Generalized degree polynomials and the chromatic symmetric function

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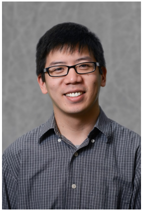


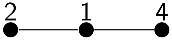
The chromatic symmetric function

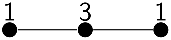
Proper colorings of graphs

A proper coloring of a graph G is a map

$$\kappa\colon V(G) \to \{1, 2, 3, \dots\}$$

such that $\kappa(u) \neq \kappa(v)$ when $\{u, v\} \in E(G)$.





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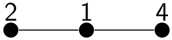

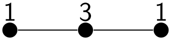

The *chromatic polynomial* of G is

$$\chi_G(k) = \#\text{ proper colorings of } G \text{ with colors } 1, \ldots, k.$$

The chromatic symmetric function

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The chromatic symmetric function (CSF) of G is defined by

$$\mathbf{X}_G = \sum_{\kappa} \prod_{v \in V(G)} x_{\kappa(v)}$$

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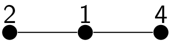

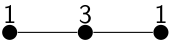

$$\mathbf{X}_G = \cdots \quad + \quad x_1x_2x_4 \quad + \quad x_1^2x_3 \quad + \quad \cdots$$

$$= 6 \sum_{i<j<k} x_ix_jx_k + \sum_{i \neq j} x_i^2 x_j.$$

Facts about the CSF

1. $\mathbf{X}_G$ generalizes χ_G: indeed,

$$\mathbf{X}_G(\underbrace{1,\dots,1}_{k},0,0,\dots) = \sum_{\kappa\colon V(G)\to[k]} 1 = \chi_G(k).$$

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4. $G \mapsto \mathbf{X}_G$ is a homomorphism of combinatorial Hopf algebras [Aguiar, Bergeron, Sottile '06].

5. $\mathbf{X}_G$ has a quasisymmetric variant, which is related to the cohomology of Hessenberg varieties [Shareshian, Wachs '16].

Our motivating question

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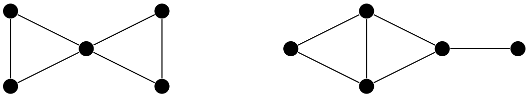
Remark. $\mathbf{X}_G$ does not distinguish graphs in general:



It also does not distinguish unicyclic graphs [Orellana, Scott '14].

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(3) The CSF determines other graph (tree) invariants:

- The subtree polynomial and connector polynomial [MMW'08].
- The generalized degree polynomial [Aliste-Prieto, Martin, Wagner, Zamora '24].

The generalized degree polynomial

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The generalized degree polynomial

We study a related invariant, the generalized degree polynomial (GDP). The GDP is determined by the CSF but does *not* distinguish trees.

Our results:

(1) Extract new kinds of tree data from the GDP such as the double-degree sequence and the leaf adjacency sequence.

(2) Introduce a variant of the GDP for polarized trees. Use this to get:

- A recursive formula for the GDP.
- Constructions for sets of trees with the same GDP.

The generalized degree polynomial

Definition of the GDP

Let T be a tree. Given $S \subseteq V(T)$, an edge $\{u, v\} \in E(T)$ is called

- an internal edge of S if $u \in S$ and $v \in S$;
- a boundary edge of S if $u \in S$ or $v \in S$, but not both.

Let $e(S)$ and $d(S)$ be the number of internal and boundary edges of S.

For example, if $S = \{v\}$, then $d(S) = \deg(v)$ and $e(S) = 0$.

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Definition

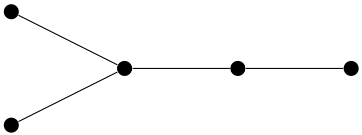
The generalized degree polynomial (GDP) of T is defined by

$$\mathrm{G}_T(x, y, z) = \sum_{S \subseteq V(T)} x^{|S|} y^{d(S)} z^{e(S)}.$$

Thus, $[x^1 y^d z^0]\, \mathrm{G}_T(x, y, z) = \#$ vertices of degree d.

