Permutations, moments, measures

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Joint work with
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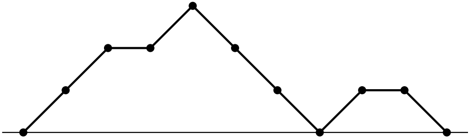
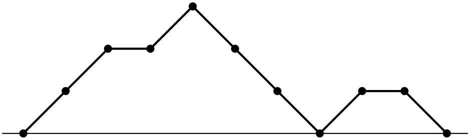
A *Motzkin path* is a sequence of up, down and level steps, starting at $(0, 0)$, ending at $(n, 0)$, never going below the x-axis:



A *Motzkin path* is a sequence of (1,1), (1,0) and (1,-1) steps, starting at $(0,0)$, ending at $(n,0)$, never going below the x-axis:



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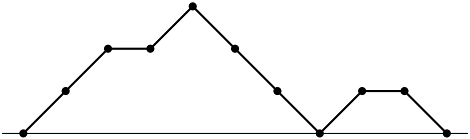
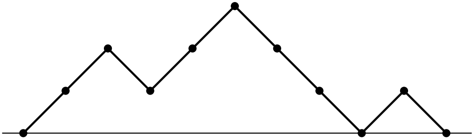


A *Dyck path* is a Motzkin path with no level steps:



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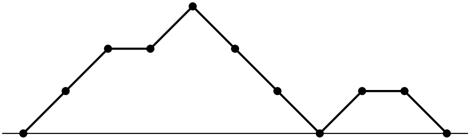

$$\frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}$$

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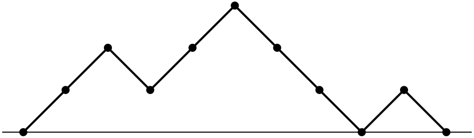
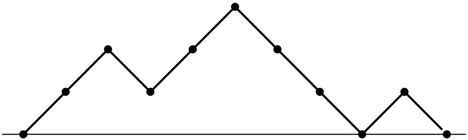


$$\frac{1-\sqrt{1-4x^2}}{2x^2}$$

The Catalan numbers count Dyck paths, whose generating function is

$$C(x) \; = \; \frac{1-\sqrt{1-4x^2}}{2x^2} = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} x^{2n}$$



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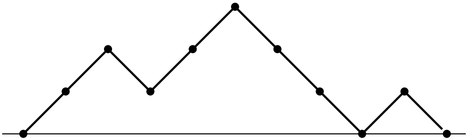
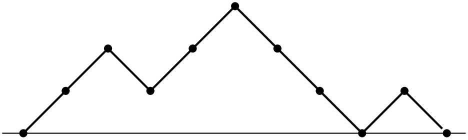
which satisfies $\quad C = 1 + x^2 C^2,$

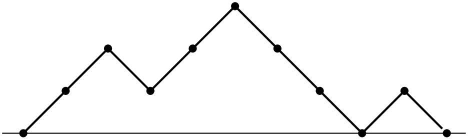
from which it follows that $\quad C(x) \;=\; \cfrac{1}{1-\cfrac{x^2}{1-\cfrac{x^2}{\ddots}}}$





Dyck path

$$\cfrac{1}{1-\cfrac{x^2}{1-\cfrac{x^2}{\ddots}}}$$



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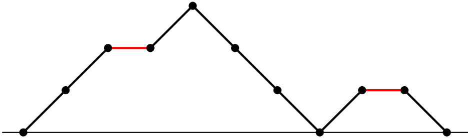
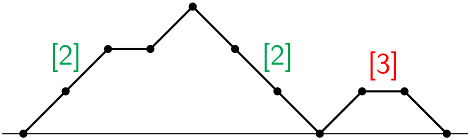


Motzkin path

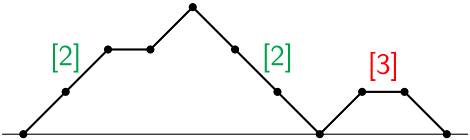
$$\cfrac{1}{1-z-\cfrac{z^2}{1-z-\cfrac{z^2}{\ddots}}}$$



Weighted Motzkin path

$$\cfrac{1}{1 - z - \cfrac{z^2}{1 - 3z - \cfrac{2^2 z^2}{\ddots \, 1 - (2n+1)z - \cfrac{n^2 z^2}{\ddots}}}} = \sum_{n \geq 0} n! \cdot z^n$$

Special case of the general correspondence by Flajolet.



