Representations from matrix varieties, and filtered RSK

Ada Stelzer
UIUC

Joint work with Abigail Price and Alexander Yong
MSU Combinatorics and Graph Theory Seminar
11 September 2024

Determinantal Varieties

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Fact

The irreducible varieties in $\mathrm{Mat}_{m,n}$ *stable under the* **GL***-action are exactly the classical determinantal varieties* $\mathfrak{X}_k$.

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Theorem (Doubilet-Rota-Stein '74)

Each coordinate ring $\mathbb{C}[\mathfrak{X}_k]$ *is a* **GL**-*representation with irreducible decomposition*

$$\mathbb{C}[\mathfrak{X}_k] \cong_{\mathbf{GL}} \bigoplus_{\ell(\lambda) \leq k} (V_\lambda(m) \boxtimes V_\lambda(n)).$$

Subgroups of GL_n $(n = 4)$

$$\begin{bmatrix} b_{11} & 0 & 0 & 0 \\ b_{21} & b_{22} & 0 & 0 \\ b_{31} & b_{32} & b_{33} & 0 \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

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These give subgroups $\mathbf{B} := B_m \times B_n$, $\mathbf{L}_{\mathbf{I}|\mathbf{J}} := L_{\mathbf{I}} \times L_{\mathbf{J}}$, and $\mathbf{T} := T_m \times T_n$ in $\mathbf{GL}$.

Matrix Schubert varieties

Fact (Fulton '92)

The irreducible $\mathbf{B}$*-stable varieties in* $\mathrm{Mat}_{m,n}$ *are the matrix Schubert varieties* $\mathfrak{X}_w \subseteq \mathrm{Mat}_{m,n}$ *(*w *a partial permutation).*

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For each w*,* $\mathbb{C}[\mathfrak{X}_w]$ *is an* $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$*-representation for some* $\mathbf{I}$ *and* $\mathbf{J}$*.*

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How does $\mathbb{C}[\mathfrak{X}_w]$ *decompose as a* $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$*-representation?*

Decomposing representations: warm-up

Think about $V := \mathbb{C}[z_1, \ldots, z_n]$, with GL_n acting by matrix-vector multiplication on the variables. How does V decompose as a GL_n representation?

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1. Decompose V as a $T_n \subseteq GL_n$ representation.
2. Figure out how to assemble T_n representations into GL_n representations.

Torus representations

$$\begin{bmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_3 & 0 \\ 0 & 0 & 0 & t_4 \end{bmatrix}$$

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Fact

The irreducible polynomial representations of T_n are 1-dimensional, indexed by tuples $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$. The T_n-action on $V_{\mathbf{a}}$ is

$$\mathbf{t} \cdot v := t_1^{a_1} t_2^{a_2} \ldots t_n^{a_n} v.$$

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The monomials $\mathbf{z}^{\mathbf{a}}$ are a basis for V with $\mathbf{t} \cdot \mathbf{z}^{\mathbf{a}} = (\mathbf{t}\mathbf{z})^{\mathbf{a}} = \mathbf{t}^{\mathbf{a}} \mathbf{z}^{\mathbf{a}}$, so

$$V \cong_{T_n} \bigoplus_{\mathbf{a} \in \mathbb{N}^n} V_{\mathbf{a}}.$$

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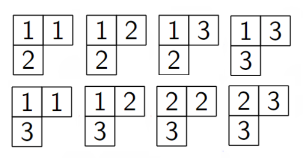
Example

The basis vectors for $V_\lambda(3)$ with $\lambda = (2, 1)$ are as follows:



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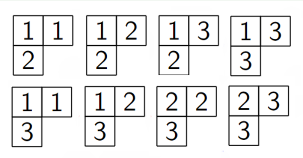

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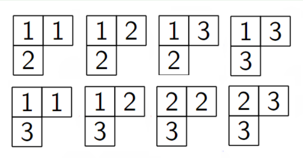