An example of the GDP

If T is the tree



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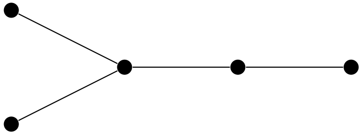

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$$\begin{aligned}\mathrm{G}_T(x,y,z) &= x^5z^4 + x^4y^3z + x^4y^2z^2 + 3x^4yz^3 \\ &+ x^3y^4 + 2x^3y^3z + x^3y^3 + 3x^3y^2z^2 + 2x^3y^2z + x^3yz^2 \\ &+ x^2y^4 + x^2y^3z + 2x^2y^3 + 2x^2y^2z + 3x^2y^2 + x^2yz \\ &+ xy^3 + xy^2 + 3xy + 1.\end{aligned}$$

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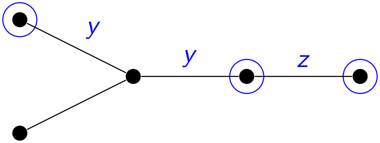

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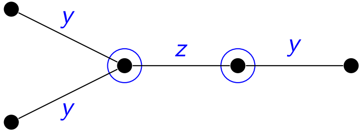

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Facts about the GDP

- The GDP is determined linearly by the chromatic symmetric function [Aliste-Prieto, Martin, Wagner, Zamora '24; Liu, T. '24].

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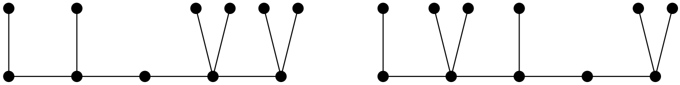
- The GDP does not distinguish trees. The smallest trees with the same GDP have 11 vertices:



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- *The leaf adjacency sequence of T: for all d and ℓ, the number of vertices of T with degree d, adjacent to ℓ leaves.*
- *The induced subgraph containing all degree-2 vertices of T, up to isomorphism.*

Proof strategy

Main idea: look at partial derivatives of $\mathrm{G}_T(x, y, z)$ with respect to z.

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Lemma

For $k \geq 1$,

$$\frac{\partial^k \mathrm{G}_T}{\partial z^k}(x, y, y^2) = \frac{k!}{y^{2k}} \mathrm{G}_T(x, y, y^2) \cdot \Gamma_T^{(k)}(x, y),$$

where

$$\Gamma_T^{(k)}(x, y) = \sum_{B \in \binom{E(T)}{k}} \prod_{v \in V(B)} \frac{xy^{\deg(v)}}{1 + xy^{\deg(v)}}.$$

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Thus, G_T determines $\Gamma_T^{(k)}(x, y)$ for all k.

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$$\Gamma_T^{(1)}(x,y) = \sum_{\{u,v\}\in E(T)} \frac{xy^{\deg(u)}}{1+xy^{\deg(u)}} \frac{xy^{\deg(v)}}{1+xy^{\deg(v)}}$$
$$= \sum_{a\le b} \frac{xy^a}{1+xy^a} \frac{xy^b}{1+xy^b} N_{a,b}(T).$$

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By expanding the rational functions $\frac{xy^a}{1+xy^a}\frac{xy^b}{1+xy^b}$ as power series, we show they are linearly independent. Thus $\Gamma_T^{(1)}(x,y)$ determines all $N_{a,b}(T)$.

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Lemma

For all integers $a \geq 2$ and $k \geq 1$, we have

$$\lim_{y \to e^{2\pi i/(a-1)}} \lim_{x \to -1/y} (1 - y^{a-1})^k (1 + xy)^k \cdot \Gamma_T^{(k)}(x^{-1}, y^{-1})$$
$$= e_k \left\{ \frac{\ell(v)}{\deg(v)-1} : \substack{\deg(v)>1 \\ \deg(v)\equiv 1 \pmod{a-1}} \right\}$$

where e_k denotes the k^{th} elementary symmetric sum and $\ell(v)$ is the number of leaves adjacent to v.