Weighted Motzkin path

$$\cfrac{1}{1-\alpha_0 z-\cfrac{\beta_1 z^2}{1-\alpha_1 z-\cfrac{\beta_2 z^2}{\ddots\, 1-\alpha_n z-\cfrac{\beta_{n+1} z^2}{\ddots}}}}$$

where $\alpha_n(\cdot)$ has $\alpha_n(\mathbf{1}) = 2n+1$ and $\beta_n(\cdot)$ has $\beta_n(\mathbf{1}) = n^2$

The Central Continued Fraction

For parameters $a, b, c, d, f, g, h, \ell, p, r, s, t, u, w \in \mathbb{R}$, let

$$\mathcal{C}(z) = \cfrac{1}{1 - \alpha_0 z - \cfrac{\beta_1 z^2}{1 - \alpha_1 z - \cfrac{\beta_2 z^2}{\ddots}}}$$

where

$$\alpha_n = u \cdot w^n + s\,[n]_{a,b} + t\,[n]_{f,g} \qquad \beta_n = p\,r\,[n]_{c,d}\,[n]_{h,\ell}$$

and $\quad [n]_{x,y} = x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}$

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The Plan: Find a bijection taking *permutations*, carrying lots of statistics, to Motzkin paths corresponding to $\mathcal{C}(z)$, using Flajolet's general correspondence.

Consider Motzkin paths labeled as follows, where $0 \leq i < k$

- Upsteps from height $k-1$ to k have labels $pc^i d^{k-1-i}$
- Downsteps from height k to $k-1$ have labels $rh^i \ell^{k-1-i}$
- Level steps at height k have labels in

$$\{u \cdot w^i\} \cup \{s\, a^i b^{k-1-i}\} \cup \{t\, f^i g^{k-1-i}\}.$$

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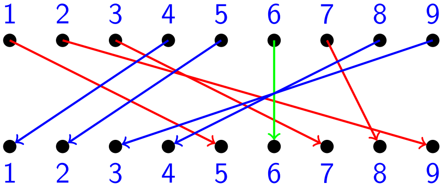
By Flajolet's correspondence, $\mathcal{C}(z)$ is the generating function for Motzkin paths thus labeled:

$$\mathcal{C}(z) = \cfrac{1}{\ddots\, 1 - (u \cdot w^n + s\,[n]_{a,b} + t\,[n]_{f,g})\, z - \cfrac{p\, r\, [n+1]_{c,d}\, [n+1]_{h,\ell}\, z^2}{\ddots}}$$

Fourteen statistics on permutations $\sigma(1)\sigma(2)\ldots\sigma(n)$, based on *excedances* and *inversions*:

$\sigma(i)$: 5 9 7 1 2 6 8 4 3
i: 1 2 3 4 5 6 7 8 9



Excedances red

Anti-excedances blue

Fixed points green

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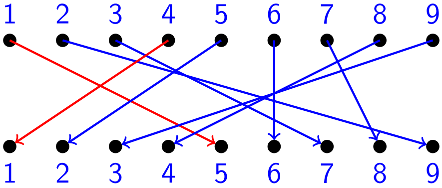

Excedances red

One of the inversions red (crossing)

Anti-excedances blue

Fixed points green

Fourteen statistics on permutations $\sigma(1)\sigma(2)\ldots\sigma(n)$, based on *excedances* and *inversions*:

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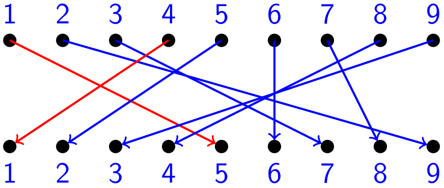

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But this gets more complicated . . .

