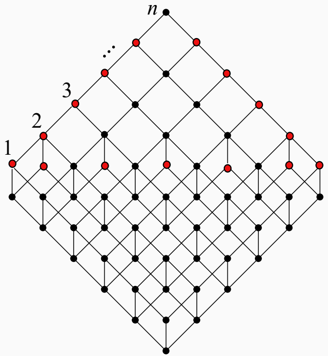
Separating Parameters for Convex Geometries I

- There are three parameters for convex geometries that are (essentially) the same.
- If P is a convex geometry, then VC-dimension is maximum down degree.

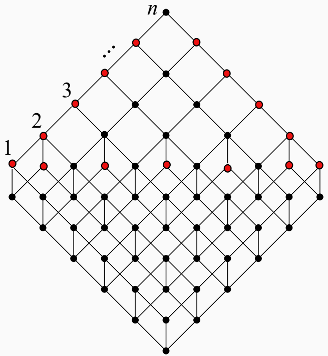
Separating Parameters for Convex Geometries I

- There are three parameters for convex geometries that are (essentially) the same.
- If P is a convex geometry, then VC-dimension is maximum down degree.
- Bandelt, Chepoi, Dress and Koolen 2006 If P is a convex geometry, then the VC-dimension of P equals $\mathrm{se}(P)$ unless the VC-dimension is 2. In this case, $\mathrm{se}(P)$ is either 1 or 2.

Knauer's Construction

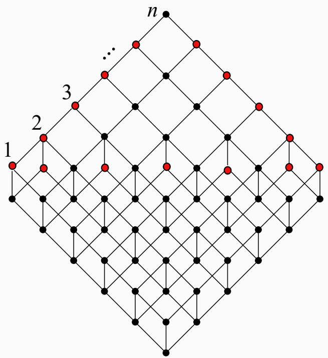


- For $n \geq 3$, the poset P_n is a convex geometry.



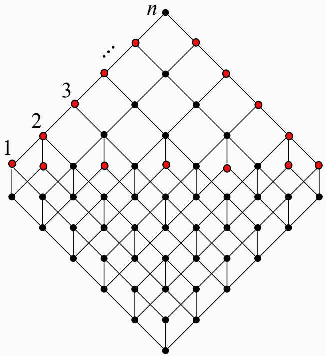
- For $n \geq 3$, the poset P_n is a convex geometry.
- It is easy to see that the dimension of P_n is 3.

Knauer's Construction



- For $n \geq 3$, the poset P_n is a convex geometry.
- It is easy to see that the dimension of P_n is 3.
- The convex dimension of P_n is $n + 1$.

Knauer's Construction



- For $n \geq 3$, the poset P_n is a convex geometry.
- It is easy to see that the dimension of P_n is 3.
- The convex dimension of P_n is $n + 1$.
- So in the class of convex geometries, convex dimension can be separated from dimension and VC-dimension.

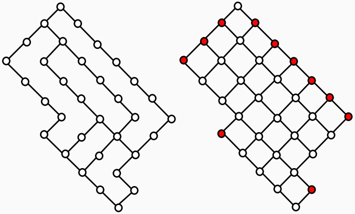
Two Natural Questions for Convex Geometries

- Can convex dimension be separated from dimension if dimension is at most 2?

Two Natural Questions for Convex Geometries

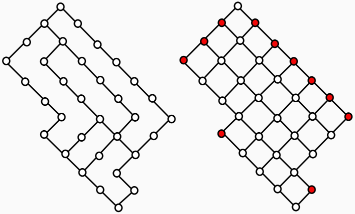
- Can convex dimension be separated from dimension if dimension is at most 2?
- Can convex dimension be separated from maximum down degree when maximum down degree is 2?

One Question has an Immediate Negative Answer



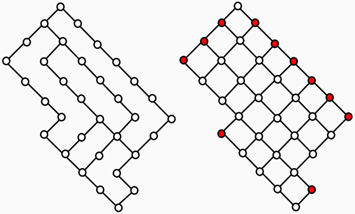
- Baker, Fishburn and Roberts 1970 If P is a finite lattice, then P has dimension at most 2 if and only if the order diagram of P is planar.

One Question has an Immediate Negative Answer



- Baker, Fishburn and Roberts 1970 If P is a finite lattice, then P has dimension at most 2 if and only if the order diagram of P is planar.
- Knauer and WTT 2023 If P is a convex geometry, and the order diagram of P is planar, then all interior faces are diamonds, and all meet-irreducible elements are on the exterior face.

