Towards a Pósa-Seymour conjecture for hypergraphs

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University of Chile

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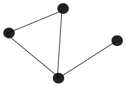
and Matías Pavez-Signé (University of Birmingham)

MSU Combinatorics and Graph Theory Seminar
January 19, 2022

GRAPHS

A graph G: 

Vertices $V(G)$, edges $E(G)$

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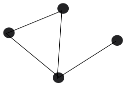

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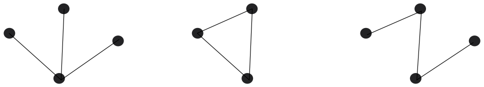
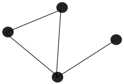


A graph G: 

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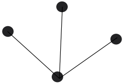
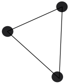
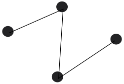
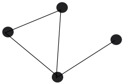
Some subgraphs of G:   

Degrees: Degree of vertex v $=$ no. of edges at v

A graph G: 

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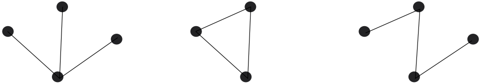
Some subgraphs of G: 

Degrees: Degree of vertex v $=$ no. of edges at v

$\delta(G) =$ minimum degree of G

$\Delta(G) =$ maximum degree of G

$d(G) =$ average degree of G

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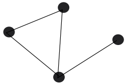

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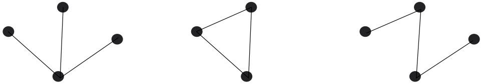

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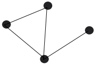
Example:



$\delta(G) = 1$, $d(G) = 2$, $\Delta(G) = 3$.

Basic Question:

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Given a graph G
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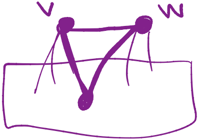
whose degree sequence satisfies certain conditions,
what subgraphs must appear in G?

An easy observation:

If $\delta(G) > \frac{1}{2}|V(G)|$, then G contains a triangle.



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Improvement:

Mantel's theorem (1907):
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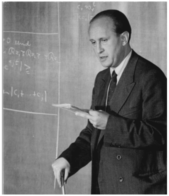

Turán's theorem (1941):
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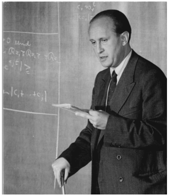

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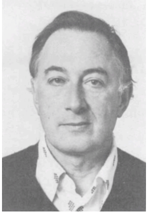

Dirac's theorem (1952):
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Hamilton cycle: Cycle going through all the vertices

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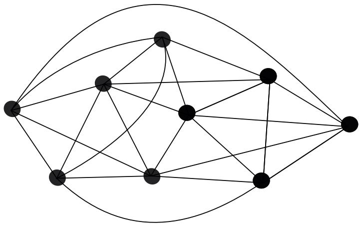

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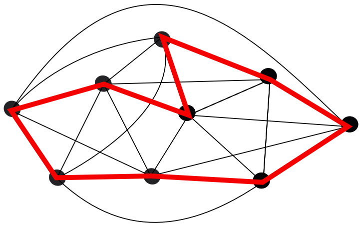
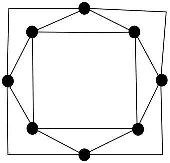


Square of a cycle: Cycle plus all 2-chords



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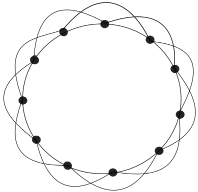

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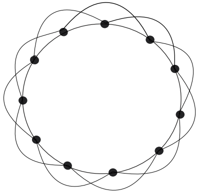

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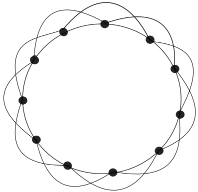

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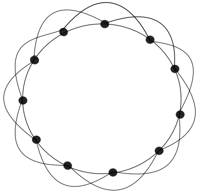

