

Posets of Ordered Set Partitions

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Joint work with
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- The d -divisible ordered set partitions : topological properties
- The action of the symmetric group \mathfrak{S}_n on the homology
- The multiplicity of the trivial representation
- The 1 mod d ordered set partitions
- Open questions

A poset P is

- 1 *bounded* if it has a unique minimal, denoted by $\hat{0}$, and a unique maximal element, denoted by $\hat{1}$;
- 2 *graded* if in any interval (x, y) , every maximal chain (with respect to inclusion) has the same length, where the *length* of a chain is one less than the number of poset elements in the chain.

If the poset P is bounded and graded, it has a *rank function* $\text{rank}(x)$, for $x \in P$, defined as the length of any maximal chain from $\hat{0}$ to x . The *rank of P* is then the rank of $\hat{1}$.

Let \bar{P} denote the proper part $P \setminus \{\hat{0}, \hat{1}\}$ of a bounded poset P .

The order complex of a poset

The *order complex* $\Delta(\bar{P})$ of a bounded poset P is the simplicial complex whose i -dimensional faces are the chains $(\hat{0} = x_0 < x_1 < \cdots < x_{i+1} < x_{i+2} = \hat{1})$ of length i in \bar{P} .

The chain $(x_1 < \cdots < x_{i+1})$ in \bar{P} corresponds to the oriented i -simplex

$$[x_1, x_2, \dots, x_{i+1}]$$

The 0-chain $\hat{0} < \hat{1}$ is the empty simplex of dimension (-1) .

The reduced homology of P is defined to be the reduced homology of $\Delta(\bar{P})$.

The Möbius number $\mu(P)$ of P is the reduced Euler characteristic of the order complex $\Delta(\bar{P})$ of P .

A group G acting in an order-preserving manner on a poset P

$$x < y \Rightarrow g \cdot x < g \cdot y$$

induces an action on $\Delta(\bar{P})$, and hence on the homology groups $\tilde{H}_i(P)$ of P .

A set partition of $[n] := \{1, 2, \dots, n\}$ is a partition into disjoint nonempty subsets, called blocks,

$$B_1 | B_2 | \dots | B_k$$

The collection of all such partitions is Π_n .

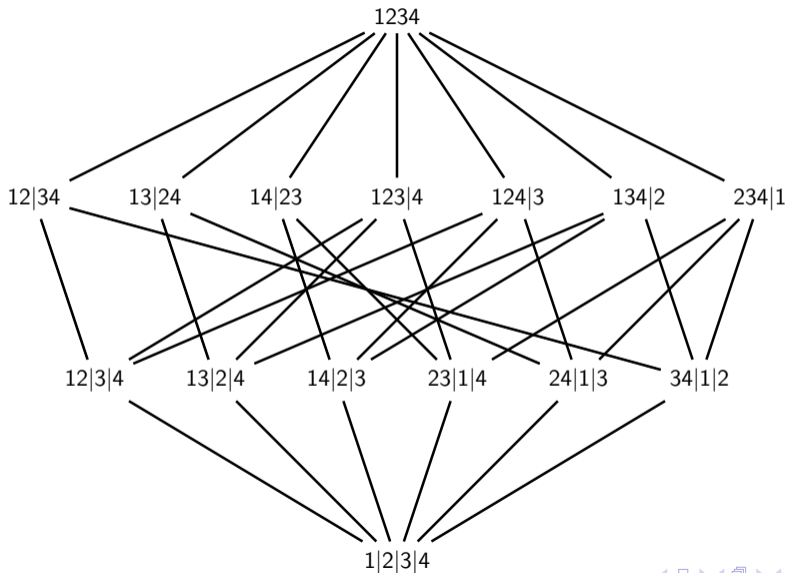
$$\Pi_2 : |12|, 1|2$$

$$\Pi_3 : 1|2|3, 12|3, 13|2, 23|1, |123|$$

The number of partitions of $[n]$ into k blocks is the

Stirling number of the second kind, $S(n, k)$.

Π_n is a lattice



Ordered set partitions

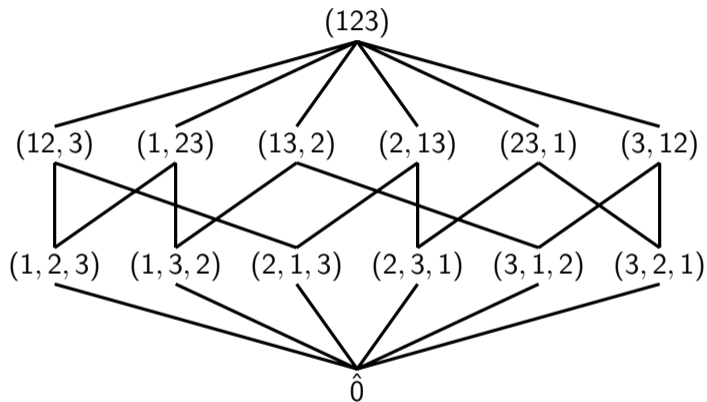
- An *ordered set partition* of the set $[n] := \{1, 2, \dots, n\}$ is a sequence of non-empty subsets $\omega = (B_1, B_2, \dots, B_k)$ whose disjoint union $\sqcup_i B_i$ equals $[n]$.
- The number of ordered set partitions of n into k blocks is $k!S(n, k)$, where $S(n, k)$ is the **Stirling number of the second kind**.