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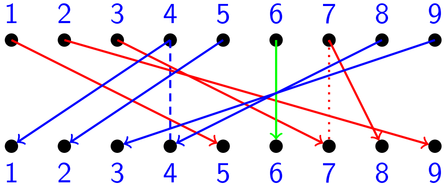
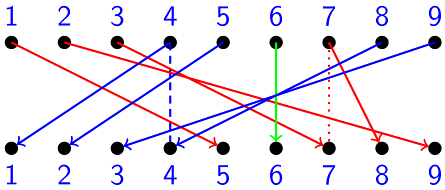


7 is a *linked* excedance: $8 = \sigma(7) > 7 > \sigma^{-1}(7) = 3$

5 9 7 1 2 6 8 4 3
1 2 3 4 5 6 7 8 9



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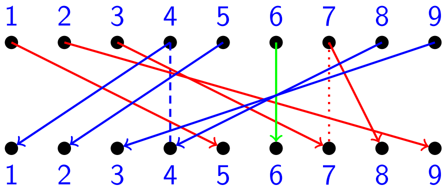
4 is a *linked* anti-excedance: $1 = \sigma(4) < 4 < \sigma^{-1}(4) = 9$

5 9 7 1 2 6 8 4 3
1 2 3 4 5 6 7 8 9



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$9 \cdots 6$ is an inversion between excedance and fixed point

1. # excedances as exc$(\sigma) := \#\{i \in [n] \mid i < \sigma(i)\}$,
2. # fixed points as fp$(\sigma) := \#\{i \in [n] \mid i = \sigma(i)\}$,
3. # anti-excedances as aexc$(\sigma) := \#\{i \in [n] \mid i > \sigma(i)\}$,
4. # linked excedances as le$(\sigma) := \#\{i \in [n] \mid \sigma^{-1}(i) < i < \sigma(i)\}$,
5. # linked anti-excedances as lae$(\sigma) := \#\{i \in [n] \mid \sigma^{-1}(i) > i > \sigma(i)\}$.
6. # inversions between excedances: ie$(\sigma) := \#\{i, j \in [n] \mid i < j < \sigma(j) < \sigma(i)\}$.
7. # inversions between excedances where the greater excedance is linked:ile$(\sigma) := \#\{i, j \in [n] \mid i < j < \sigma(j) < \sigma(i)$ and $\sigma^{-1}(j) < j\}$.
8. # restricted non-inversions between excedances:nie$(\sigma) := \#\{i, j \in [n] \mid i < j < \sigma(i) < \sigma(j)\}$.
9. # restricted non-inversions between excedances where the rightmost excedance is linked: nile$(\sigma) := \#\{i, j \in [n] \mid i < j < \sigma(i) < \sigma(j)$ and $\sigma^{-1}(j) < j\}$.
10. # inversions between anti-excedances: iae$(\sigma) := \#\{i, j \in [n] \mid j > i > \sigma(i) > \sigma(j)\}$.
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4. # linked excedances as $\text{le}(\sigma) := \#\{i \in [n] \mid \sigma^{-1}(i) < i < \sigma(i)\}$,
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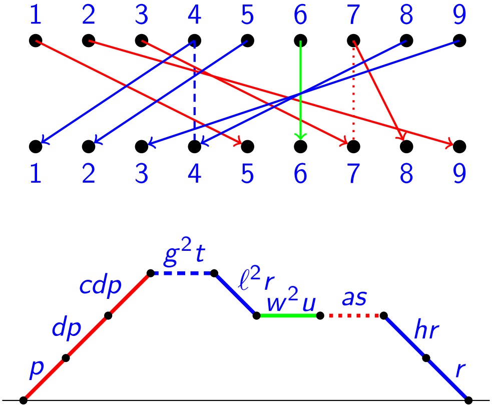
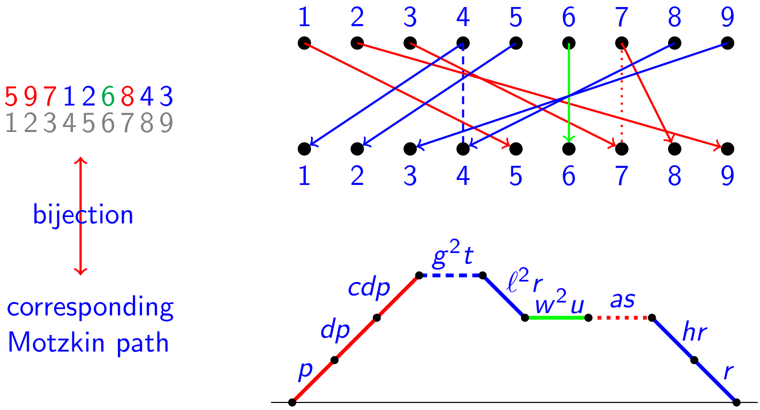
5 9 7 1 2 6 8 4 3
1 2 3 4 5 6 7 8 9