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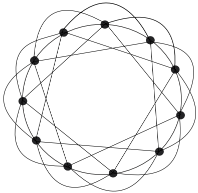

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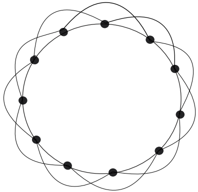

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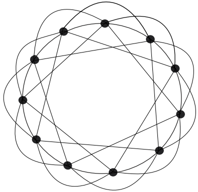

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Generalization by Seymour (1974):
If $\delta(G) \geq \frac{r-1}{r}|V(G)|$, then G contains the $(r-1)$th power of a Hamilton cycle.

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True for all $r \geq 3$ and all large enough graphs ($|V(G)| \geq n_0(r)$).

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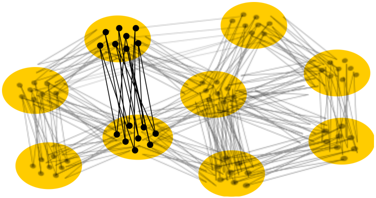
Tool for proof:



Regularity Lemma (Szemerédi 1978): Every large graph admits a partition of its vertex set so that edges between classes are random-like.

DETOUR: Trees

Dirac: $\delta(G) \geq \frac{|V(G)|}{2} \Rightarrow G$ has a Hamilton cycle (also Hamilton path)

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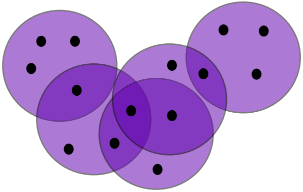
- Can be slightly improved
- Can drop condition on $\Delta(T)$ if we add additional conditions on G:

Thm Reed, St. (2019+): Each graph G on n vertices with $\delta(G) \geq \frac{2n}{3}$ and $\Delta(G) \geq n-1$ contains every tree T of order n.

HYPERGRAPHS

An *r-uniform hypergraph,* or *r-graph* for short, has vertex set V and its (hyper)edges are r-subsets of V.



Cycles in hypergraphs:

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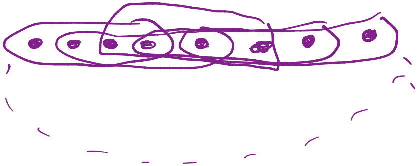
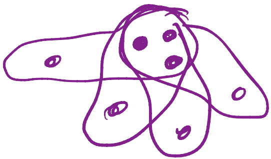
Berge cycles, Loose cycles, Tight cycles



Codegrees:

$\delta_{r-1}(H) \geq k$ means every $(r-1)$-tuple of vertices of H belongs to at least k hyperedges.



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If H is an r-graph on n vertices and $\delta_{r-1}(H) \geq \frac{n}{2}$, then H has a Hamilton tight cycle.

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How about squares/powers of Hamilton cycles?

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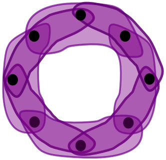

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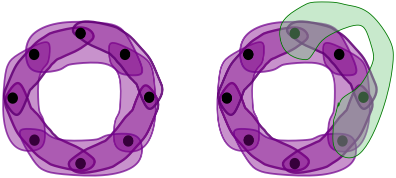

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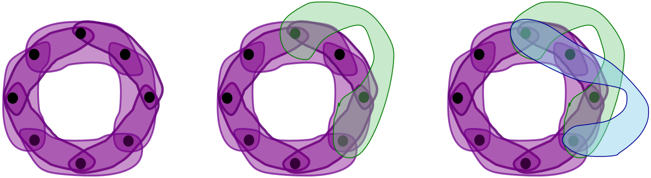