- There is a partial order on the ordered set partitions of $[n]$, defined by the cover relation merging *adjacent* blocks: (B_1, B_2, \dots, B_k) is covered by all elements of the form

$$(B_1, \dots, B_{i-1}, B_i \sqcup B_{i+1}, B_{i+2}, \dots, B_k) \text{ for } 1 \leq i < k.$$

- Appending a bottom element makes the ordered set partitions Ω_n a lattice (the **face lattice** of the **permutahedron**), graded of rank n .

The lattice of ordered set partitions Ω_3



12 maximal chains, two \mathfrak{S}_3 -orbits, six \mathfrak{S}_2 -orbits

The d -divisible ordered set partition lattice $\Omega_{dm,d}$

Fix $d, m \geq 1$. Then $\Omega_{dm,d}$ is the subposet of Ω_{dm} consisting of all set partitions whose block sizes are divisible by d .

Ex. The 2-divisible partitions of $\{1, 2, 3, 4\}$ are

$$\hat{0}, (12, 34), (13, 24), (14, 23), (23, 14), (24, 13), (34, 12), \text{ and } (1234) = \hat{1}$$

The rank function is $\text{rank}(x) = m - k + 1$ where k is the number of blocks of x .

$\Omega_{dm,d}$ is a lattice of rank m , but not semimodular for $m \geq 3$, so **not geometric**:
semimodular: $x \vee y$ covers y implies x covers $x \wedge y$.

Let $u = (B_1, B_2 \sqcup B_3, B_4, \dots)$ and $v = (B_2, B_1 \sqcup B_3, B_4, \dots)$ be elements covering the atom $(B_1, B_2, B_3, B_4, \dots)$ and the atom $(B_2, B_1, B_3, B_4, \dots)$ in $\Omega_{dm,d}$ respectively.

Then $u \vee v = (B_1 \sqcup B_2 \sqcup B_3, B_4, \dots) = (B_2 \sqcup B_1 \sqcup B_3, B_4, \dots)$ covers u and v , but notice:

$$u \wedge v = \hat{0},$$

so neither of the rank 2 elements u or v covers $u \wedge v$.

A bounded and graded poset P is Cohen-Macaulay if the homology of any interval $[x, y]$ in P vanishes in all except possibly the top dimension.

In that case, the *Betti number* $\dim \tilde{H}_{top}(P)$ is given by $|\mu(P)|$, where $\mu(P)$ is the Möbius number of the poset P .

Conditions that guarantee the Cohen-Macaulay property for posets:

- 1 **Shellability** of $\Delta(\bar{P})$ (a sequential “gluing” condition on the maximal faces of $\Delta(\bar{P})$);
- 2 **Lexicographic shellability** of P . This is an edge-labelling of the Hasse diagram of P which induces a shelling order on the maximal chains of P , and hence on the maximal faces of $\Delta(\bar{P})$;

Lexicographic shellability (due to Björner, inspired by the R -labellings of Stanley) comes in two flavours: **EL shellability** and **CL shellability**.

The case $d = 1$: the ordered set partition lattice

Ω_n is the face lattice of a convex polytope, the *permutahedron*.

Theorem (Björner-Wachs 1983)

The face lattice of a convex polytope is CL-shellable.

Since shellable \Rightarrow Cohen-Macaulay, we have:

Corollary

The ordered set partition lattice Ω_n is Cohen-Macaulay.

The d -divisible case $\Omega_{dm,d}$

No polytope for $\Omega_{dm,d}$ when $d \geq 2$. Check: $\mu(\Omega_{2d,2}) = (2d!)/2^d - 1 \neq 1$.

Let \mathcal{B}_n be the Boolean lattice of subsets of $[n]$, and let $\mathcal{B}_{dm,d}$ be the subposet of \mathcal{B}_{dm} consisting of subsets of size divisible by d .

Proposition

The face lattice of the order complex of $\bar{\mathcal{B}}_{dm,d}$ is poset-isomorphic to the dual $\Omega_{dm,d}^$ of the lattice $\Omega_{dm,d}$.*

Proof: The chain of subsets $(\hat{0} \subset X_1 \subset \cdots \subset X_r \subset \hat{1})$ in $\mathcal{B}_{dm,d}$ maps to the ordered set partition $x = (B_1, \dots, B_{r+1})$, where

$$B_1 = X_1, B_2 = X_2 \setminus X_1, \dots, B_{r+1} = [dm] \setminus X_r.$$

- Every $|X_i|$ is divisible by d , so x is a d -divisible ordered set partition.
- The empty chain maps to the top element $[dm]$.
- The poset of chains is the face lattice of $\Delta(\bar{\mathcal{B}}_{dm,d})$. □

Notice: This map commutes with the action of the symmetric group.