bijection

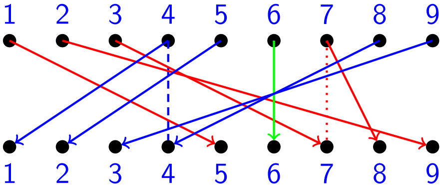
corresponding Motzkin path





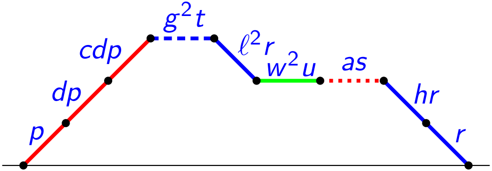
Weight of labeled Motzkin path, wt(M): Product of its labels

5 9 7 1 2 6 8 4 3
1 2 3 4 5 6 7 8 9



↕ bijection

corresponding Motzkin path



wt: $a \cdot c \cdot d^2 \cdot g^2 \cdot h \cdot \ell^2 \cdot p^3 \cdot r^3 \cdot s \cdot t \cdot u \cdot w^2$

Weight of labeled Motzkin path, wt(M): Product of its labels

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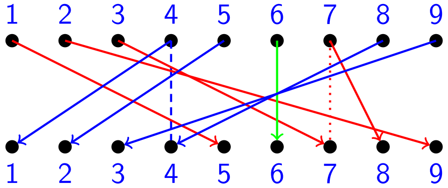

bijection

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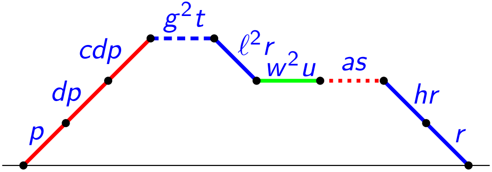

wt: $a \cdot c \cdot d^2 \cdot g^2 \cdot h \cdot \ell^2 \cdot p^3 \cdot r^3 \cdot s \cdot t \cdot u \cdot w^2$

Above wt is one term in $[z^9]\,\mathcal{C}(z)$

The *weight* of a labeled Motzkin path M, $\mathrm{wt}(M)$, is the product of its labels.

Theorem: There is a bijection $\eta : \mathcal{S}_n \to \mathcal{M}_n$ such that if $M = \eta(\sigma)$ then $\mathrm{wt}(M)$ equals

$$\begin{aligned}\mathrm{stat}(\sigma) = \quad & a^{\mathrm{ile}(\sigma)} b^{\mathrm{nile}(\sigma)} c^{\mathrm{ie}(\sigma)-\mathrm{ile}(\sigma)} d^{\mathrm{nie}(\sigma)-\mathrm{nile}(\sigma)} \\ & \times f^{\mathrm{ilae}(\sigma)} g^{\mathrm{nilae}(\sigma)} h^{\mathrm{iae}(\sigma)-\mathrm{ilae}(\sigma)} \ell^{\mathrm{niae}(\sigma)-\mathrm{nilae}(\sigma)} \\ & \times p^{\mathrm{exc}(\sigma)-\mathrm{le}(\sigma)} r^{\mathrm{aexc}(\sigma)-\mathrm{lae}(\sigma)} s^{\mathrm{le}(\sigma)} t^{\mathrm{lae}(\sigma)} u^{\mathrm{fp}(\sigma)} w^{\mathrm{iefp}(\sigma)}\end{aligned}$$

Corollary: $\mathcal{C}(z) = \sum_{n \geq 0} \sum_{\sigma \in \mathcal{S}_n} \mathrm{stat}(\sigma) z^n.$