Fact

If T is an SSYT with content $\mathbf{a}$ (meaning a_1 1's, a_2 2's, etc.), then for any $\mathbf{t} \in T_n \subseteq GL_n$ we have $\mathbf{t} \cdot T = \mathbf{t}^{\mathbf{a}} T$. Thus

$$V_\lambda(n) \cong_{T_n} \bigoplus_{\mathbf{a}} V_{\mathbf{a}}^{\oplus c_{\mathbf{a}}^{\lambda}},$$

where $c_{\mathbf{a}}^{\lambda}$ is the number of SSYT of shape λ and content $\mathbf{a}$.

GL_n-decompositions from T_n-representations

Fact

Given a GL_n-representation V and decompositions

$$V \cong_{GL_n} \bigoplus_{\lambda} V_\lambda(n)^{\oplus c_\lambda^V} \cong_{T_n} \bigoplus_{\mathbf{a} \in \mathbb{N}^n} V_{\mathbf{a}}^{\oplus c_{\mathbf{a}}^V},$$

the numbers $c_{\mathbf{a}}^V$ uniquely determine the c_λ^V.

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Given a GL_n-representation V and decompositions

$$V \simeq_{GL_n} \bigoplus_{\lambda} V_\lambda(n)^{\oplus c_\lambda^V} \simeq_{T_n} \bigoplus_{\mathbf{a}\in\mathbb{N}^n} V_{\mathbf{a}}^{\oplus c_{\mathbf{a}}^V},$$

the numbers $c_{\mathbf{a}}^V$ uniquely determine the c_λ^V.

To compute the c_λ^V for V, find a T_n-representation basis $\mathfrak{B}$ to compute the $c_{\mathbf{a}}^V$, then define a "nice" map from $\mathfrak{B}$ to SSYT.

Completing the warm-up

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Monomials $\mathbf{z}^{\mathbf{a}}$ correspond bijectively to 1-row tableaux:

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Thus we obtain

$$\mathbb{C}[z_1, \ldots, z_n] \cong_{GL_n} V_\emptyset(n) \oplus V_{\square}(n) \oplus V_{\square\square}(n) \oplus \cdots =: \bigoplus_{d \in \mathbb{N}} V_{(d)}(n).$$

Alternate interpretation: Hilbert series

$$\bigoplus_{\mathbf{a}\in\mathbb{N}^n} V_{\mathbf{a}} \cong_{T_n} \mathbb{C}[z_1,\ldots,z_n] \cong_{GL_n} \bigoplus_{d\in\mathbb{N}} V_{(d)}(n).$$

Each $V_{\mathbf{a}}$ is spanned by the unique monomial with multidegree $\mathbf{a}$, whereas $V_{(d)}(n)$ is spanned by all monomials of total degree d.

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Main takeaway: we can compute the equivariant Hilbert series from the multigraded one combinatorially! This is important when computing via *degenerations* that don't preserve the GL_n-action.

The challenge

A set map $\phi : \mathfrak{B} \to SSYT$ is "nice" if for each partition λ, $|\phi^{-1}(T)|$ is constant over tableaux T of shape λ.

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The warm-up map ϕ sends monomials to 1-row tableaux based on the column indices of their variables, e.g.

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This time ϕ is not nice. For example,

$$\phi^{-1}\left(\boxed{1}\boxed{1}\right) = \{z_{11}^2, z_{11}z_{21}, z_{21}^2\},$$
$$\phi^{-1}\left(\boxed{1}\boxed{2}\right) = \{z_{11}z_{12}, z_{11}z_{22}, z_{21}z_{12}, z_{21}z_{22}\}.$$

Crystal bases

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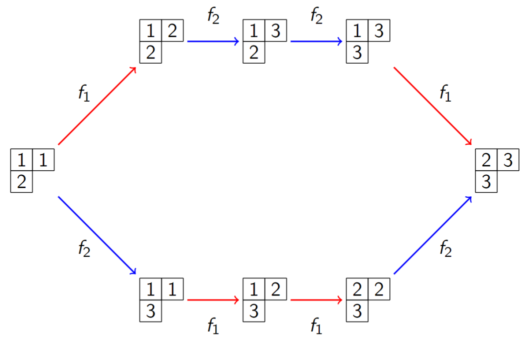
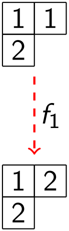
Example

Take GL_3 and $\lambda = (2, 1)$. Each f_i changes an i to an $(i + 1)$.