Barycentric subdivision

The order complex of the face lattice of a simplicial complex Δ is the barycentric subdivision sd of Δ .

$$\Delta \rightarrow \mathcal{L}(\Delta) \rightarrow \Delta(\mathcal{L}(\Delta)) = \text{sd } \Delta$$

$$P \rightarrow \Delta(\bar{P}) \rightarrow \mathcal{L}(\Delta(\bar{P})) \stackrel{\text{def}}{=} \text{sd} P \rightarrow \Delta(\overline{\text{sd} P}) \stackrel{\text{FACT}}{=} \text{sd} \Delta(\bar{P}).$$

Here $\mathcal{L}(\Delta(\bar{P}))$ is the face lattice of $\Delta(\bar{P})$, with an artificial $\hat{1}$ appended.

Since the order complex of the face lattice of $\Delta(\bar{P})$ is the barycentric subdivision sd of $\Delta(\bar{P})$, there is an \mathfrak{S}_{dm} -equivariant homeomorphism

$$\text{sd } \Delta(\bar{\mathcal{B}}_{dm,d}) \simeq \Delta(\bar{\Omega}_{dm,d}^*)$$

Putting together a lot of shellability theory...

$\mathcal{B}_{dm,d}$ is a **rank-selected** subposet of the Boolean lattice \mathcal{B}_{dm} , for the ranks that are divisible by d .

- (Baclawski, Björner 1980): Rank-selection preserves the Cohen-Macaulay property, and shellability.
- (Björner-Wachs 1983): The face lattice of a shellable simplicial complex admits a *recursive atom ordering*, and hence is shellable and therefore Cohen-Macaulay. In fact:

Proposition (Sagan-S 2025)

There is an explicit recursive atom ordering for the lattice $\Omega_{dm,d}$.

- (Björner-Wachs 1983): **CL**-shellability is equivalent to having a recursive atom ordering.

Hence $\Delta(\overline{\Omega}_{dm,d})$ is (shellable and) Cohen-Macaulay.

Question: Does $\Omega_{dm,d}$ have an **EL**-labelling?

Further conclusions

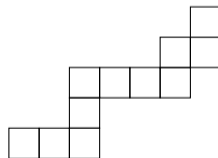
Well-known structure of homology modules for rank-selected subsets of Boolean lattice (since $\Delta(\bar{\mathcal{B}}_n)$ is the Coxeter complex for the root system A_{n-1}):

Theorem (Solomon 1968, Stanley 1982)

For the rank-selected subposet $\mathcal{B}_n(S)$ corresponding to the rank-set S , the \mathfrak{S}_n -module structure of $\tilde{H}(\mathcal{B}_n(S))$ is given by the **Specht module** associated to the ribbon $\text{Rib}(S)$ determined by $S = \{s_1 < s_2 < \dots < s_k\}$, with **bottom-to-top** row lengths

$$s_1, s_2 - s_1, \dots, s_k - s_{k-1}, n - s_k.$$

Ex. The ribbon for the rank-set $\{3, 4, 8, 10\}$ in \mathcal{B}_{11} is



The special case $S = \{1, 2, \dots, n-1\}$ confirms the well-known fact that the top homology module of \mathcal{B}_n is the sign representation.

The \mathfrak{S}_{dm} -homeomorphism

$$\Delta(\overline{\Omega}_{dm,d}^*) \simeq \text{sd } \Delta(\overline{\mathcal{B}}_{dm,d})$$

gives an \mathfrak{S}_{dm} -isomorphism of homology modules

$$\tilde{H}(\Omega_{dm,d}) \cong \tilde{H}(\mathcal{B}_{dm,d}) \cong \mathcal{S}^{\text{Rib}(d,2d,\dots,(m-1)d)},$$

the Specht module associated to the ribbon $\text{Rib}(d, 2d, \dots, (m-1)d)$.

The Betti number $\dim \tilde{H}(\Omega_{dm,d})$ is the **generalised Euler number**

$$\mathcal{E}_{dm,d} = |\{\pi \in \mathfrak{S}_{dm} \mid \text{Des } \pi = \{d, 2d, \dots, (m-1)d\}\}|,$$

where the descent set of a permutation σ is $\text{Des}(\sigma) = \{i \mid \sigma(i) > \sigma(i+1)\}$.

What about the **rank-selected** subsets of $\Omega_{dm,d}$?

Whitney homology

If P is a poset with least element $\hat{0}$, then the i th Whitney homology of P is related to the usual order homology of P by group-equivariant isomorphisms establishing that

$$WH_i(P) = \bigoplus_{x \in P} \tilde{H}_{i-2}(\hat{0}, x), 0 \leq i \leq r.$$

where r is the length of the longest chain in P .

Proposition (S 1994: Acyclicity of Whitney homology)

Let P be a bounded poset, and let G be a group of automorphisms of P . Then each Whitney homology module is a G -module. Let r be the length of the longest chain in P . Then $WH_r(P) = \tilde{H}_{r-2}(P)$ and as a virtual sum of G -modules one has

$$WH_r(P) - WH_{r-1}(P) + \cdots + (-1)^{r-1} WH_1(P) + (-1)^r WH_0(P) = 0,$$

where $WH_0(P)$ is the trivial module.

$$WH_i(P) = \bigoplus_{x:\text{rank}(x)=i} \tilde{H}(\hat{0}, x) \text{ and}$$

$$\tilde{H}_{r-2}(P) = WH_{r-1}(P) - WH_{r-2}(P) + \cdots + (-1)^{r-1} WH_0(P).$$

$$WH_j^*(P) = \bigoplus_{x:\text{rank}(x)=r-j} \tilde{H}(x, \hat{1}) \text{ and}$$

$$\tilde{H}_{r-2}(P) = WH_{r-1}^*(P) - WH_{r-2}^*(P) + \cdots + (-1)^{r-1} WH_0^*(P)$$