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In short: Weight of Motzkin path goes to 14-parameter statistic on corresponding permutation

There are several related bijections in earlier literature by

Françon-Viennot 1979
Foata-Zeilberger 1990
Biane 1993
de Médicis-Viennot 1994
Simion-Stanton 1994
Clarke-Steingrímsson-Zeng 1996
Randrianarivony 1998
Elizalde 2018

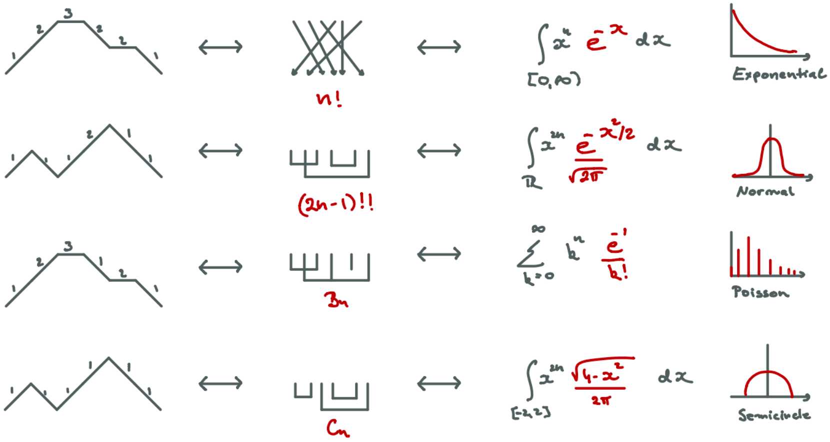
Our results generalize most of these, some modulo a bijection interchanging excedances and descents.

In a contemporaneous (yet unpublished) paper, Sokal and Zeng present a framework similar to ours, but with an additional four statistics, including some originally defined by Corteel.

Of the above, only Biane, Elizalde and Sokal-Zeng separate fixed points from anti-excedances, as we do. This leads to greater symmetry in the continued fraction, and to results not otherwise obtainable.

The number sequences arising from $\mathcal{C}$ enumerate many different combinatorial structures, such as permutations, perfect matchings and set partitions.

These basic examples happen to be moment sequences of important distributions from probability theory.



Some refinements of these objects also have meaning in probability theory.

Which structures give something probabilistically meaningful?

Moment sequences

A sequence $a_0, a_1, a_2, \ldots$ is a moment sequence of a positive measure on the real line *if and only if* all principal minors of

$$\begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ & & \vdots & \\ a_n & a_{n+1} & \cdots & a_{2n} \end{pmatrix}$$

are non-negative for any n. (Hamburger, a 100 years ago)

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Can get strong lower bounds on growth rates of moment sequences (Haagerup–Haagerup–Ramirez-Solano, Elvey Price, Clisby–Conway–Guttmann)

Moment sequences

$$\sum_{n\geq 0} m_n z^n \quad = \quad \mathcal{C}(z) = \cfrac{1}{1-\alpha_0 z - \cfrac{\beta_1 z^2}{1-\alpha_1 z - \cfrac{\beta_2 z^2}{\ddots}}}$$

$$\alpha_n = u\cdot w^n + s\,[n]_{a,b} + t\,[n]_{f,g} \qquad \beta_n = p\,r\,[n]_{c,d}\,[n]_{h,\ell}$$

Theorem: For $a,b,c,d,f,g,h,\ell,p,r,s,t,u,w \in \mathbb{R}$ with $pr>0$ and c,d,h,ℓ satisfying

$$c=-d \quad \textbf{or} \quad h=-\ell \quad \textbf{or}$$
$$(c>-d \text{ and } h>-\ell) \quad \textbf{or} \quad (c<-d \text{ and } h<-\ell),$$

the sequence (m_n) is the moment sequence of some probability measure on $\mathbb{R}$. In particular if all non-negative and $pr>0$.

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With mild conditions on the parameters of $\mathcal{C}(z)$, which are easy to check, we get moment sequences.

