Promotion, rotation, and web invariant polynomials

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joint work with Rebecca Patrias and Oliver Pechenik

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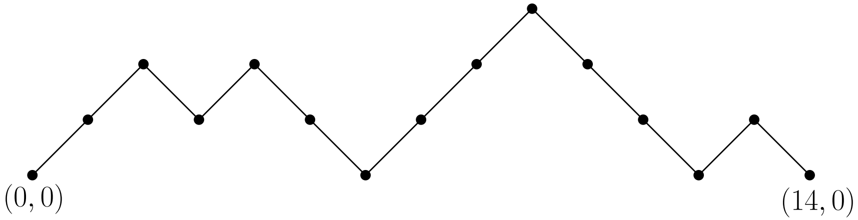
Promotion, rotation, and web invariant polynomials

1. Combinatorial objects and actions
2. Invariant polynomials
3. Invariant polynomials - new generalization
4. More combinatorial objects and actions

Promotion, rotation, and web invariant polynomials

Objects - Dyck paths



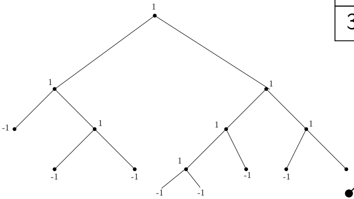
Theorem

Dyck paths from $(0,0)$ *to* $(2n,0)$ *are counted by the nth* **Catalan number**: $C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}$

The Catalan numbers for $n = 0, 1, 2, \ldots, 10$ are:

1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796

Catalan objects



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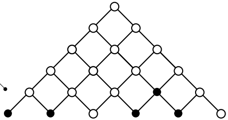
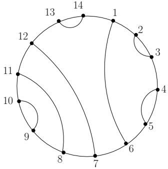
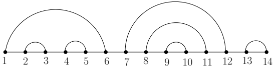
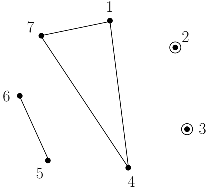
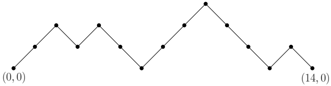


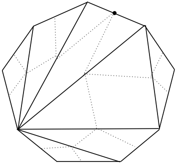
1 1 2 4 4 4 7











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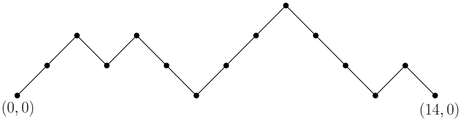
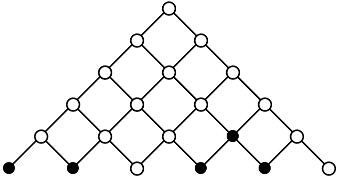
Objects - Bijections





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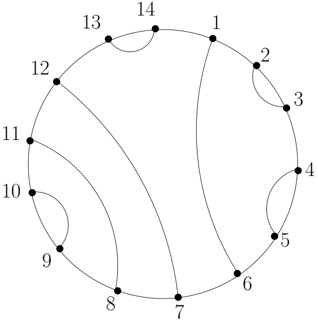

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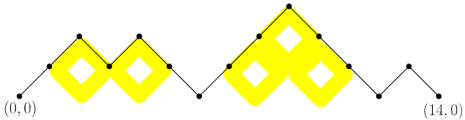

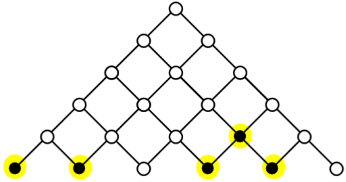

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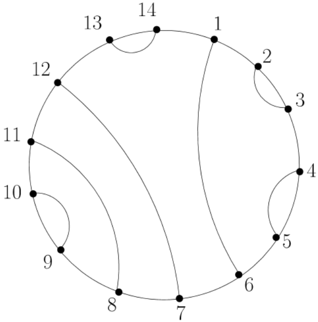

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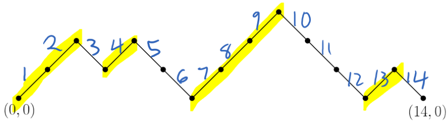

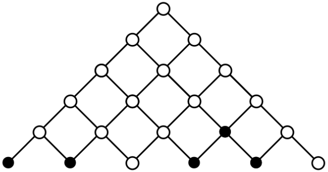

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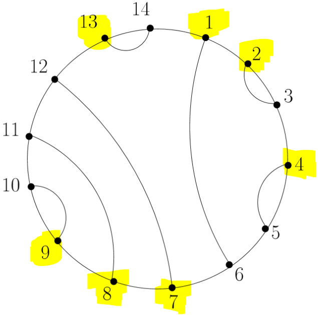

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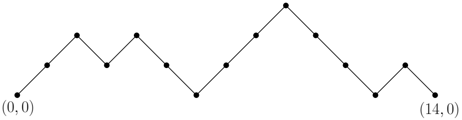

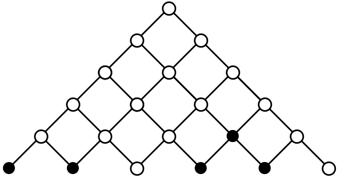

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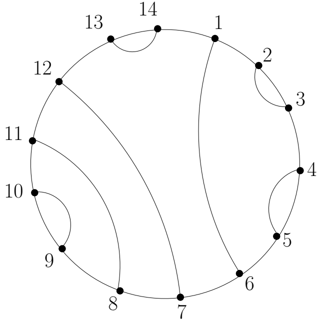
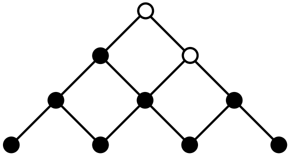


Actions



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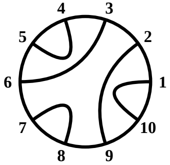

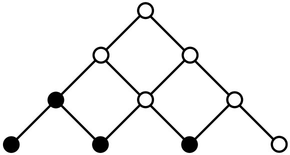

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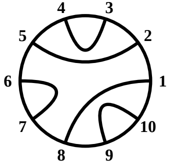

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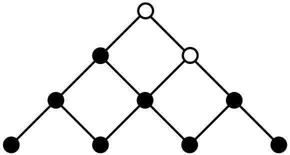

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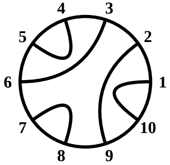
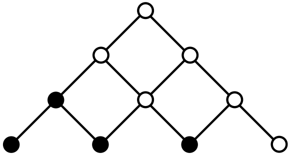


Rotation



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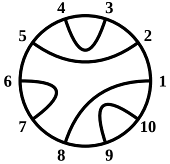