One Question has an Immediate Negative Answer



- Baker, Fishburn and Roberts 1970 If P is a finite lattice, then P has dimension at most 2 if and only if the order diagram of P is planar.
- Knauer and WTT 2023 If P is a convex geometry, and the order diagram of P is planar, then all interior faces are diamonds, and all meet-irreducible elements are on the exterior face. It follows that the convex dimension is at most 2.

Answering the Second Question I

- When m is an integer, let $[m] = \{1, \ldots, m\}$ when $m \geq 1$, and let $[m] = \emptyset$ when $m \leq 0$.

Answering the Second Question I

- When m is an integer, let $[m] = \{1, \ldots, m\}$ when $m \geq 1$, and let $[m] = \emptyset$ when $m \leq 0$.
- Let k and n be integers with $1 \leq k \leq n - 2$. Then let $P(k, n)$ denote the family of all sets $A \subseteq [n]$ such that if $|A| = k + i - 1$, then $[i - 1] \subseteq A$.

Answering the Second Question I

- When m is an integer, let $[m] = \{1, \dots, m\}$ when $m \geq 1$, and let $[m] = \emptyset$ when $m \leq 0$.
- Let k and n be integers with $1 \leq k \leq n-2$. Then let $P(k,n)$ denote the family of all sets $A \subseteq [n]$ such that if $|A| = k+i-1$, then $[i-1] \subseteq A$.
- Note that $A \in P(k,n)$ whenever $A \subset [n]$ and $|A| \leq k$.

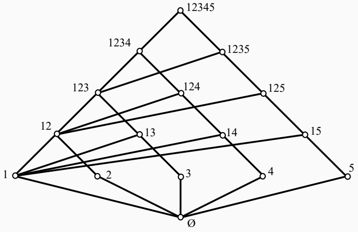
Answering the Second Question I

- When m is an integer, let $[m] = \{1, \ldots, m\}$ when $m \geq 1$, and let $[m] = \emptyset$ when $m \leq 0$.
- Let k and n be integers with $1 \leq k \leq n-2$. Then let $P(k,n)$ denote the family of all sets $A \subseteq [n]$ such that if $|A| = k+i-1$, then $[i-1] \subseteq A$.
- Note that $A \in P(k,n)$ whenever $A \subset [n]$ and $|A| \leq k$.
- $\{1,2,3,6,11\}$ and $\{1,2,6,10,11\}$ belong to $P(3,12)$, while $\{1,3,6,10,11\}$ does not.

Answering the Second Question I

- When m is an integer, let $[m] = \{1, \ldots, m\}$ when $m \geq 1$, and let $[m] = \emptyset$ when $m \leq 0$.
- Let k and n be integers with $1 \leq k \leq n-2$. Then let $P(k, n)$ denote the family of all sets $A \subseteq [n]$ such that if $|A| = k + i - 1$, then $[i-1] \subseteq A$.
- Note that $A \in P(k, n)$ whenever $A \subset [n]$ and $|A| \leq k$.
- $\{1, 2, 3, 6, 11\}$ and $\{1, 2, 6, 10, 11\}$ belong to $P(3, 12)$, while $\{1, 3, 6, 10, 11\}$ does not.
- Exercise If $1 \leq k \leq n-2$, then $P(k, n)$ is a convex geometry.

An Example of a Convex Geometry III



- This is the convex geometry $P(1,5)$.

Answering the Second Question II

- If k and n are integers with $1 \leq k \leq n-2$, then the following statements hold:
  1. If $1 \leq k < k' \leq n-2$, then $P(k,n)$ is a subposet of $P(k',n)$.
  2. $\operatorname{cdim}(P(k,n)) = \binom{n-1}{k}$.
  3. $\operatorname{maxdd}(P(k,n)) = \operatorname{se}(P(k,n)) = k+1$.
  4. $\dim(P(1,n)) = 1 + \lfloor \lg n \rfloor$,
  5. $\dim(P(k,n)) \leq (k+1)2^{k+2} \log n$.