The takeaway

- $\Gamma_T^{(k)}$ is not linear in G_T (or $\mathbf{X}_T$), since its formula involves division
- Thus, the tree data we recover does not come linearly from $\mathbf{X}_T$ (or in some cases even polynomially!)

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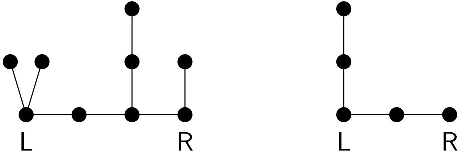
- *Question*: What other kinds of tree data are determined by G_T?
- What can be said about "higher-order" GDPs $\mathrm{G}_T^{(m)}$, where we instead sum over $m > 1$ disjoint subsets of vertices of T?

Polarized graphs and the generalized degree polynomial

Polarized graphs

Definition

A polarized graph is a graph A with two distinguished vertices, called the left pole $L(A)$ and right pole $R(A)$ (which can be the same).



Concatenation of polarized graphs

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Let A and B be polarized graphs. Their concatenation *AB is obtained from the disjoint union $A \sqcup B$ by identifying $R(A)$ and $L(B)$, with $L(AB) = L(A)$ and $R(AB) = R(B)$.*

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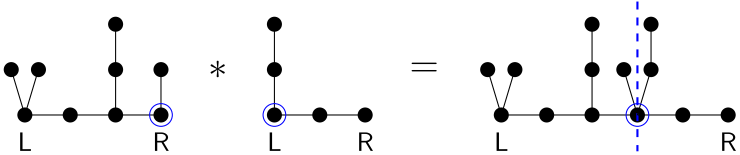

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$$\mathrm{G}_A^{1} = \sum_{\substack{L(A)\notin S \\ R(A)\notin S}} x^{|S|} y^{d(S)} z^{e(S)}, \qquad \mathrm{G}_A^{\mathrm{R}} = \sum_{\substack{L(A)\notin S \\ R(A)\in S}} x^{|S|} y^{d(S)} z^{e(S)},$$

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Then the generalized degree polynomial of A is the 2×2 matrix

$$\mathbf{G}_A = \begin{bmatrix} \mathrm{G}_A^{1} & \mathrm{G}_A^{\mathrm{R}} \\ \mathrm{G}_A^{\mathrm{L}} & \mathrm{G}_A^{\mathrm{LR}} \end{bmatrix} \in \mathrm{Mat}_{2\times 2}(\mathbf{k}).$$

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Example

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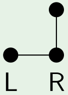

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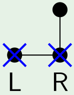

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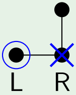

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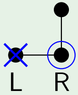

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$$\mathrm{G}_A^{1} = xy + 1, \qquad \mathrm{G}_A^{\mathrm{R}} = xy(xz + y),$$
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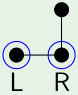

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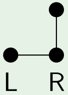

then

$$\mathrm{G}^{1}_{A} = xy + 1, \qquad \mathrm{G}^{\mathrm{R}}_{A} = xy(xz + y),$$
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so

$$\mathbf{G}_A = \begin{bmatrix} xy + 1 & xy(xz + y) \\ xy(xy + 1) & x^2 z(xz + y) \end{bmatrix}.$$

Concatenation and the polarized GDP

Proposition

For polarized graphs A and B,

$$\mathbf{G}_{AB} = \mathbf{G}_A \begin{bmatrix} 1 & 0 \\ 0 & 1/x \end{bmatrix} \mathbf{G}_B .$$

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Algebraic interpretation: The map

$$A \mapsto \mathbf{G}_A \begin{bmatrix} 1 & 0 \\ 0 & 1/x \end{bmatrix}$$

defines a homomorphism $\mathcal{R} \to \mathrm{Mat}_{2\times 2}(\mathbf{k})$, where $\mathcal{R}$ is the $\mathbf{k}$-algebra formally spanned by polarized trees with concatenation as multiplication.