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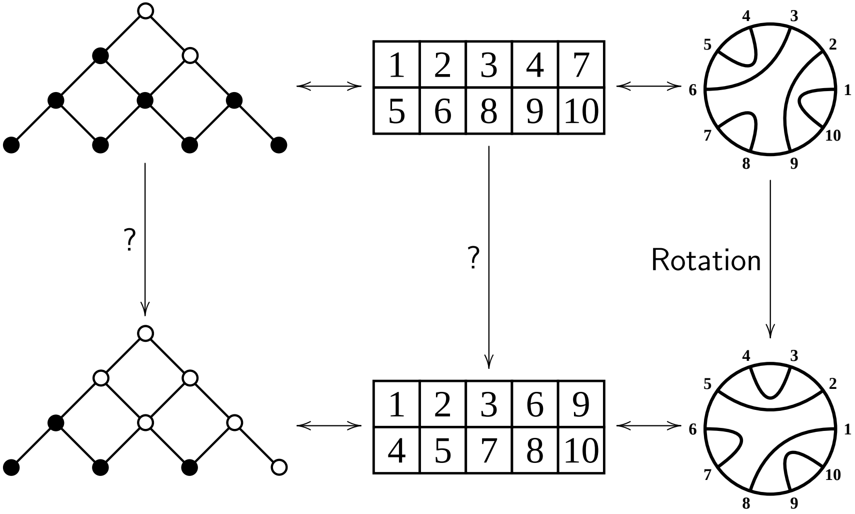

1 2 3 4 7
5 6 8 9 10
4 3
5 2
6 1
7 10
8 9
?
?
Rotation
1 2 3 6 9
4 5 7 8 10
4 3
5 2
6 1
7 10
8 9

Standard Young Tableaux

Definition

A **standard Young tableau** is a collection of n boxes of partition shape λ filled with positive the integers $\{1, 2, \ldots, n\}$ such that the rows are increasing from left to right and columns are increasing from top to bottom.

1	2	4	9	10
3	5	6	8	
7	11			

Promotion

1	2	3	4	7
5	6	8	9	10

Promotion

	2	3	4	7
5	6	8	9	10

Promotion

2		3	4	7
5	6	8	9	10

Promotion

2	3		4	7
5	6	8	9	10

Promotion

2	3	4		7
5	6	8	9	10

Promotion

2	3	4	7	
5	6	8	9	10

Promotion

2	3	4	7	10
5	6	8	9	

Promotion

2	3	4	7	10
5	6	8	9	11

Promotion

1	2	3	6	9
4	5	7	8	10

Promotion

1	2	3	4	7
5	6	8	9	10

Promotion

1	2	3	4	7
5	6	8	9	10

Promotion

1	2	3	4	7
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1	2	3	4	7
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1	2	3	4	7
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Promotion

1	2	3	5	7
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Promotion

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1	2	3	6	9
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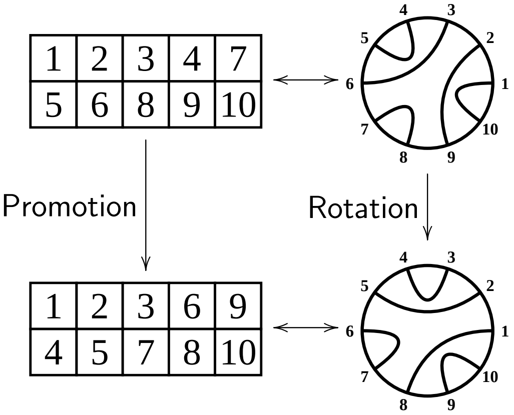
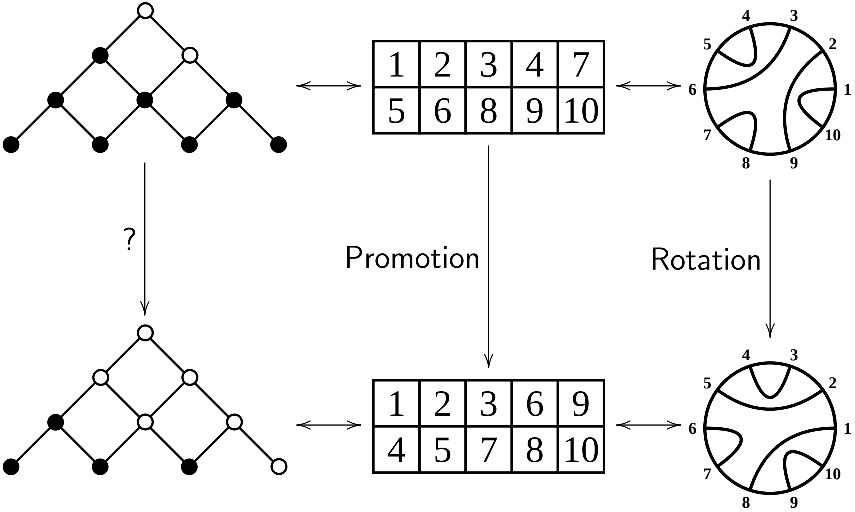
Theorem (D. White)

There is an **equivariant bijection** *between promotion on $2 \times n$ standard Young tableaux and rotation on non-crossing matchings of $2n$. So promotion has order $2n$.*



Actions - Equivariant Bijections



Actions - Equivariant Bijections

Proposition (Williams-S. 2012)

There is an **equivariant bijection** *between promotion on* $2 \times n$ *standard Young tableaux and toggling left to right on order ideals of the triangular poset* $\Phi^+(A_{n-1})$.

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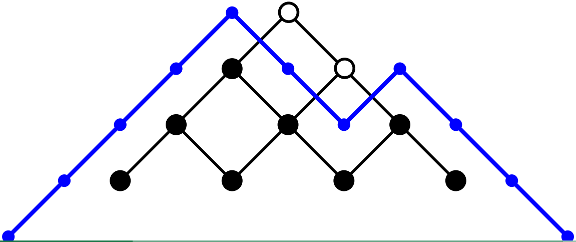

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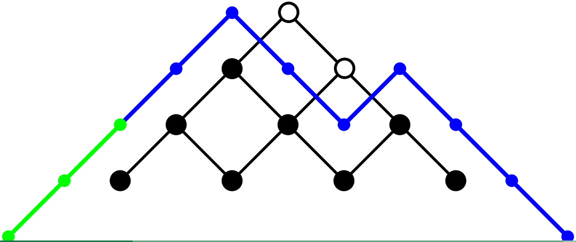

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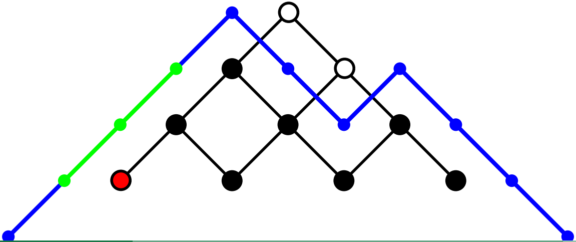

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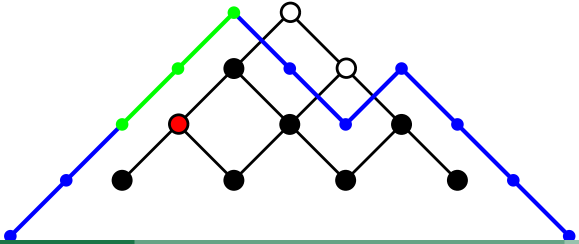