Brief symmetric functions dictionary (Frobenius characteristics):

homogeneous symmetric function $h_n \leftrightarrow$ trivial repn of \mathfrak{S}_n

elementary symmetric function $e_n \leftrightarrow$ sign repn of \mathfrak{S}_n

Schur function $s_\lambda \leftrightarrow \mathfrak{S}_n$ -irreducible indexed by the integer partition λ of n

Let $\beta_{dm,d}$ be the Frobenius characteristic of the top homology $\tilde{H}_{m-2}(\Omega_{dm,d})$.

Determining the homology modules of $\Omega_{dm,d}$

Note: the **atoms** of $\Omega_{dm,d}$ are the partitions with **all block sizes equal to d** .

The **rank function** is $\text{rank}(\omega) = m - k + 1$, for the partition $\omega = (B_1, \dots, B_k)$.

$\Omega_{dm,d}$ has a helpful **recursive** structure which will allow us to determine the homology representations:

- For $\hat{0} \neq \pi < \omega \leq \hat{1}$ in $\Omega_{dm,d}$, the interval $[\pi, \omega]$ is isomorphic to the **Boolean lattice** $\mathcal{B}_{\text{rank}(\omega) - \text{rank}(\pi)}$.
- The interval $[\hat{0}, \pi]$ has the structure of a *reduced* product of smaller ordered partition posets.

Similar to product structure of intervals $[\hat{0}, \pi]$ in the unordered partition lattice Π_n .

(S, 1994): Let P_i be a bounded poset with minimum element $\hat{0}_i$, $i = 1, 2$. The *reduced product* $P_1 \dot{\times} P_2$, is the subposet of the usual product poset $P_1 \times P_2$ obtained by removing all elements (x_1, x_2) such that $x_i = \hat{0}_i$ for exactly one i . Moreover there is a Künneth formula for the homology modules.

Whitney homology for $\Omega_{dm,d}$

Write $\alpha \vDash m$ and $\ell(\alpha) = k$ for a **composition** $\alpha = (\alpha_1, \dots, \alpha_k)$ of m , i.e., a k -tuple of positive integers summing to m .

Theorem (Sagan-S 2025: Whitney homology)

$$\text{ch } WH_k^*(\Omega_{dm,d}) = \sum_{\substack{\alpha \vDash m \\ \ell(\alpha) = k+1}} h_{d\alpha_1} \cdots h_{d\alpha_{k+1}}, \quad 0 \leq k \leq m-1, \quad \text{ch}(WH_m^*(\Omega_{dm,d})) = \beta_{dm,d}.$$

$$\text{ch } WH_{m-k+1}(\Omega_{dm,d}) = \sum_{\substack{\alpha \vDash m \\ \ell(\alpha) = k}} \beta_{d\alpha_1,d} \cdots \beta_{d\alpha_k,d}, \quad 1 \leq k \leq m, \quad \text{ch}(WH_0(\Omega_{dm,d})) = h_{dm}.$$

$$\beta_{dm,d} = \sum_{j=1}^m (-1)^{m-j} \sum_{\substack{\alpha \vDash m \\ \ell(\alpha) = j}} \prod_{i=1}^j h_{d\alpha_i} \quad (\text{an alternating sum of permutation}$$

$$\text{modules}) = \sum_{j=2}^m (-1)^j \sum_{\substack{\alpha \vDash m \\ \ell(\alpha) = j}} \prod_{i=1}^j \beta_{d\alpha_i,d} + (-1)^{m+1} h_{dm} \quad (\text{a recursive formula})$$

When $d = 1$, $\beta_{dm,d} = e_m$ and these expressions are equivalent to the expansion of the elementary symmetric function e_m in terms of the h_k .

The representations for $d = 1$

$\Delta(\Omega_m)$ is a sphere, so we have the following observation.

Proposition (Stanley 1982)

Let $S \subseteq [m - 1]$ be a subset of ranks of Ω_m , and let \bar{S} be its complement in $[m - 1]$.
Then

$$\tilde{H}(\Omega_m(S)) \cong \tilde{H}(\Omega_m(\bar{S})) \otimes \text{sgn}_{\mathfrak{S}_m}.$$

Proof: This follows by Alexander duality in a sphere, since the order complex of Ω_m is homotopy equivalent to the sphere \mathbb{S}^{m-2} . \square

The **corank**-selected representations for $d = 1$

- Subsets below correspond to **coranks**, i.e. $m - \text{rank}$.
- Partitions at **corank** k have $k + 1$ blocks.
- $\beta_m(\emptyset) = h_m$, $\beta_m(\{m - 1\}) = h_1^m - h_m$.
- Alexander duality says $\beta_m(S) = \beta_m(\bar{S}) \otimes \text{sgn}$

For $m = 4$ it suffices to list all the singleton **corank** sets:

$$\beta_4(\{1\}) = 2h_2h_1 + h_2^2 - h_4 \quad \beta_4(\{2\}) = 3h_2h_1^2 - h_4 \quad \beta_4(\{3\}) = h_1^4 - h_4$$