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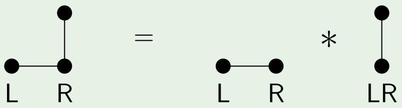

$$\begin{bmatrix} xy+1 & xy(xz+y) \\ xy(xy+1) & x^2z(xz+y) \end{bmatrix} = \begin{bmatrix} 1 & xy \\ xy & x^2z \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{x} \end{bmatrix} \begin{bmatrix} xy+1 & 0 \\ 0 & xz+y \end{bmatrix}.$$

The rooted GDP

We can also define the GDP of a rooted graph P, with root $r(P)$:

$$\mathbf{G}_P = \begin{bmatrix} \mathrm{G}^1_P \\ \mathrm{G}^{\mathrm{r}}_P \end{bmatrix} \in \mathbf{k}^2$$

where $\mathrm{G}^1_P = \displaystyle\sum_{r(P)\notin S} x^{|S|} y^{d(S)} z^{e(S)}$ and $\mathrm{G}^{\mathrm{r}}_P = \displaystyle\sum_{r(P)\in S} x^{|S|} y^{d(S)} z^{e(S)}$.

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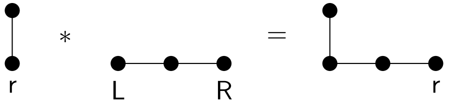
This also behaves well with respect to concatenation (treating $\mathbf{G}_P$ as a row vector when on the left, and a column vector when on the right).



A recurrence for the ordinary GDP

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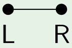
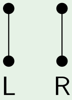
Example

The polarized GDPs of the three graphs

$A =$ 

$B =$ 

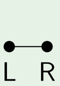
$C =$ 

satisfy

$$\mathbf{G}_A = \frac{x(z-y^2)}{(xz+y)(xy+1)}\,\mathbf{G}_B + \frac{y}{(xz+y)(xy+1)}\,\mathbf{G}_C.$$

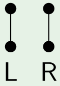
A recurrence for the ordinary GDP

Example



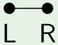
$$= \frac{x(z-y^2)}{(xz+y)(xy+1)}$$

$$+ \frac{y}{(xz+y)(xy+1)}$$



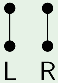
A recurrence for the ordinary GDP

Example

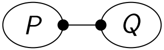


$= \frac{x(z-y^2)}{(xz+y)(xy+1)}$

$+ \frac{y}{(xz+y)(xy+1)}$



For rooted trees P and Q, this implies that



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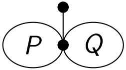

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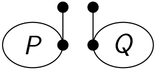

.

Thus, we get a recurrence relation for the ordinary GDP (which reduces to a recurrence found by [Aliste-Prieto, Martin, Wagner, Zamora '24]).

Ordinary and polarized GDPs

The reversal phenomenon

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Definition

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- A polarized forest is *singular* if its poles are in different components.
- Here $\bullet$ is the only polarized graph with one vertex, so $\mathbf{G}_\bullet = \left[\begin{smallmatrix} 1 & 0 \\ 0 & x \end{smallmatrix}\right]$.

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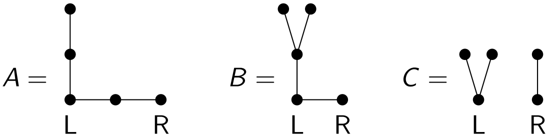
The smallest interesting example of this is



The reversal phenomenon

Theorem (Liu, T. '25+)

Suppose A, B, and C satisfy R.P. Then for any $T_1, T_2, \ldots, T_k \in \{A, B\}$,

$$(T_1 T_2 \cdots T_k)\#C \qquad \textit{and} \qquad (T_k \cdots T_2 T_1)\#C$$

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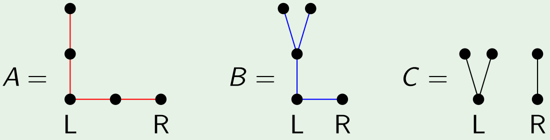
In fact, a converse statement holds: if $(AB)\#C$ and $(BA)\#C$ have the same GDP, then A, B, and C satisfy R.P.