- If k and n are integers with $1 \leq k \leq n-2$, then the following statements hold:
  1. If $1 \leq k < k' \leq n-2$, then $P(k,n)$ is a subposet of $P(k',n)$.
  2. $\mathrm{cdim}(P(k,n)) = \binom{n-1}{k}$.
  3. $\mathrm{maxdd}(P(k,n)) = \mathrm{se}(P(k,n)) = k+1$.
  4. $\dim(P(1,n)) = 1 + \lfloor \lg n \rfloor$,
  5. $\dim(P(k,n)) \leq (k+1)2^{k+2} \log n$.
- The family $\{P(1,n) : n \geq 3\}$ shows that dimension and convex dimension can be separated from VC-dimension, even when VC-dimension is 2.

Boolean Dimension

- Let $\sigma = (L_1, \ldots, L_n)$ be a sequence of linear orders on the ground set of a poset P. We say σ is a Boolean realizer of P if there is a set τ of 0–1 strings of length n such that $x < y$ in P if and only if $q(x, y, \sigma) \in \tau$.

- Let $\sigma = (L_1, \ldots, L_n)$ be a sequence of linear orders on the ground set of a poset P. We say σ is a Boolean realizer of P if there is a set τ of 0–1 strings of length n such that $x < y$ in P if and only if $q(x, y, \sigma) \in \tau$.
- The Boolean dimension of P, denoted $\mathrm{bdim}(P)$, is the least n such that P has a Boolean realizer of length n.

Boolean Dimension

- Let $\sigma = (L_1, \ldots, L_n)$ be a sequence of linear orders on the ground set of a poset P. We say σ is a Boolean realizer of P if there is a set τ of 0–1 strings of length n such that $x < y$ in P if and only if $q(x, y, \sigma) \in \tau$.
- The Boolean dimension of P, denoted $\mathrm{bdim}(P)$, is the least n such that P has a Boolean realizer of length n.
- A Dushnik-Miller realizer is a Boolean realizer, as evidenced by the set $\tau = \{(1, 1, \ldots, 1)\}$.

- Let $\sigma = (L_1, \ldots, L_n)$ be a sequence of linear orders on the ground set of a poset P. We say σ is a Boolean realizer of P if there is a set τ of 0–1 strings of length n such that $x < y$ in P if and only if $q(x, y, \sigma) \in \tau$.
- The Boolean dimension of P, denoted $\mathrm{bdim}(P)$, is the least n such that P has a Boolean realizer of length n.
- A Dushnik-Miller realizer is a Boolean realizer, as evidenced by the set $\tau = \{(1, 1, \ldots, 1)\}$.
- $\mathrm{bdim}(P) \leq \dim(P)$ for all posets P.

Boolean Dimension

- Let $\sigma = (L_1, \ldots, L_n)$ be a sequence of linear orders on the ground set of a poset P. We say σ is a Boolean realizer of P if there is a set τ of 0–1 strings of length n such that $x < y$ in P if and only if $q(x, y, \sigma) \in \tau$.
- The Boolean dimension of P, denoted $\mathrm{bdim}(P)$, is the least n such that P has a Boolean realizer of length n.
- A Dushnik-Miller realizer is a Boolean realizer, as evidenced by the set $\tau = \{(1, 1, \ldots, 1)\}$.
- $\mathrm{bdim}(P) \leq \dim(P)$ for all posets P.
- $\dim(P) = \mathrm{bdim}(P)$ when $\dim(P) \leq 3$.

Boolean Dimension

- Let $\sigma = (L_1, \ldots, L_n)$ be a sequence of linear orders on the ground set of a poset P. We say σ is a Boolean realizer of P if there is a set τ of 0–1 strings of length n such that $x < y$ in P if and only if $q(x, y, \sigma) \in \tau$.
- The Boolean dimension of P, denoted $\mathrm{bdim}(P)$, is the least n such that P has a Boolean realizer of length n.
- A Dushnik-Miller realizer is a Boolean realizer, as evidenced by the set $\tau = \{(1, 1, \ldots, 1)\}$.
- $\mathrm{bdim}(P) \leq \dim(P)$ for all posets P.
- $\dim(P) = \mathrm{bdim}(P)$ when $\dim(P) \leq 3$.
- $\mathrm{bdim}(S_n) = 4$ for all $n \geq 4$.

Fractional Dimension

- Let $\mathcal{E}(P)$ be the set of all linear extensions of a poset P, and let f be a function assigning to each $L \in \mathcal{E}(P)$ a non-negative real number $f(L)$.