Defining crystal operators



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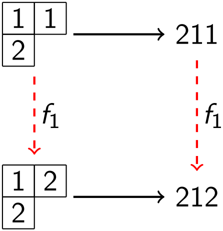

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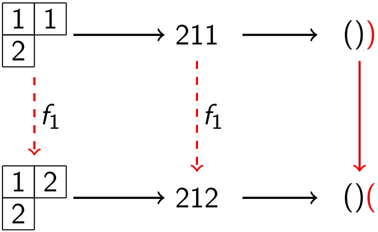

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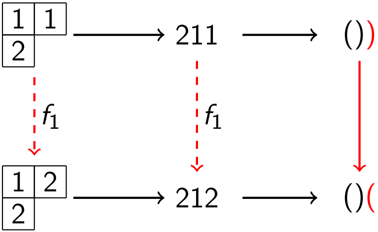

This defines crystal operators on all words, not just tableaux.

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We will repeatedly apply this strategy, building up to a map filterRSK$_{\mathbf{I|J}}$ decomposing $\mathbb{C}[\mathfrak{X}_w]$ as a $\mathbf{L_{I|J}}$-representation.

Monomials and matrices

We will consider the polynomial ring $\mathbb{C}[Z] := \mathbb{C}[\mathrm{Mat}_{m,n}]$ as a G-representation for $G = GL_n$, $L_{\mathbf{J}}$, $\mathbf{GL}$, and $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$.

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Definition

The *column word* of $M \in \mathrm{Mat}_{m,n}(\mathbb{Z}_{\geq 0})$:

$$z_{11}z_{12}z_{21}^2z_{22}^3 \leftrightarrow \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \xrightarrow{\text{col}} 1211222.$$

Crystal operators on $\mathrm{Mat}_{m,n}(\mathbb{Z}_{\geq 0})$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \xrightarrow{f_1^{\mathrm{col}}} \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$$

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\begin{array}{ccc}
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\Big\downarrow f_1^{\text{col}} & & \Big\downarrow f_1 \\
\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} & \xrightarrow{\text{col}} & 221
\end{array}
$$

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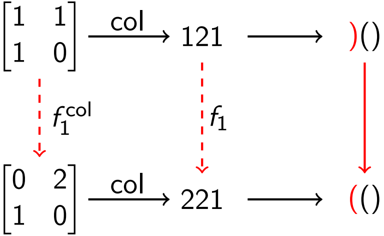

$\mathbb{C}[Z]$ as a GL_n-representation

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Corollary

The composition $\phi = \text{tab} \circ \text{col}$ *is a local isomorphism computing the decomposition of* $\mathbb{C}[Z]$ *as a* GL_n*-representation.*

$\mathbb{C}[Z]$ as an $L_{\mathbf{J}}$-representation

Now view $\mathbb{C}[Z]$ as an $L_{\mathbf{J}}$-representation. Irreducible $L_{\mathbf{J}}$-representations are (tensor) products of V_λ's, one per block.

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Definition

The **J**-*filtered column word* of $M \in \mathrm{Mat}_{m,n}(\mathbb{Z}_{\geq 0})$:

$$\left[\begin{array}{c|cc} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow{\mathrm{col}_{\mathbf{J}}} (11, 32232) \quad (\mathbf{J} = \{0, 1, 3\}).$$

This defines a new, restricted crystal structure on $\mathfrak{B}$.