By Alexander duality, $\beta_4(\{1, 2\}) = e_1^4 - e_4 = 3h_2h_1^2 - 2h_3h_1 - h_2^2 + h_4$.

$$\beta_5(\{1\}) = 2h_3h_2 + 2h_4h_1 - h_5 \quad \beta_5(\{1, 2\}) = 3h_3h_1^2 + 3h_2^2h_1 - 2h_4h_1 - 2h_3h_2 + h_5$$

$$\beta_5(\{2\}) = 3h_3h_1^2 + 3h_2^2h_1 - h_5 \quad \beta_5(\{1, 3\}) = 8h_2h_1^3 - 2h_4h_1 - 2h_3h_2 + h_5$$

$$\beta_5(\{3\}) = 4h_2h_1^3 - h_5 \quad \beta_5(\{1, 4\}) = 3h_1^5 - 2h_4h_1 - 2h_3h_2 + h_5$$

$$\beta_5(\{4\}) = h_1^5 - h_5$$

By Alexander duality,

$$\beta_5(\{2, 4\}) = 8e_2e_1^3 - 2e_4e_1 - 2e_3e_2 + e_5 = 5h_1^5 - 3h_2^2h_1 - 3h_3h_1^2 + h_5.$$

Theorem (Sagan-S 2025)

Let $T \neq \emptyset$ be a subset of the **coranks** $[m-1]$ of $\Omega_{dm,d}$, with largest element **max** T . The Frobenius characteristic $\beta_{dm,d}(T)$ of \mathfrak{S}_{dm} acting on the top homology of the **corank**-selected subposet $\Omega_{dm,d}(T)$ is a polynomial in $\{h_{id} : 1 \leq i \leq m\}$ determined by the following recurrence:

$$\beta_{dm,d}(T) + \beta_{dm,d}(T \setminus \{\mathbf{max} T\}) = \delta(T) \sum_{\substack{\alpha \vdash m \\ \ell(\alpha) = \mathbf{max} T + 1}} h_{d\alpha_1} h_{d\alpha_2 \dots},$$

where $\delta(T) = |\{\sigma \in \mathfrak{S}_{\mathbf{max} T} : \sigma \text{ has descent set } T \setminus \{\mathbf{max} T\}\}|$, and $\beta_{dm,d}(\emptyset) = h_{dm}$.

Partitions at corank k have $k+1$ blocks.

A pleasing consequence

Define the homomorphism $\Psi_d : h_k \mapsto h_{dk}$ in the ring of polynomials $\mathbb{Z}[h_1, h_2, \dots]$. The homology modules of $\Omega_{d_m, d}$ are determined by those of Ω_m as follows:

Corollary (Sagan-S 2025)

Let f be the Frobenius characteristic of the \mathfrak{S}_m -action on the homology of any corank-selected subposet $\Omega_m(T)$ of Ω_m (respectively the \mathfrak{S}_m -action on the maximal chains of any corank-selected subposet of Ω). Write f as a polynomial in $\{h_1, h_2, \dots\}$. Then for any $d \geq 1$, the Frobenius characteristic of the corresponding \mathfrak{S}_{d_m} -action on the same module associated to the poset $\Omega_{d_m, d}$ is given by $\Psi_d(f)$.

In Ω_5 we had $\beta_5(\{1, 3\}) = 8h_2h_1^3 - 2h_4h_1 - 2h_3h_2 + h_5$.

In $\Omega_{5d, d}$ we have $\beta_{5d}(\{1, 3\}) = 8h_{2d}h_d^3 - 2h_{4d}h_d - 2h_{3d}h_{2d} + h_{5d}$.

The action on the maximal chains of $\Omega_{dm,d}$

Proposition (Sagan-S 2025)

The \mathfrak{S}_{dm} -action on the maximal chains of $\Omega_{dm,d}$ has Frobenius characteristic

$$(m-1)! h_d^m.$$

Let $T \subseteq [m-1]$ be a subset of **coranks**.

Let $b_m(T)$ (resp. $b'_m(T)$) denote the multiplicity of the trivial representation of \mathfrak{S}_{dm} (resp. \mathfrak{S}_{dm-1}) in the homology of the **corank**-selected subposet $\Omega_{dm,d}(T)$.

Proposition (Sagan-S 2025)

For $T \subseteq [m-1]$ the numbers $\{b_m(T)\}$ and $\{b'_m(T)\}$ are **independent of d** and refine the factorials $(m-1)!$ and $m!$ respectively.

$$\sum_{T \subseteq [m-1]} b_m(T) = (m-1)!,$$

$$\sum_{T \subseteq [m-1]} b'_m(T) = m!$$

The multiplicity of the trivial representation

For the rank-selected homology of the unordered partition lattice Π_n , the multiplicities of the trivial representation of \mathfrak{S}_n and \mathfrak{S}_{n-1} refine the Euler numbers E_{n-1} and E_n , respectively, by subsets of $[n-2]$. (Stanley 1982 and S 1994).

For $\Omega_{dm,d}$ and any d , refine the factorials $(m-1)!$ and $m!$ by subsets of $[m-1]$.