Fractional Dimension

- Let $\mathcal{E}(P)$ be the set of all linear extensions of a poset P, and let f be a function assigning to each $L \in \mathcal{E}(P)$ a non-negative real number $f(L)$.
- We say f is a fractional realizer of P if for all $x, y \in P$ with $x \not< y$ in P, $\sum\{f(L) : L \in \mathcal{E}(P), x \geq y \text{ in } L\} \geq 1$.

Fractional Dimension

- Let $\mathcal{E}(P)$ be the set of all linear extensions of a poset P, and let f be a function assigning to each $L \in \mathcal{E}(P)$ a non-negative real number $f(L)$.
- We say f is a fractional realizer of P if for all $x, y \in P$ with $x \not< y$ in P, $\sum\{f(L) : L \in \mathcal{E}(P), x \geq y \text{ in } L\} \geq 1$.
- The fractional dimension of P is the least real number t for which there is a fractional realizer f with $\sum_{L\in\mathcal{E}(P)} f(L) \leq t$.

Fractional Dimension

- Let $\mathcal{E}(P)$ be the set of all linear extensions of a poset P, and let f be a function assigning to each $L \in \mathcal{E}(P)$ a non-negative real number $f(L)$.
- We say f is a fractional realizer of P if for all $x, y \in P$ with $x \not< y$ in P, $\sum\{f(L) : L \in \mathcal{E}(P), x \geq y \text{ in } L\} \geq 1$.
- The fractional dimension of P is the least real number t for which there is a fractional realizer f with $\sum_{L \in \mathcal{E}(P)} f(L) \leq t$.
- If $(L_1, \ldots, L_n)$ is a Dushnik-Miller realizer of P, then setting $f(L) = 1$ if $L = L_i$ for some $i \in [n]$; otherwise $f(L) = 0$, shows that $\operatorname{fdim}(P) \leq \dim(P)$.

Fractional Dimension

- Let $\mathcal{E}(P)$ be the set of all linear extensions of a poset P, and let f be a function assigning to each $L \in \mathcal{E}(P)$ a non-negative real number $f(L)$.
- We say f is a fractional realizer of P if for all $x, y \in P$ with $x \not< y$ in P, $\sum\{f(L) : L \in \mathcal{E}(P), x \geq y \text{ in } L\} \geq 1$.
- The fractional dimension of P is the least real number t for which there is a fractional realizer f with $\sum_{L \in \mathcal{E}(P)} f(L) \leq t$.
- If $(L_1, \ldots, L_n)$ is a Dushnik-Miller realizer of P, then setting $f(L) = 1$ if $L = L_i$ for some $i \in [n]$; otherwise $f(L) = 0$, shows that $\mathrm{fdim}(P) \leq \dim(P)$.
- Fact $\mathrm{fdim}(S_n) = n$ for all $n \geq 2$.

Fractional Dimension

- Let $\mathcal{E}(P)$ be the set of all linear extensions of a poset P, and let f be a function assigning to each $L \in \mathcal{E}(P)$ a non-negative real number $f(L)$.
- We say f is a fractional realizer of P if for all $x, y \in P$ with $x \not< y$ in P, $\sum\{f(L) : L \in \mathcal{E}(P), x \geq y \text{ in } L\} \geq 1$.
- The fractional dimension of P is the least real number t for which there is a fractional realizer f with $\sum_{L \in \mathcal{E}(P)} f(L) \leq t$.
- If $(L_1, \ldots, L_n)$ is a Dushnik-Miller realizer of P, then setting $f(L) = 1$ if $L = L_i$ for some $i \in [n]$; otherwise $f(L) = 0$, shows that $\mathrm{fdim}(P) \leq \dim(P)$.
- Fact $\mathrm{fdim}(S_n) = n$ for all $n \geq 2$.
- Brightwell and Scheinerman If $\mathrm{se}(P) = 1$, then $\mathrm{fdim}(P) < 4$.

Local Dimension

- Let $\sigma = (M_1, \ldots, M_t)$ be a sequence of linear extensions of subposets of a poset P. We say that σ is a local realizer of P if there is some $i \in [t]$ with $x \geq y$ in M_i whenever $x \not\leq y$ in P.