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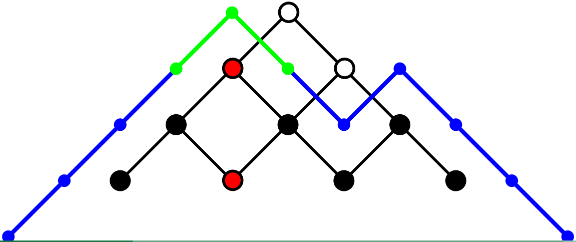

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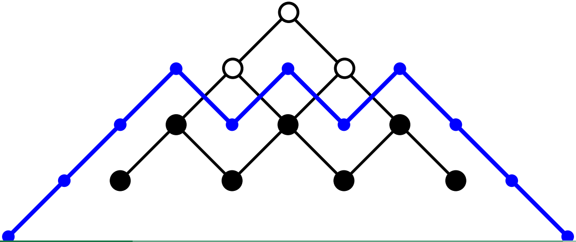

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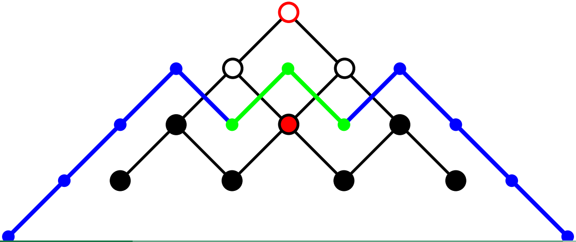

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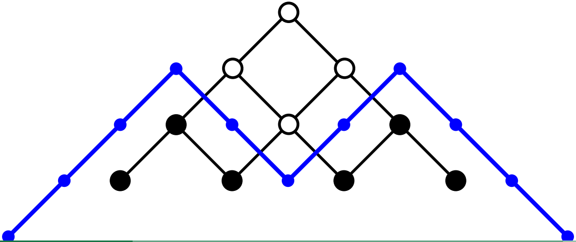

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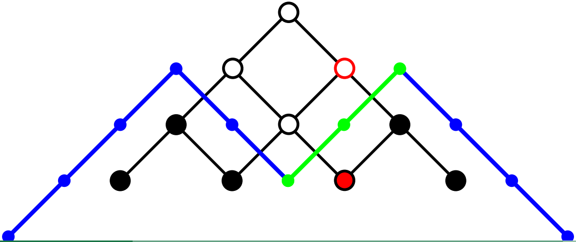

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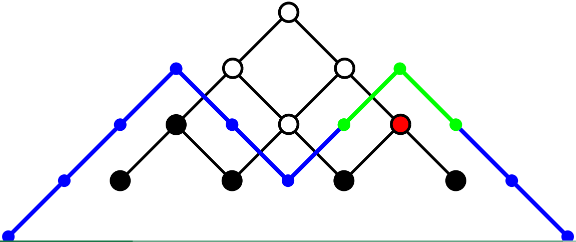

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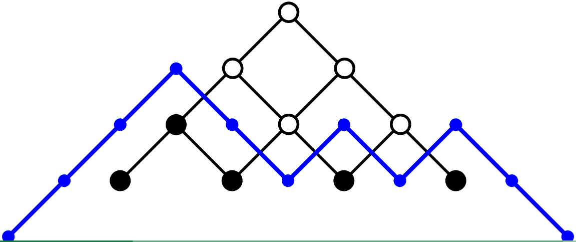

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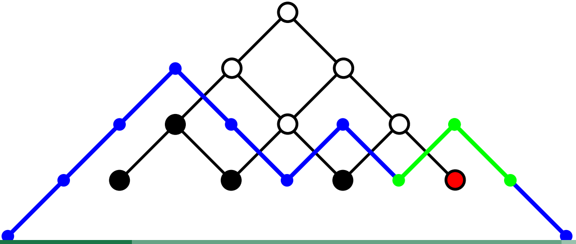

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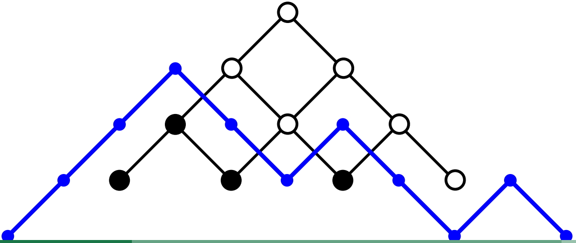

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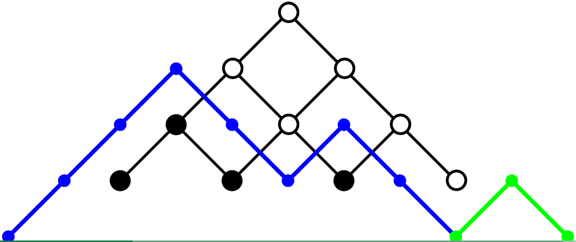

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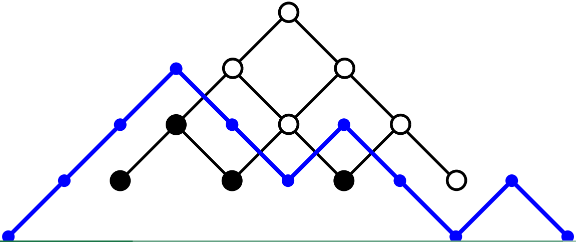

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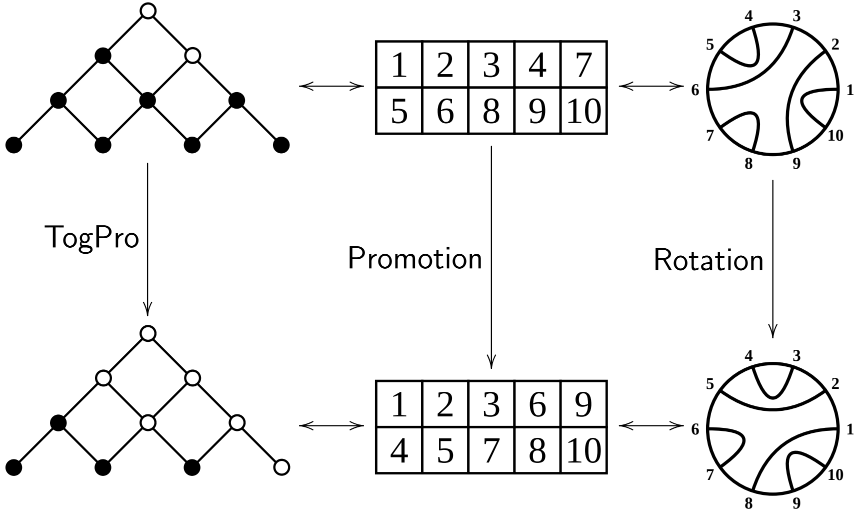
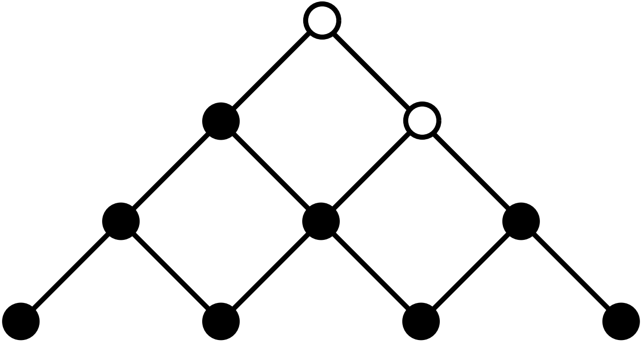