Theorem (Sagan-S 2025)

- (a) If $m-1 \notin T$, then $b_m(T) = |\{\sigma \in \mathfrak{S}_{m-1} : \text{Des } \sigma = T\}|$.
Hence $b_m(T) \geq 1$ in this case.
- (b) If $m-1 \in T$, then $b_m(T) = 0$.

We explain this **topologically** by analysing the **quotient complex** $\Delta(\Omega_{dm,d})/\mathfrak{S}_{dm}$.

Two refinements of the factorial function

$$\sum_{T \subseteq [m-1]} b_m(T) = (m-1)!,$$

$$\sum_{T \subseteq [m-1]} b'_m(T) = m!$$

The refinement of $(m-1)!$ by subsets of $[m-1]$ simply tracks the descent set of a permutation in \mathfrak{S}_{m-1} , since $b_m(T) = |\{\sigma \in \mathfrak{S}_{m-1} : \text{Des } \sigma = T\}|$.

The **second refinement**, of $m!$ by subsets of $[m-1]$, appears to be new.

Proposition (Sagan-S 2025)

The numbers $b'_m(T)$ satisfy the recurrence

$$b'_m(T) + b'_m(T \setminus \{\max T\}) = \delta(T) \binom{m-1}{\max T} (1 + \max T)$$

where $\delta(T) = |\{\sigma \in \mathfrak{S}_{\max T} : \sigma \text{ has descent set } T \setminus \{\max T\}\}|$.

The refinement of $m!$ by subsets of $[m-1]$

$$\sum_{T \subseteq [m-1]} b'_m(T) = m!$$

Theorem (Sagan-S 2025)

$b'_m(T) = 0$ if and only if T contains both the coranks $m-2, m-1$.

More precisely, we have:

- (a) If $\{m-2, m-1\} \subseteq T$, then $b'_m(T) = 0$.
- (b) If $m-1 \in T$, $m-2 \notin T$, then $b'_m(T) \geq 1$, and $b'_m(T) = (m-1)b_{m-1}(T \setminus \{m-1\})$.
- (c) If $m-1 \notin T$, then $b'_m(T) \geq 1$, and $b'_m(T) = mb_m(T) - (m-1)b_{m-1}(T)$.

- Is there a combinatorial interpretation for $b'_m(T)$?
- Is there a topological explanation for these relationships?

Multiplicity of trivial representation in rank-selected homology of Ω_m

m, T	$\mathbf{b}_m(\mathbf{T})$	$\mathbf{b}'_m(\mathbf{T})$
$T \subseteq [m-1]$	$\sum = (m-1)!$	$\sum = m!$
3, \emptyset	1	1
3, {1}	1	3
3, {2}	0	2
3, {1,2}	0	0
4, \emptyset	1	1
4, {1}	2	5
4, {2}	2	8
4, {3}	0	3
4, {1,2}	1	4
4, {1,3}	0	3
4, {2,3}	0	0
4, {1,2,3}	0	0

m, T	$\mathbf{b}_m(\mathbf{T})$	$\mathbf{b}'_m(\mathbf{T})$
$T \subseteq [m-1]$	$\sum = (m-1)!$	$\sum = m!$
5, \emptyset	1	1
5, {1}	3	7
5, {2}	5	17
5, {3}	3	15
5, {4}	0	4
5, {1,2}	3	11
5, {1,3}	5	25
5, {1,4}	0	8
5, {2,3}	3	15
5, {2,4}	0	8
5, {3,4}	0	0
5, {1,2,3}	1	5
5, {1,2,4}	0	4
5, {1,3,4}	0	0
5, {2,3,4}	0	0
5, {1,2,3,4}	0	0

Theorem (Richard Stanley, 1982)

The number of \mathfrak{S}_n -orbits of maximal chains is E_{n-1} , the $(n-1)$ th Euler number.

Generating function for E_n : $\sum_{n \geq 0} E_n \frac{x^n}{n!} = \tan x + \sec x$

E_{2n-1} is the n th tangent number, E_{2n} is the n th secant number.

E_n is the number of alternating permutations $\sigma \in \mathfrak{S}_n$:

$$\sigma(1) > \sigma(2) < \sigma(3) > \dots$$

$$E_n = 1, 1, 2, 5, 16, 61, 272, 1385, \dots, n \geq 1.$$

Multiplicity of the trivial representation in the unordered partition lattice

For the partition lattice Π_n , we obtain a refinement of the Euler number E_{n-1} into **nonnegative integers** indexed by **subsets** of $[n-2]$:

Proposition (Stanley 1982)

$$E_{n-1} = \sum_{T \subseteq [n-2]} b_n(T)$$

There is a second refinement of the Euler numbers!