All sequences mentioned from now on are moment sequences arising from $\mathcal{C}(z)$.

$$\mathcal{C}(z) = \cfrac{1}{\ddots \; 1 - (u \cdot w^n + s\,[n]_{a,b} + t\,[n]_{f,g})\, z - \cfrac{p\,r\,[n+1]_{c,d}\,[n+1]_{h,\ell}\, z^2}{\ddots}}$$

With $s = qx$, $p = x$, all other parameters $= 1$, we get

$$\mathcal{C}(z) = \sum_{n \geq 0} \sum_{\sigma \in \mathcal{S}_n} x^{\mathrm{des}(\sigma)} q^{\mathrm{occ}_{321}(\sigma)} z^n,$$

where occ_{321} is #occurrences of the consecutive pattern 321

occurrence: 356412   not consecutive: 356412

First shown by Elizalde 2018, using a different continued fraction.

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$\mathrm{Av}_{321}(n) = \#$ n-permutations *avoiding* consecutive pattern 321

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If $b, d, g, \ell = q$, $s = xq$, $p, u = x$, others $= 1$:

$$\mathcal{C}(z) = \sum_{n \geq 0} \sum_{\sigma \in \mathcal{S}_n} x^{\mathrm{des}(\sigma)+1} q^{\mathrm{occ}_{2\text{-}31}(\sigma)} z^n.$$

where $\mathrm{occ}_{2\text{-}31}$ is #occurrences of the *vincular* pattern 2–31

2–31 occurrence: 416523   62 not adjacent: 416523

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Two more cases: Catalan and Bell numbers, both moment sequences

1–2–3 1–23

The only 3-pattern whose avoiders don't give a moment sequence is the consecutive pattern 132 (equivalently 213, 231, 312).

This is the only 3-pattern whose avoidance is not captured in $\mathcal{C}(z)$.

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What tools from probability/analysis would it let us use?

Specializations of $\mathcal{C}(z)$ also capture a large part of the q-Askey scheme of orthogonal polynomials, here interpreted in terms of the simple concepts of excedances and inversions in permutations.

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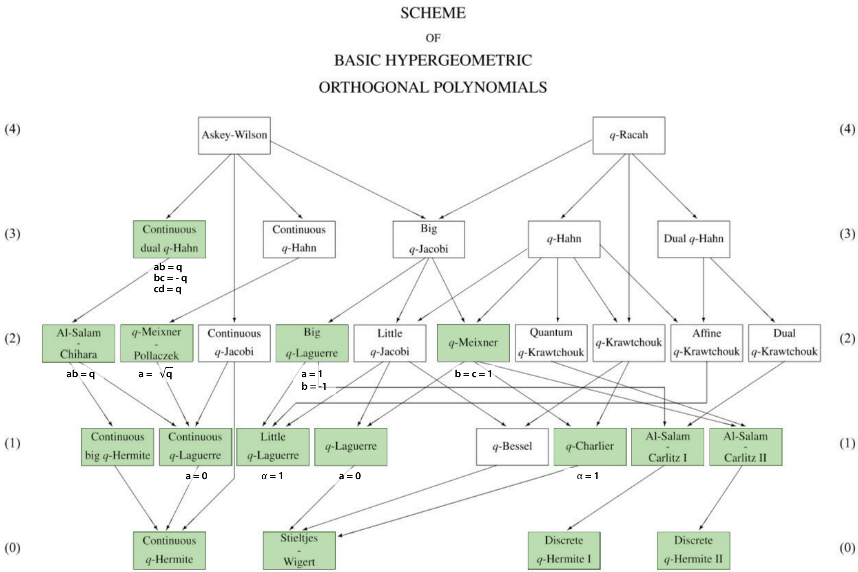
Corteel and Williams have a combinatorial interpretation with statistics on different objects (staircase tableaux) for all polynomials that specialize from the Askey-Wilson family.