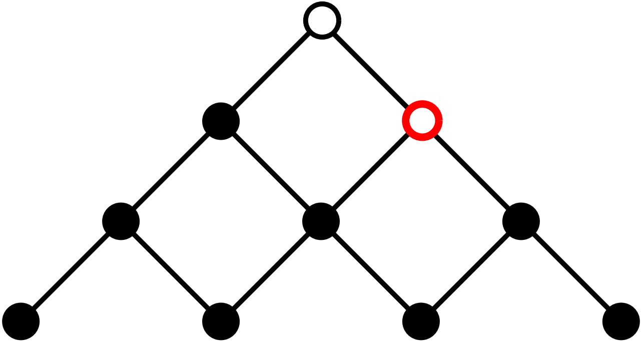
Rowmotion

An order ideal



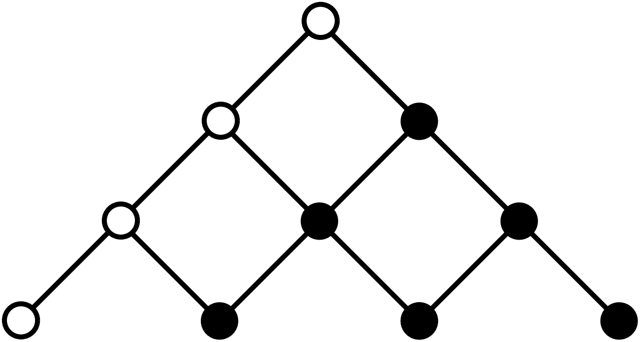
Rowmotion

Find the **minimal** elements of P not in the order ideal.



Rowmotion

Use them to generate a new order ideal.



Promotion and rowmotion are conjugate actions

Theorem (Cameron-Fon-der-Flaass 1995)

Rowmotion can also be computed by toggling from top to bottom.

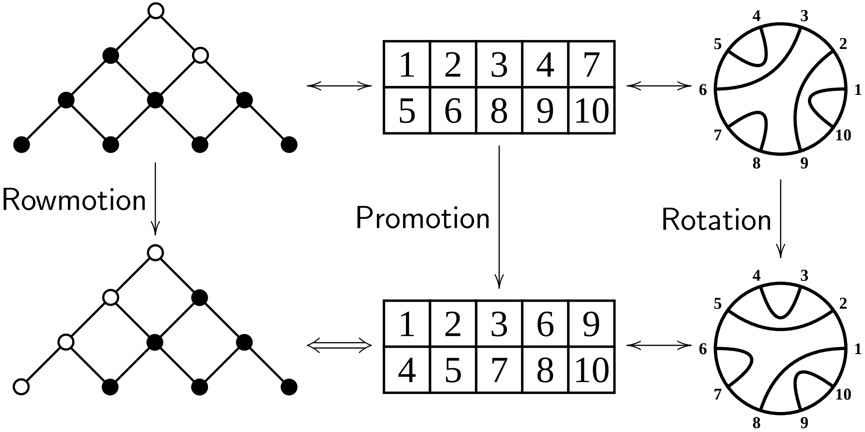
Theorem (Williams-S. 2012)

In any ranked poset, there is an equivariant bijection between the order ideals under rowmotion (toggle top to bottom) and promotion (toggle left to right).

In an equivariant bijection
the orbit structure is preserved.

Corollary (Williams-S. 2012)

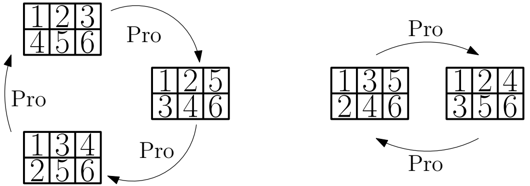
There is an equivariant bijection between promotion on $2 \times n$ standard Young tableaux and rowmotion on order ideals of $\Phi^+(A_{n-1})$. So rowmotion has order $2n$.



The cyclic sieving phenomenon

Definition (V. Reiner, D. Stanton, D. White 2004)

Given a set S, a polynomial $f(q)$, and a bijective action g of order n, the triple $(S, f(q), g)$ exhibits the *cyclic sieving phenomenon* if $f(\zeta^d)$, where $\zeta = e^{2\pi i/n}$, counts the elements of S fixed under g^d.



$f(q) = \frac{6!_q}{4!_q 3!_q} = 1 + q^2 + q^3 + q^4 + q^6 \qquad \zeta = e^{2\pi i/6} = e^{\pi i/3}$

$f(\zeta^1) = f(\zeta^4) = f(\zeta^5) = 0$, so 0 elements fixed under Pro^1, Pro^4, Pro^5.

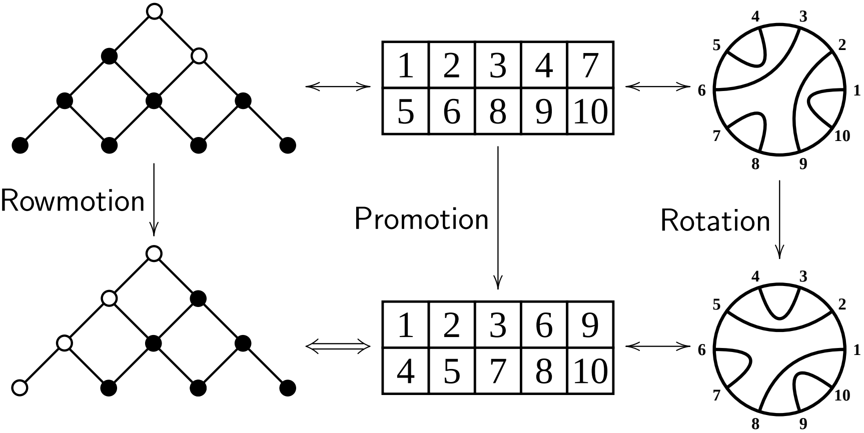
$f(\zeta^2) = 2$, so 2 elements are fixed under Pro^2.

$f(\zeta^3) = f(-1) = 3$, so 3 elements are fixed under Pro^3.

$f(\zeta^6) = f(1) = 5$, so 5 elements are fixed under Pro^6.