Local Dimension

- Let $\sigma = (M_1, \ldots, M_t)$ be a sequence of linear extensions of subposets of a poset P. We say that σ is a local realizer of P if there is some $i \in [t]$ with $x \geq y$ in M_i whenever $x \not\leq y$ in P.
- The local dimension of P, denoted $\mathrm{ldim}(P)$, is the least n such that P has a local realizer σ such that all elements of P appear in at most n different extensions of σ.

Local Dimension

- Let $\sigma = (M_1, \ldots, M_t)$ be a sequence of linear extensions of subposets of a poset P. We say that σ is a local realizer of P if there is some $i \in [t]$ with $x \geq y$ in M_i whenever $x \not\leq y$ in P.
- The local dimension of P, denoted $\mathrm{ldim}(P)$, is the least n such that P has a local realizer σ such that all elements of P appear in at most n different extensions of σ.
- A Dushnik-Miller realizer is a local realizer. Therefore, $\mathrm{ldim}(P) \leq \dim(P)$ for all posets P.

Local Dimension

- Let $\sigma = (M_1, \ldots, M_t)$ be a sequence of linear extensions of subposets of a poset P. We say that σ is a local realizer of P if there is some $i \in [t]$ with $x \geq y$ in M_i whenever $x \not\leq y$ in P.
- The local dimension of P, denoted $\mathrm{ldim}(P)$, is the least n such that P has a local realizer σ such that all elements of P appear in at most n different extensions of σ.
- A Dushnik-Miller realizer is a local realizer. Therefore, $\mathrm{ldim}(P) \leq \dim(P)$ for all posets P.
- If $\dim(P) \leq 3$, then $\mathrm{ldim}(P) = \dim(P)$.

Local Dimension

- Let $\sigma = (M_1, \ldots, M_t)$ be a sequence of linear extensions of subposets of a poset P. We say that σ is a local realizer of P if there is some $i \in [t]$ with $x \geq y$ in M_i whenever $x \not\leq y$ in P.
- The local dimension of P, denoted $\mathrm{ldim}(P)$, is the least n such that P has a local realizer σ such that all elements of P appear in at most n different extensions of σ.
- A Dushnik-Miller realizer is a local realizer. Therefore, $\mathrm{ldim}(P) \leq \dim(P)$ for all posets P.
- If $\dim(P) \leq 3$, then $\mathrm{ldim}(P) = \dim(P)$.
- $\mathrm{ldim}(S_n) = n$ for all $n \geq 3$.

Concepts of Dimension for Convex Geometries

- For fixed k, both the Boolean dimension and the local dimension of $P(k,n)$ tends to infinity.

Concepts of Dimension for Convex Geometries

- For fixed k, both the Boolean dimension and the local dimension of $P(k, n)$ tends to infinity.
- For fixed k, the fractional dimension of $P(k, n)$ is less than 2^k.

Concepts of Dimension for Convex Geometries

- For fixed k, both the Boolean dimension and the local dimension of $P(k,n)$ tends to infinity.
- For fixed k, the fractional dimension of $P(k,n)$ is less than 2^k.
- In the class of convex geometries, both Boolean dimension and local dimension can be separated from fractional dimension and VC-dimension, even when the VC-dimension is 2.

Separating Dimension Paramters for Posets

- **Fact** Dimension and fractional dimension are separated from Boolean dimension, and local dimension by the family of standard examples.

Separating Dimension Paramters for Posets

- Fact Dimension and fractional dimension are separated from Boolean dimension, and local dimension by the family of standard examples.
- Barrera-Cruz, Prag, Smith, Taylor, WTT and Wang Dimension, Boolean dimension and local dimension are separated from fractional dimension by the family of posets with standard example number 1.

Separating Dimension Paramters for Posets

- Fact Dimension and fractional dimension are separated from Boolean dimension, and local dimension by the family of standard examples.
- Barrera-Cruz, Prag, Smith, Taylor, WTT and Wang Dimension, Boolean dimension and local dimension are separated from fractional dimension by the family of posets with standard example number 1.
- WTT and Walczak Each of Boolean dimension and local dimension can be separated from the other.

Open Questions for Convex Geometries

- In the class of convex geometries, we have been unable to separate dimension from either of Boolean dimension and local dimension.

Open Questions for Convex Geometries

- In the class of convex geometries, we have been unable to separate dimension from either of Boolean dimension and local dimension.
- Also, we are unable to separate Boolean dimension and local dimension in either direction.