An example of the reversal phenomenon

Example

Since



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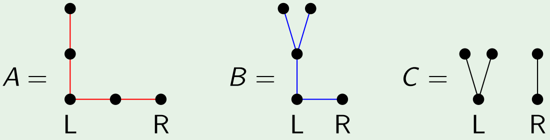

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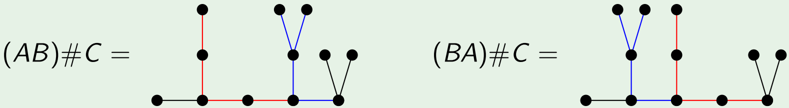

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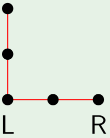
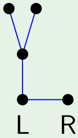
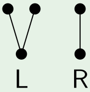
$A =$ 

$B =$ 

$C =$ 

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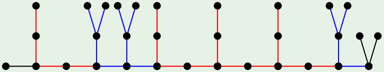
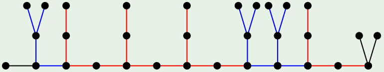
$(ABBAAAB)\#C =$ 

$(BAAABBA)\#C =$ 

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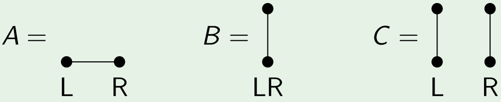
Example

We saw that the graphs



from earlier have linearly dependent GDPs, so they satisfy C.P.

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The gory details:

- For integer compositions $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\beta = (\beta_1, \ldots, \beta_\ell)$, the *concatenation* $\alpha\beta = (\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_\ell)$, and the *near-concatenation* $\alpha \odot \beta = (\alpha_1, \ldots, \alpha_k + \beta_1, \ldots, \beta_\ell)$.
- Define $\alpha \circ \beta = \beta^{\odot\alpha_1}\beta^{\odot\alpha_2} \cdots \beta^{\odot\alpha_k}$. It turns out that $\circ$ is associative.
- Finally, define a tree $T_\alpha = (A^{\alpha_1 - 1}BA^{\alpha_2 - 1}B \cdots BA^{\alpha_k - 1})\#C$.
- Theorem: Suppose $\alpha = \alpha^{(1)} \circ \cdots \circ \alpha^{(r)}$ and $\beta = \beta^{(1)} \circ \cdots \circ \beta^{(r)}$, where $\beta^{(i)}$ is either $\alpha^{(i)}$ or its reverse for all i. Then $\mathrm{G}_{T_\alpha} = \mathrm{G}_{T_\beta}$.

The composition phenomenon

Here is the smallest case of C.P. that does not also come from R.P.:

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If A, B, C satisfy C.P., then

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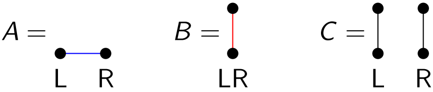
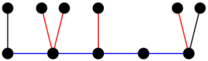
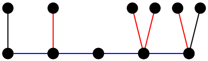


these two trees are



and



which is the 11-vertex example we saw earlier!

Numerology

n	11	12	13	14	15	16	17	18	19	20	21	22	23	Total
Pairs w/ same GDP	1	1	1	5	1	7	19	15	23	56	22	90	160	401
Composition	1		1	5	1	4	18	11	17	48	21	82	154	363
Reversal		1				3	1	2	6	8	1	7	3	32
Unexplained								**2**				**1**	**3**	**6**

Thank you!