Corollary (Williams-S. 2012)

There is an equivariant bijection between promotion on $2 \times n$ standard Young tableaux and rowmotion on order ideals of $\Phi^+(A_{n-1})$.
So this is an instance of the cyclic sieving phenomenon.



Promotion, rotation, and web invariant polynomials

Algebraic question

Suppose $X = \{x_1, x_2, \ldots, x_n\}, Y = \{y_1, y_2, \ldots, y_n\}$.

Which polynomials in $\mathbb{C}[X, Y]$ are invariant under

$$SL_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; ad - bc = 1 \right\}?$$

Algebraic question

Suppose $X = \{x_1, x_2, \ldots, x_n\}$, $Y = \{y_1, y_2, \ldots, y_n\}$.

Which polynomials in $\mathbb{C}[X, Y]$ are invariant under
$SL_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; ad - bc = 1 \right\}$?

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}$$
$$= \begin{pmatrix} ax_1 + by_1 & ax_2 + by_2 & \cdots & ax_n + by_n \\ cx_1 + dy_1 & cx_2 + dy_2 & \cdots & cx_n + dy_n \end{pmatrix}$$

Example of a polynomial in $\mathbb{C}[X, Y]^{SL_2(\mathbb{C})}$

One such polynomial is: $x_1 y_2 - x_2 y_1$.

Under the change of variables

$$\begin{array}{ll} x_1 \mapsto ax_1 + by_1 & x_2 \mapsto ax_2 + by_2 \\ y_1 \mapsto cx_1 + dy_1 & y_2 \mapsto cx_2 + dy_2 \end{array},$$

this polynomial equals:

$$\begin{aligned}
(ax_1 + by_1)(cx_2 + dy_2) - (ax_2 + by_2)(cx_1 + dy_1) \\
=& acx_1x_2 + bcx_2y_1 + adx_1y_2 + bdy_1y_2 \\
&- (acx_1x_2 + bcx_1y_2 + adx_2y_1 + bdy_1y_2) \\
=& bc(x_2y_1 - x_1y_2) + ad(x_1y_2 - x_2y_1) \\
=& (ad - bc)(x_1y_2 - x_2y_1) \\
=& x_1y_2 - x_2y_1.
\end{aligned}$$

Algebraic question

Suppose $X = \{x_1, x_2, \ldots, x_n\}$, $Y = \{y_1, y_2, \ldots, y_n\}$.

Which polynomials in $\mathbb{C}[X, Y]$ are invariant under

$$SL_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; ad - bc = 1 \right\}?$$

Algebraic question

Suppose $X = \{x_1, x_2, \ldots, x_n\}$, $Y = \{y_1, y_2, \ldots, y_n\}$.

Which polynomials in $\mathbb{C}[X, Y]$ are invariant under
$SL_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; ad - bc = 1 \right\}$?

$\mathbb{C}[X, Y]^{SL_2}$ is spanned by products of 2×2 minors of

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}.$$

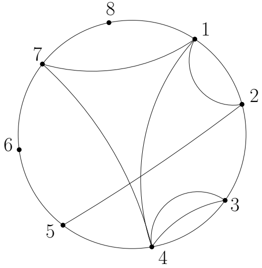
Algebraic interpretation of tableaux

Interpret a tableau as a product of minors corresponding to the columns.

1	1	1	2	3	3	4
2	4	7	5	4	4	7

$\leftrightarrow$

$$\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \begin{vmatrix} x_1 & x_4 \\ y_1 & y_4 \end{vmatrix} \begin{vmatrix} x_1 & x_7 \\ y_1 & y_7 \end{vmatrix} \begin{vmatrix} x_2 & x_5 \\ y_2 & y_5 \end{vmatrix} \begin{vmatrix} x_3 & x_4 \\ y_3 & y_4 \end{vmatrix}^2 \begin{vmatrix} x_4 & x_7 \\ y_4 & y_7 \end{vmatrix}$$



Tableaux form a basis

Theorem (Standard monomial theory)

A basis for the degree $2n$ homogeneous part of $\mathbb{C}[X, Y]^{SL_2}$ is given by those products of matrix minors corresponding to semistandard tableaux of shape (n, n).

So the dimension is given by the hook-content formula.

Theorem (Standard monomial theory)

A basis for the degree $2n$ homogeneous part of $\mathbb{C}[X, Y]^{SL_2}$ with degree 1 in each pair of variables $\{x_i, y_i\}$ (the Specht module $S^{(n,n)}$) is given by those products of matrix minors corresponding to standard Young tableaux of shape (n, n).

So the dimension is given by the nth Catalan number.

Tableaux form a basis

What happens if we let S_n act on $\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}$ by $(12\cdots n)$ (cycling the columns)?

Tableaux form a basis

What happens if we let S_n act on $\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}$ by $(12\cdots n)$ (cycling the columns)?

This cycles the numbers in the tableau:

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 6 & 9 \\ \hline 4 & 5 & 7 & 8 & 10 \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|c|}\hline 2 & 3 & 4 & 7 & 10 \\ \hline 5 & 6 & 8 & 9 & 1 \\ \hline \end{array}$$

Tableaux form a basis

What happens if we let S_n act on $\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{pmatrix}$ by $(12 \cdots n)$ (cycling the columns)?

This cycles the numbers in the tableau:

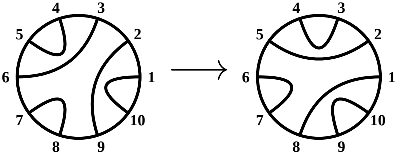
$$\begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 6 & 9 \\ \hline 4 & 5 & 7 & 8 & 10 \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|c|c|} \hline 2 & 3 & 4 & 7 & 10 \\ \hline 5 & 6 & 8 & 9 & 1 \\ \hline \end{array}$$

Is there another basis that behaves better under this action?

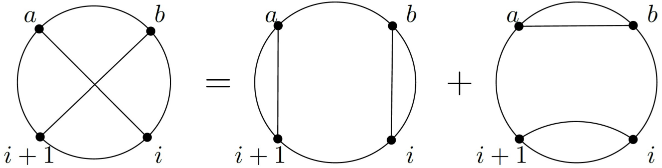
Web basis of noncrossing matchings

Theorem (Theory of SL_2 webs)

A basis for the Specht module $S^{(n,n)}$ is given by those products of matrix minors corresponding to noncrossing matchings of $2n$. The long cycle $(12\cdots n)$ acts by rotation of diagrams.



We also have a reasonable uncrossing rule:



Three row generalization

$\mathbb{C}[X, Y, Z]^{SL_3}$ is spanned by products of 3×3 minors of

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \\ z_1 & z_2 & \cdots & z_n \end{pmatrix}.$$

Theorem (Standard monomial theory)

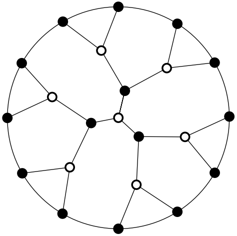
Standard Young tableaux of shape $\lambda = (n, n, n) = n^3$ index a basis for the degree $3n$ homogeneous part of $\mathbb{C}[X, Y, Z]^{SL_3}$ with degree 1 in each triple of variables $\{x_i, y_i, z_i\}$ (the Specht module S^λ).