Let $\mathbf{b}'_n(\mathbf{T})$ denote the multiplicity of the trivial representation of $\mathfrak{S}_{n-1} \times \mathfrak{S}_1$ on the rank-selected homology of $\Pi_n(T)$. Then

Theorem (S, 1994)

The number of orbits of $\mathfrak{S}_{n-1} \times \mathfrak{S}_1$ acting on the maximal chains of Π_n is the Euler number E_n , and hence

$$E_n = \sum_{T \subseteq [n-2]} \mathbf{b}'_n(\mathbf{T}).$$

Fundamental **three-term plethystic recurrence** for the \mathfrak{S}_n -action on the **homology** of $\Pi_n(T)$:

Theorem (S, 1994)

Let $T = \{s_1 < s_2 < \dots < s_r\}$, $r \geq 2$ be a subset of the nontrivial ranks $[1, n-2]$ of Π_n , and let $T - s_1$ denote the subset $\{s_2 - s_1, s_3 - s_1, \dots, s_r - s_1\}$ of ranks in Π_{n-s_1} . Then

$$\beta_n(\mathbf{T}) + \beta_n(\mathbf{T} \setminus \{\mathbf{s}_1\}) = \beta_{(n-s_1)}(\mathbf{T} - \mathbf{s}_1) [\sum_{i \geq 1} \mathbf{h}_i] |_{\deg n}.$$

For the \mathfrak{S}_n -action on the rank-selected maximal chains, one has

$$\alpha_n(T) = \alpha_{(n-s_1)}(T - s_1) [\sum_{i \geq 1} h_i] |_{\deg n}.$$

Here $\beta_n(T)$, $\alpha_n(T)$ are the Frobenius characteristics of the \mathfrak{S}_n -action on the **rank-selected homology** and on the **rank-selected maximal chains** of $\Pi_n(T)$. The square brackets denote the plethysm operation.

What is known about $b_n(\mathbf{T})$ and $b'_n(\mathbf{T})$

Theorem (S, 1994: Relating $b_n(T)$ and $b'_n(\mathbf{T})$)

Let T be a subset of $[n - 2]$ and suppose $1 \notin T$. Let $T - 1$ denote the subset of $[n - 3]$ obtained by subtracting 1 from each element of T . Then

$$b_n(T \cup \{1\}) + b_n(T) = b'_{n-1}(\mathbf{T} - 1).$$

Theorem

(Hanlon 1981): $b_n(\{1, 2, \dots, r\}) = 0$, $r \geq 1$.

(Hanlon & Hersh 2003) proved the following two conjectures of (S, 1994):

- $b'_n(T) > 1$ unless $T = [1, r]$, in which case it equals 1.
- If $1 \notin T$ then $b_n(T) \neq 0$.

(S, 1994): Explicit formulas for $b_n(T)$ are known in some cases, e.g. when

$$T = [a, b] \text{ or } T = [1, r] \cup \{j\}, j \geq r.$$

Open questions (from 1994!) for Π_n

For the unordered partition lattice Π_n :

- 1 What do $b_n(\mathcal{T})$ and $\mathbf{b}'_n(\mathbf{T})$ count?
- 2 Is there a combinatorial explanation for the refinements of E_{n-1} and E_n by subsets of $[n-2]$?
(Some statistic, indexed by subsets, on the set of alternating permutations?)
- 3 Is there a topological explanation for the three-term recurrence

$$b_n(\mathcal{T} \cup \{1\}) + b_n(\mathcal{T}) = \mathbf{b}'_{n-1}(\mathbf{T} - \mathbf{1})?$$

Multiplicity of trivial representation in rank-selected homology of Π_n

$E_n=1,1,2,5,16,61,\dots$

n, T	$\mathbf{b}_n(\mathbf{T})$	$\mathbf{b}'_n(\mathbf{T})$
$T \subseteq [n-2]$	$\Sigma = E_{n-1}$	$\Sigma = E_n$
4, \emptyset	1	1
4, {1}	0	1
4, {2}	1	2
4, {1,2}	0	1
5, \emptyset	1	1
5, {1}	0	1
5, {2}	1	3
5, {3}	1	3
5, {1,2}	0	1
5, {1,3}	1	3
5, {2,3}	1	3
5, {1,2,3}	0	1

n, T	$\mathbf{b}_n(\mathbf{T})$	$\mathbf{b}'_n(\mathbf{T})$
$T \subseteq [n-2]$	$\Sigma = E_{n-1}$	$\Sigma = E_n$
6, \emptyset	1	1
6, {1}	0	1
6, {2}	1	3
6, {3}	2	5
6, {4}	2	4
6, {1,2}	0	1
6, {1,3}	1	5
6, {1,4}	1	5
6, {2,3}	1	5
6, {2,4}	3	10
6, {3,4}	1	6
6, {1,2,3}	0	1
6, {1,2,4}	0	3
6, {1,3,4}	2	6
6, {2,3,4}	1	4
6, {1,2,3,4}	0	

The 1 mod d ordered set partitions $\check{\check{\Omega}}_{1+dm,d}$

Let $\check{\check{\Omega}}_{1+dm,d}$ be the subposet of ordered set partitions Ω_{1+dm} of the set $[1 + dm]$ with all block sizes congruent to 1 modulo d , with a minimal element $\hat{0}$ appended.

- $\check{\check{\Omega}}_{1+dm,d}$ is a bounded and graded poset of rank $m + 1$.
- all blocks of an atom have size 1.
- The cover relation must merge $dk + 1$ consecutive blocks.

E.g. for $d = 2$, the set partition $(1, 234, 56789, 10, 11)$ is covered by $(123456789, 10, 11)$, $(1, 23456789, 10, 11)$ and $(1, 234, 56789, 10, 11)$.