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Remark

If $L_{\mathbf{J}} = T_n$, $\phi = \mathrm{tab} \circ \mathrm{col}_{\mathbf{J}}$ *is the warm-up map!*

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Theorem

The map $M \mapsto \mathrm{col}_{\mathbf{J}}(M)$ is a local isomorphism. Thus $\phi = \mathrm{tab} \circ \mathrm{col}_{\mathbf{J}}$ is a local isomorphism computing the decomposition of $\mathbb{C}[Z]$ as an $L_{\mathbf{J}}$-representation.

$\mathbb{C}[Z]$ as a **GL**-representation

Recall that $\mathbf{GL} := GL_m \times GL_n$.

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The maps $\text{tab} \circ \text{row}$ and $\text{tab} \circ \text{row}_{\mathbf{I}}$ compute the decompositions of $\mathbb{C}[Z]$ as a GL_m- or $L_{\mathbf{I}}$-representation respectively.

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Theorem (Danilov-Koshevoi '05, van Leeuwen '06)

The product map $M \mapsto (\text{tab}(\text{row}(M))|\text{tab}(\text{col}(M)))$ *is a local isomorphism computing the* **GL***-decomposition of* $\mathbb{C}[Z]$.

This product map is the *Robinson-Schensted-Knuth map* RSK.

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Proving the theorem reduces to showing that row crystal moves on M do not alter $\mathrm{tab}(\mathrm{col}(M))$.

$\mathbb{C}[Z]$ as a **GL**-representation

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For example, $\mathrm{tab}(\mathrm{col}(M)) = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \\ \cline{1-1}\end{array}$ in each matrix below:

$$\begin{bmatrix}1&1&0\\1&0&0\\0&0&0\end{bmatrix} \xrightarrow{f_1^{\mathrm{row}}} \begin{bmatrix}0&1&0\\2&0&0\\0&0&0\end{bmatrix} \xrightarrow{f_2^{\mathrm{row}}} \begin{bmatrix}0&1&0\\1&0&0\\1&0&0\end{bmatrix} \xrightarrow{f_2^{\mathrm{row}}} \begin{bmatrix}0&1&0\\0&0&0\\2&0&0\end{bmatrix} \xrightarrow{f_1^{\mathrm{row}}} \begin{bmatrix}0&0&0\\0&1&0\\2&0&0\end{bmatrix}$$

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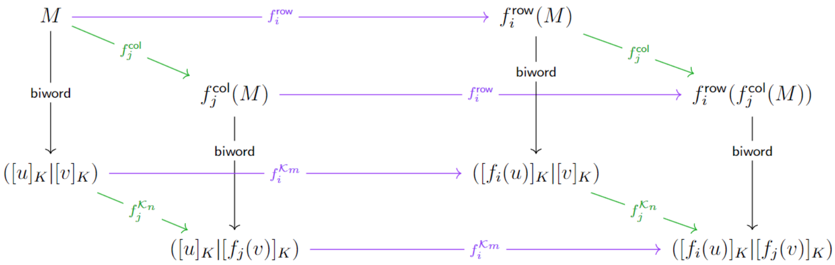
$\mathbb{C}[Z]$ as a **GL**-representation

More abstractly, the proof shows the following cube commutes:



$\mathbb{C}[Z]$ as a $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$-representation

Recall that $\mathbf{L}_{\mathbf{I}|\mathbf{J}} := L_{\mathbf{I}} \times L_{\mathbf{J}} \subseteq \mathbf{GL}$.