1	2	4	7
3	6	8	9
5	10	11	12

Three row generalization

Theorem (Petersen-Pylyavskyy-Rhoades, 2009)

SL_3 webs index a basis for the Specht module $S^{(n,n,n)}$. The long cycle $(12\cdots n)$ acts by rotation of diagrams.



What about SL_m? Standard Young tableaux of shape n^m index a basis for the Specht module S^{n^m}, but no one knows a web basis that interacts well with tableaux combinatorics. (There are non-diagramatic bases that respect the S_n action.)

Promotion, rotation, and web invariant polynomials

A parabolic subgroup of $SL_m(\mathbb{C})$

Which polynomials in $\mathbb{C}[X_1, \ldots, X_m]$ are invariant under

$$SL_m(\mathbb{C})^* = \left\{ \begin{pmatrix} a_{11} & a_{12} & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mm} \end{pmatrix} \middle| \det = 1 \right\}?$$

$\mathbb{C}[X_1, X_2, \ldots, X_m]^{SL_m^*}$ is spanned by products of $m \times m$ and 2×2 top-justified minors of $\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix}$.

Tableaux form a basis

Theorem (Standard monomial theory)

Standard Young tableaux of pennant shape $(n, n, 1^{m-2})$ index a basis for the degree $2n + m - 2$ homogeneous part of $\mathbb{C}[X_1, X_2, \ldots, X_m]^{SL_m^*}$ with degree 1 in each m-tuple of variables $\{x_{1i}, x_{2i}, \ldots, x_{mi}\}$ (the Specht module $S^{(n,n,1^{m-2})}$).

1	4	7
2	5	9
3		
6		
8		
10		

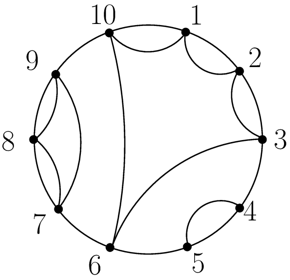
$$\leftrightarrow \begin{vmatrix} x_{11} & x_{12} & x_{13} & x_{16} & x_{18} & x_{110} \\ x_{21} & x_{22} & x_{23} & x_{26} & x_{28} & x_{210} \\ x_{31} & x_{32} & x_{33} & x_{36} & x_{38} & x_{310} \\ x_{41} & x_{42} & x_{43} & x_{46} & x_{48} & x_{410} \\ x_{51} & x_{52} & x_{53} & x_{56} & x_{58} & x_{510} \\ x_{61} & x_{62} & x_{63} & x_{66} & x_{68} & x_{610} \end{vmatrix} \begin{vmatrix} x_{14} & x_{15} \\ x_{24} & x_{25} \end{vmatrix} \begin{vmatrix} x_{17} & x_{19} \\ x_{27} & x_{29} \end{vmatrix}$$

$$M_{1,2,3,4,5,6}^{1,2,3,6,8,10} \cdot M_{1,2}^{4,5} \cdot M_{1,2}^{7,9}$$

A bijection

1	4	7
2	5	9
3		
6		
8		
10		

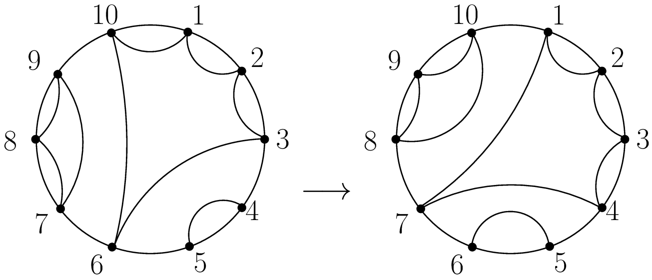
$\longleftrightarrow$



Web basis of noncrossing partitions

Theorem (Rhoades 2017, Kim-Rhoades 2021+, Patrias-Pechenik-S. 2021+)

Noncrossing partitions of $2n+m-2$ into n parts with no singletons index a basis for the Specht module $S^{(n,n,1^{m-2})}$. The long cycle $(12\cdots n)$ acts by rotation of diagrams.



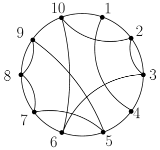
In our new proof of this theorem, we define an explicit polynomial in the X variables for each noncrossing partition and show this is a basis.

Web invariant polynomials

Given a set partition π, how do we define its polynomial $[\pi]$?

Our polynomial will be a signed sum over *Reiner-Shimozono tableaux*.

$$\pi = \{\{2,3,6,10\},\{5,7,8,9\},\{1,4\}\}$$



2	5	1
3	7	4
6		
	8	
10		
	9	

$$\begin{vmatrix} x_{12} & x_{13} & x_{16} & x_{110} \\ x_{22} & x_{23} & x_{26} & x_{210} \\ x_{32} & x_{33} & x_{36} & x_{310} \\ x_{52} & x_{53} & x_{56} & x_{510} \end{vmatrix} \begin{vmatrix} x_{15} & x_{17} & x_{18} & x_{19} \\ x_{25} & x_{27} & x_{28} & x_{29} \\ x_{45} & x_{47} & x_{48} & x_{49} \\ x_{65} & x_{67} & x_{68} & x_{69} \end{vmatrix} \begin{vmatrix} x_{11} & x_{14} \\ x_{21} & x_{24} \end{vmatrix}$$

$$(-1)^{\mathrm{inv}(T)}\mathrm{RS}(T) = (-1)^7 M^{2,3,6,10}_{1,2,3,5} \cdot M^{5,7,8,9}_{1,2,4,6} \cdot M^{1,4}_{1,2}$$

Web invariant polynomials

Suppose $\pi = \{\{2,3,6,10\},\{5,7,8,9\},\{1,4\}\}$. Then $\mathcal{RS}(\pi)$ is:

2	5	1
3	7	4
6		
10		
	8	
	9	

2	5	1
3	7	4
6		
	8	
10		
	9	

2	5	1
3	7	4
6		
	8	
	9	
10		

2	5	1
3	7	4
	8	
6		
10		
	9	

2	5	1
3	7	4
	8	
6		
	9	
10		

2	5	1
3	7	4
	8	
	9	
6		
10		

$$[\pi] = \sum_{T \in \mathcal{RS}(\pi)} (-1)^{\mathrm{inv}\, T}\, \mathrm{RS}(T) = \sum_{T \in \mathcal{RS}(\pi)} \mathrm{sgn}(T)\, \mathrm{RS}(T)$$

$$= M^{2,3,6,10}_{1,2,3,4} \cdot M^{5,7,8,9}_{1,2,5,6} \cdot M^{1,4}_{1,2} - M^{2,3,6,10}_{1,2,3,5} \cdot M^{5,7,8,9}_{1,2,4,6} \cdot M^{1,4}_{1,2}$$