SCHEME
OF
BASIC HYPERGEOMETRIC
ORTHOGONAL POLYNOMIALS
(4)
Askey-Wilson
q-Racah
(3)
Continuous dual q-Hahn
Continuous q-Hahn
Big q-Jacobi
q-Hahn
Dual q-Hahn
ab = q
bc = - q
cd = q
(2)
Al-Salam - Chihara
q-Meixner - Pollaczek
Continuous q-Jacobi
Big q-Laguerre
Little q-Jacobi
q-Meixner
Quantum q-Krawtchouk
q-Krawtchouk
Affine q-Krawtchouk
Dual q-Krawtchouk
ab = q
a = √q
a = 1
b = -1
b = c = 1
(1)
Continuous big q-Hermite
Continuous q-Laguerre
Little q-Laguerre
q-Laguerre
q-Bessel
q-Charlier
Al-Salam - Carlitz I
Al-Salam - Carlitz II
a = 0
α = 1
a = 0
α = 1
(0)
Continuous q-Hermite
Stieltjes - Wigert
Discrete q-Hermite I
Discrete q-Hermite II

Generalizations

Via simple substitutions of parameters, many of the permutation statistics carried by $\mathcal{C}(z)$ generalize to the k-*colored permutations* $\mathcal{S}_n^k$ — each letter gets one of k colors — in particular the signed permutations of the type B Coxeter groups.

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Let c_i be the color of the i-th letter.

An *excedance* in a colored permutation $a_1 a_2 \ldots a_n$ is an i such that

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Proposition: With $s, p = kx,\; t, r = ky,\; u = (k-1)x + q$, and all other parameters set to 1, we get

$$\mathcal{C}(z) \;=\; \sum_{n \ge 0} \sum_{\sigma \in \mathcal{S}_n^k} x^{\mathrm{exc}(\sigma)} y^{\mathrm{aexc}(\sigma)} q^{\mathrm{fix}(\sigma)} z^n .$$

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Easy to refine this to distinguish linked/unlinked (anti-)excedances, because the colors embed naturally in $\mathcal{C}(z)$.

- An *inversion* is a pair (i, j) where $i < j$ and

$$c_i > c_j \quad \text{OR} \quad (c_i = c_j \;\; \text{AND} \;\; a_i > a_j)$$

With

$a, c, h, r = q, \quad b, f, d, \ell, t = q^2, \quad g, w = 0, \quad p, u = 1, \quad s = 2q,$
we get the distribution of inversions over $\mathcal{S}_n$ from $\mathcal{C}$:

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Unclear whether that can be extended to $\mathcal{S}_n^k$ via $\mathcal{C}$ and whether other Euler-Mahonian pairs can be obtained from $\mathcal{C}$.

Coloring only fixed points

Because fixed points live independently in $\mathcal{C}(z)$, the following generalization is obvious:

k-arrangements: Permutations with k-colored fixed points

- 0-arrangements are derangements (no fixed points)
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But they have many nice properties, and doubtless many more to be discovered.

Proposition: Let $A_k(n)$ be the number of k-arrangements of $[n]$. Then

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What does that count?

In a colored permutation, let c_i be the color of the letter in place i.

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- The *major index* of π is the sum of its descents

Encoding k-arrangements

Replacing fixed points colored i (resp. $i < k$) by $-i$ gives the *derangement (resp. permutation) form* of a k-arrangement.

Conjecture: des has the same distribution on the derangement and permutation forms for k-arrangements of $[n]$.

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Proved by Fu-Han-Lin. Surprisingly non-trivial.

Another encoding of k-arrangements

Given a k-arrangement as a permutation π with fixed points colored with $\{1, 2, \ldots, k\}$, let its non-fixed points have color 0 and regard π as a k-colored permutation in $\mathcal{S}_n^k$.

Conjecture: In this encoding inv and maj are equidistributed. Also, des has the same distribution as it does on the permutation or derangement form.

Problem: There is a modification $\mathcal{C}'$ of the continued fraction $\mathcal{C}$ that captures the distributions of statistics on the colored permutations. Is there a restriction of $\mathcal{C}'$ that carries the corresponding statistics on k-arrangements?

A very open problem

Sokal and Zeng have a continued fraction with another four parameters, carrying statistics on alignments and crossings in permutations, first defined by Corteel.

Is it possible to add further parameters carrying even more permutation statistics?

In particular, is it possible to expand these continued fractions to encompass all of the q-Askey scheme?

Thanks!

N. Blitvić and E. Steingrímsson: Permutations, moments, measures

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