- The number of blocks is always also congruent to 1 modulo d .
- The rank of $\omega = (B_1, \dots, B_{dk+1})$ is $m - k + 1$.
- The corank of $\omega = (B_1, \dots, B_{dk+1})$ is k .
- If $\omega = (B_1, \dots, B_{dk+1})$ and $|B_i| = \alpha_i$, then $[\hat{0}, \omega] = \check{\check{\Omega}}_{\alpha_1} \times \check{\check{\Omega}}_{\alpha_2} \times \dots \times \check{\check{\Omega}}_{\alpha_{dk+1}}$, a reduced product.

$\check{\Omega}_{1+dm,d}$ is not a lattice!

$\check{\Omega}_{1+dm,d}$ is NOT a lattice for $d \geq 2$:

Consider the following partitions, each with one block of size $d + 1$ and all other blocks singletons.

$$\pi = (1, [2, d + 2], d + 3, \dots, dm + 1) \text{ and } \omega = (1, 2, [3, d + 3], d + 4, \dots, dm + 1).$$

Then π and ω have no least upper bound, because they are both covered by

$$\begin{aligned} \pi' &= ([1, 2d + 1], 2d + 2, \dots, dm + 1) \text{ and} \\ \omega' &= (1, [2, 2d + 2], 2d + 3, \dots, dm + 1). \end{aligned}$$

A **run** in a set of positive integers S is a maximal subset of S consisting of **consecutive** integers.

Ex. Let $S = \{2, 3, 4, 6, 8, 9\}$. There are 3 runs.

Let $T = \{1, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13\}$. The 3 runs have sizes 1, 6 and 4.

Define a subposet of the Boolean lattice \mathcal{B}_n as follows.

$\mathcal{R}_n^d := \{S \subseteq [n] \mid \text{every run in } S \text{ has length divisible by } d\}$.

Proposition (Sagan-S 2025)

For $\hat{0} \neq \pi < \omega \leq \hat{1}$ in $\check{\Omega}_{1+dm,d}$, the interval $[\pi, \omega]$ is isomorphic to the poset of runs

$$\mathcal{R}_{\text{rank}(\omega) - \text{rank}(\pi)}^d.$$

The Möbius function of $\check{\Omega}_{1+dm,d} - I$

Define the d -Catalan number $C_{n,d}$, for $n \geq 0, d \geq 1$, to be

$$C_{n,d} := \frac{1}{n(d-1)+1} \binom{dn}{n}$$

Then $C_{n,1} = 1$ and $C_{n,2} = C_n$, the usual Catalan number.

The generating function $C_d(q) := \sum_{n \geq 0} C_{n,d} q^n$ satisfies the functional equation

$$C_d(q) = 1 + q(C_d(q))^d.$$

The numbers $C_{n,d}$ count certain kinds of Dyck paths called d -Dyck paths.

Theorem (Sagan-S 2025)

Let $\hat{0} \neq \pi \leq \omega \leq \hat{1}$ in $\check{\Omega}_{dm+1,d}$ and let $d \geq 1$. Also let $\text{rank}(\omega) - \text{rank}(\pi) = r$. Then

$$\mu(\pi, \omega) = (-1)^r C_{r,d}.$$

The Möbius function of $\check{\Omega}_{dm+1,d}$ – II

Define the *remainder 1 Euler numbers* by

$$\check{\xi}_n^{(d)} := \mu(\check{\Omega}_{n,d}) \quad (\check{\xi}_n^{(d)} = 0 \text{ if } n \text{ is not congruent to } 1 \pmod{d}).$$

Extend to ordered set partitions $\omega = (B_1, \dots, B_\ell)$ by setting $\check{\xi}_\omega^{(d)} := \check{\xi}_{|B_1|}^{(d)} \cdots \check{\xi}_{|B_\ell|}^{(d)}$.

Using properties of the reduced product and the recursive definition of the Möbius function, we have:

Theorem (Sagan-S 2025)

Let $n = 1 + dm$ for $m, d \geq 1$. Let $\omega \in \check{\Omega}_n$ and let $\ell(\omega)$ be the number of blocks of ω . Then

- 1 $\mu(\hat{0}, \omega) = (-1)^{\ell(\omega)-1} \check{\xi}_\omega^{(d)}$
- 2 $\check{\xi}_n^{(d)} = \sum_{\omega \in \check{\Omega}_n} (-1)^{\lceil \ell(\omega)/d \rceil} C_{\lfloor \ell(\omega)/d \rfloor, d}$.

There are other interesting generating function identities involving the *remainder 1 Euler numbers* and the generating function for the d -Catalan numbers.

Some open questions

- (d -divisible case) Does $\Omega_{dm,d}$ have an EL-labelling?
- What is the quotient complex $\Delta(\Omega_{dm,d})/\mathfrak{S}_{dm-1} \times \mathfrak{S}_1$?
- What is the topology of the runs poset \mathcal{R}_n^d ?
- ($1 \bmod d$ case) Is $\check{\Omega}_{1+dm,d}$ shellable? Cohen-Macaulay? (Does not appear to have an RAO.)
- The Lefschetz module of $\check{\Omega}_{1+dm,d}$ appears to be a true \mathfrak{S}_{1+dm} -module.

THANK YOU TO THE ORGANISERS

and

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