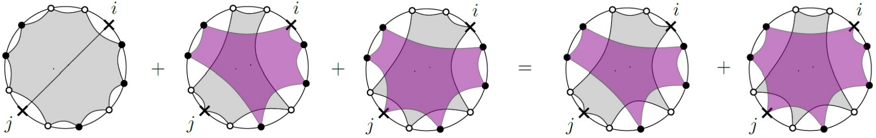
$$+ M^{2,3,6,10}_{1,2,3,6} \cdot M^{5,7,8,9}_{1,2,4,5} \cdot M^{1,4}_{1,2} + M^{2,3,6,10}_{1,2,4,5} \cdot M^{5,7,8,9}_{1,2,3,6} \cdot M^{1,4}_{1,2}$$

$$- M^{2,3,6,10}_{1,2,4,6} \cdot M^{5,7,8,9}_{1,2,3,5} \cdot M^{1,4}_{1,2} + M^{2,3,6,10}_{1,2,5,6} \cdot M^{5,7,8,9}_{1,2,3,4} \cdot M^{1,4}_{1,2}.$$

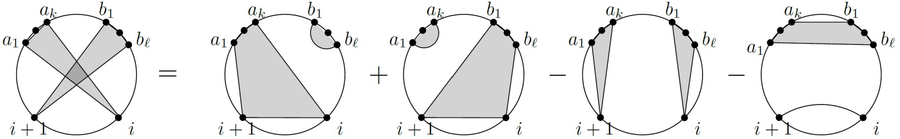
Proof that this is a basis

A polynomial relation for changing block sizes:

$$\left[\{A \cup B, I \cup J\}\right] + \left[\{A \cup I, B \cup J\}\right] + \left[\{A \cup J, B \cup I\}\right] = \left[\{A, B \cup I \cup J\}\right] + \left[\{A \cup I \cup J, B\}\right]$$

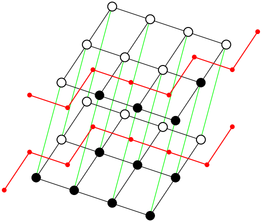


This specializes to an uncrossing rule:



Promotion, rotation, and web invariant polynomials

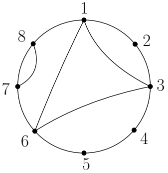
Actions - Equivariant Bijections

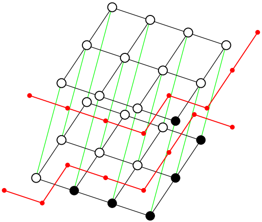


$\longleftrightarrow$

1	3	7
3	6	8

$\longleftrightarrow$

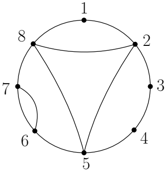




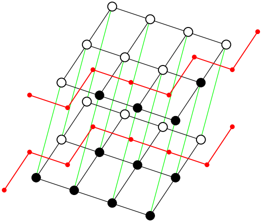
$\Longleftrightarrow$

2	5	6
5	7	8

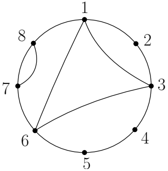
$\longleftrightarrow$



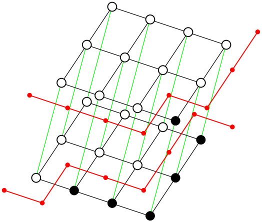
Actions - Equivariant Bijections



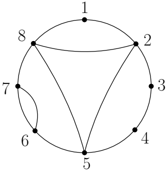
1	3	7
3	6	8



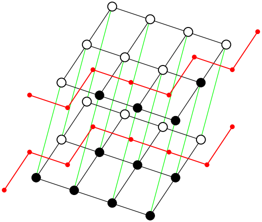
Rotation



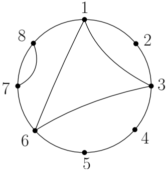
2	5	6
5	7	8



Actions - Equivariant Bijections



1	3	7
3	6	8



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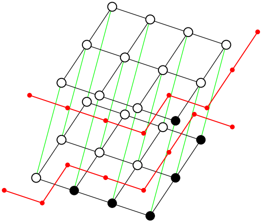
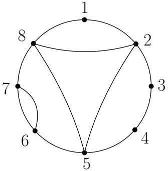
Rotation



2	5	6
5	7	8



K-promotion

1	3	7
3	6	8

K-promotion

	3	7
3	6	8

K-promotion

3		7
	6	8

K-promotion

3	6	7
6		8

K-promotion

3	6	7
6	8	

K-promotion

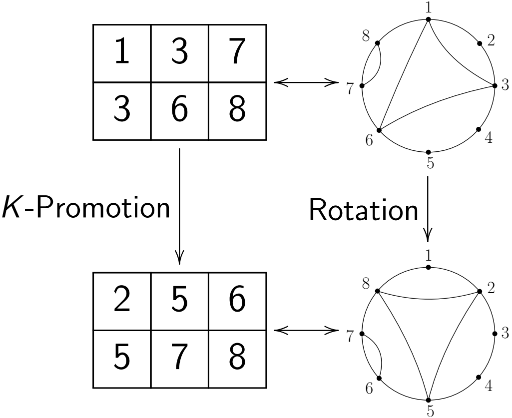
3	6	7
6	8	9

K-promotion

2	5	6
5	7	8

Theorem (O. Pechenik, 2014)

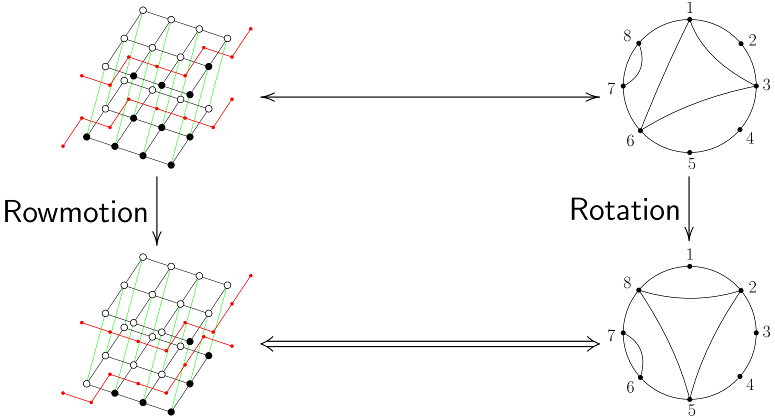
There is an **equivariant bijection** *between K-promotion on packed $2 \times n$ increasing tableaux with entries at most $2n + m - 2$ and rotation on non- crossing partitions of $2n + m - 2$ into n parts with no singletons. So K-promotion has order $2n + m - 2$ and exhibits the cyclic sieving phenomenon.*



Actions - equivariant bijections

Theorem (Williams-S. 2012)

There is an equivariant bijection between rowmotion on order ideals of $\mathbf{a} \times \mathbf{b} \times \mathbf{2}$ and rotation on noncrossing partitions of $a+b+1$ into $b+1$ blocks. So rowmotion has order $a+b+1$ and exhibits the CSP.