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Definition

For $M \in \mathrm{Mat}_{m,n}(\mathbb{Z}_{\geq 0})$, let

$$\mathrm{filterRSK}_{\mathbf{I}|\mathbf{J}}(M) := (\mathrm{tab}(\mathrm{row}_{\mathbf{I}}(M))|\mathrm{tab}(\mathrm{col}_{\mathbf{J}}(M))).$$

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Theorem (Price-S.-Yong '24)

$\mathrm{filterRSK}_{\mathbf{I}|\mathbf{J}}$ *is a local isomorphism computing the decomposition of* $\mathbb{C}[Z]$ *as a* $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$*-representation.*

Gröbner degeneration

Any variety $\mathfrak{X} \subseteq \mathrm{Mat}_{m,n}$ has a basis $\mathfrak{B}_{\mathfrak{X}} \subseteq \mathrm{Mat}_{m,n}(\mathbb{Z}_{\geq 0})$ of *standard monomials* for $\mathbb{C}[\mathfrak{X}]$, computed via *Gröbner degeneration*.

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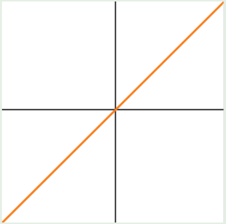

$y - x = 0$

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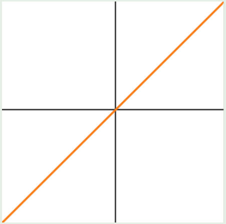

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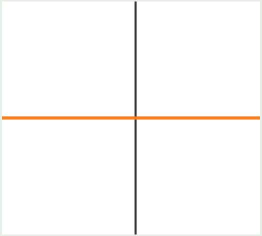

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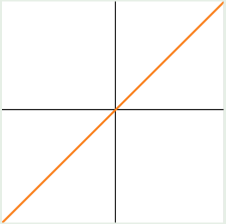

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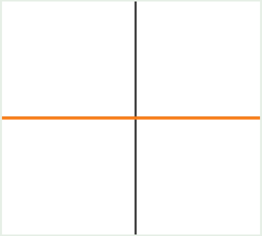

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With $\mathfrak{X} = \{(x, y) | y - x = 0\}$, $\mathfrak{B}_{\mathfrak{X}} = \{1, x, x^2, \dots\}$.

Standard monomials and determinantal varieties

Any variety $\mathfrak{X} \subseteq \mathrm{Mat}_{m,n}$ has a basis $\mathfrak{B}_{\mathfrak{X}} \subseteq \mathrm{Mat}_{m,n}(\mathbb{Z}_{\geq 0})$ of *standard monomials* for $\mathbb{C}[\mathfrak{X}]$, computed via *Gröbner degeneration*.

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Theorem (Price-S.-Yong '24)

If $\mathfrak{X}$ is $\mathbf{L_{I|J}}$-stable and $\mathfrak{B}_{\mathfrak{X}}$ is closed under the crystal operators, then $\mathrm{filterRSK}_{\mathbf{I|J}}$ is a local isomorphism computing the decomposition of $\mathbb{C}[\mathfrak{X}]$ as a $\mathbf{L_{I|J}}$-representation.

We call a variety $\mathfrak{X}$ satisfying the hypotheses of the theorem $\mathbf{L_{I|J}}$-*bicrystalline*.

The main theorem

The bases $\mathfrak{B}_{\mathfrak{X}}$ are known for the **GL**-stable varieties $\mathfrak{X}_k$ (Sturmfels '90) and **B**-stable varieties $\mathfrak{X}_w$ (Knutson-Miller '05).

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Recall that every **B**-stable $\mathfrak{X} \subseteq \mathrm{Mat}_{m,n}$ is $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$-stable for some **I**, **J**.

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Theorem (Price-S.-Yong '24)

If a **B***-stable variety* $\mathfrak{X}$ *is* $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$*-stable, then it is* $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$*-bicrystalline.*

In particular, $\mathrm{filterRSK}_{\mathbf{I}|\mathbf{J}}$ decomposes the coordinate ring of any matrix Schubert variety $\mathfrak{X}_w$.

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- Give an $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$-equivariant minimal free resolution of $\mathbb{C}[\mathfrak{X}_w]$.

The End

Thank you!