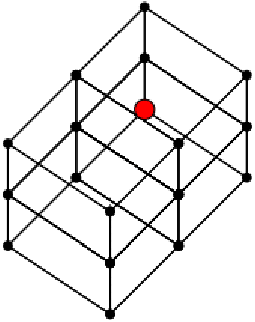


Actions - equivariant bijections

Theorem (K. Dilks, O. Pechenik, S. 2017)

There is an equivariant bijection between K-promotion on $a \times b$ **increasing tableaux** *with entries at most $a + b + c - 1$ and toggling back to front on* $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$.

1	2	4
4	5	6

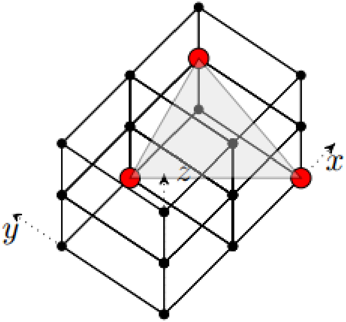


Actions - equivariant bijections

Theorem (K. Dilks, O. Pechenik, S. 2017)

There is an equivariant bijection between K-promotion on $a \times b$ **increasing tableaux** *with entries at most $a + b + c - 1$ and toggling back to front on* $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$.

1	2	4
4	5	6



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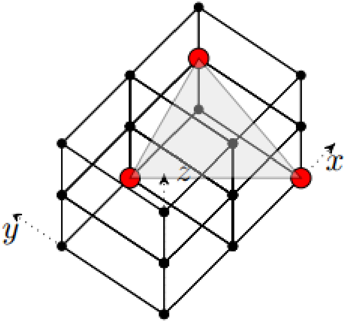
1	3	4
4	5	6



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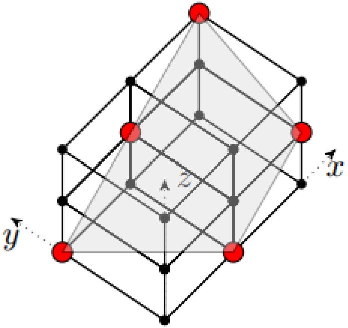

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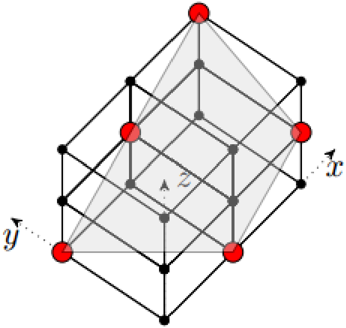
1	3	4
3	5	6



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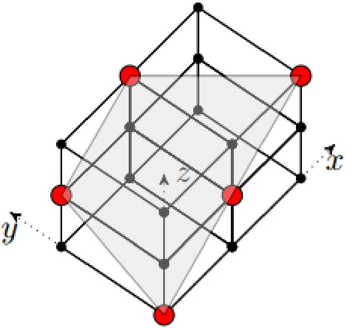
1	3	4
3	5	6



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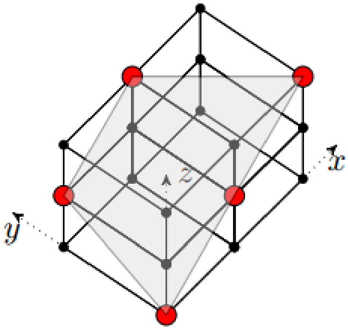
1	3	5
3	4	6



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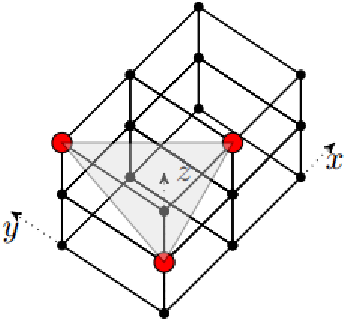
1	3	5
3	4	6



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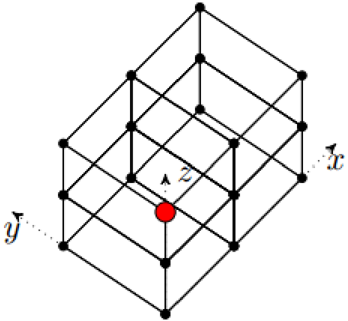
1	3	5
3	4	6



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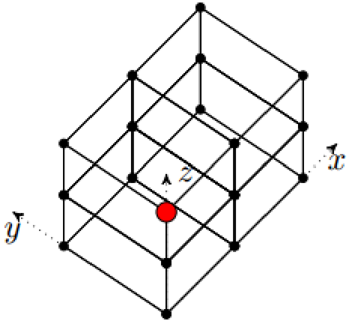
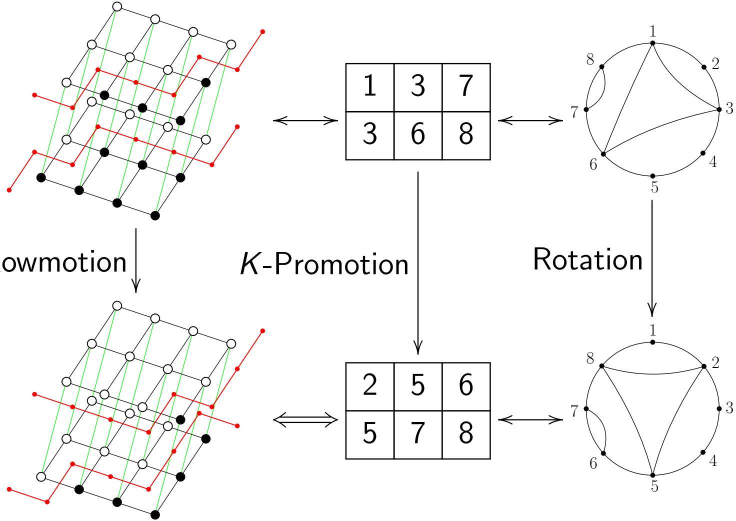
1	3	5
3	4	7



Actions - Equivariant Bijections



Promotion, rotation, and web invariant polynomials

1. Combinatorial objects and actions
2. Invariant polynomials
3. Invariant polynomials - new generalization
4. More combinatorial objects and actions

THANKS

- R. Patrias, O. Pechenik, J. Striker, Web invariants for noncrossing partitions and a geometric realization of Rhoades' skein modules, In preparation.
- J. Striker and N. Williams, Promotion and rowmotion, *Eur. J. Combin.* **33** (2012), no. 8, 1919–1942.
- K. Dilks, O. Pechenik, and J. Striker, Resonance in orbits of plane partitions and increasing tableaux, *J. Combin. Series A*, **148** (2017) 244–274.
- J. Striker, Dynamical algebraic combinatorics: promotion, rowmotion, and resonance, *Notices of the AMS*, **64** (2017), no. 6, 543–549.