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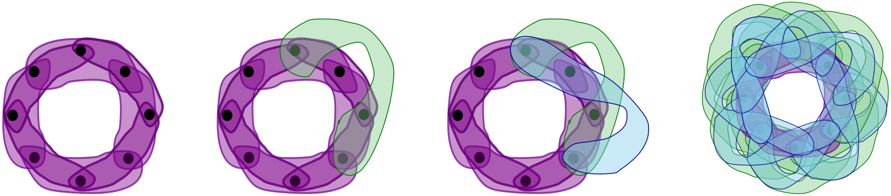

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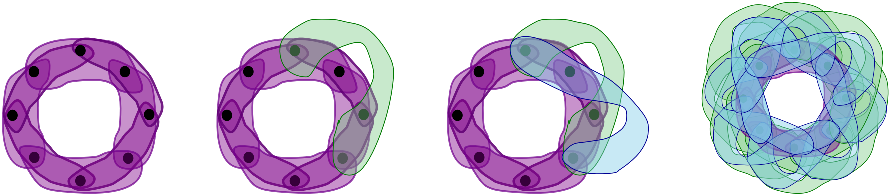

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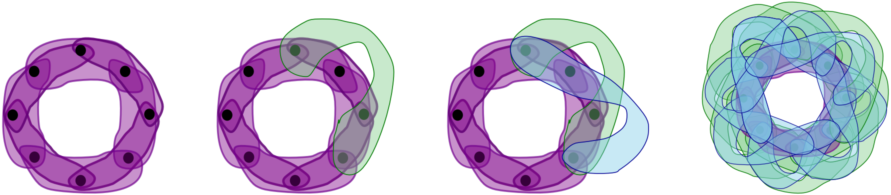

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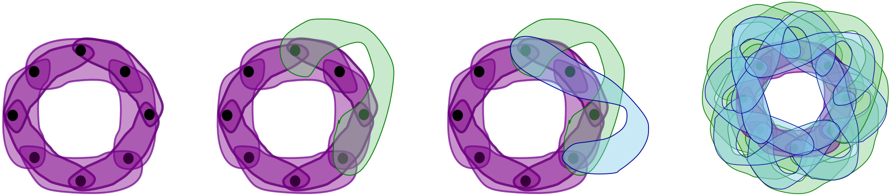

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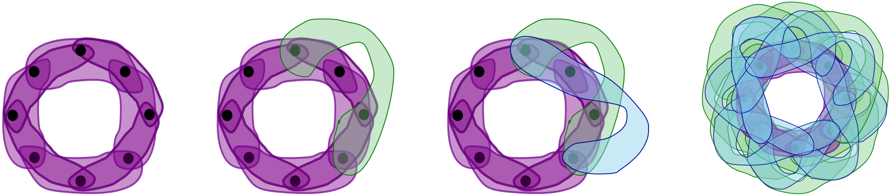

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If H is a 3-graph on n vertices and $\delta_2(G) \geq \frac{4}{5}n + o(n)$, then H contains the square of a tight Hamilton cycle.

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Trees

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Theorem (Komlós, Sárközy, Szemerédi 1995) $\forall \varepsilon, \Delta\ \exists n_0$ s.t. each graph G on $n \geq n_0$ vertices with $\delta(G) \geq (\frac{1}{2} + \varepsilon)n$ contains every tree T of order n and $\Delta(T) \leq \Delta$.

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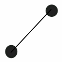

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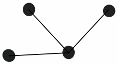

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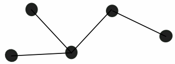

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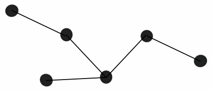

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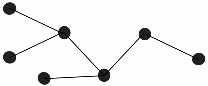

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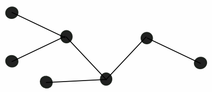

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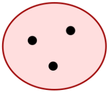

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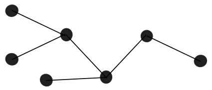

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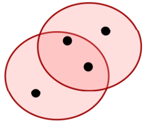

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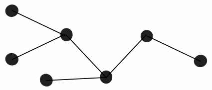

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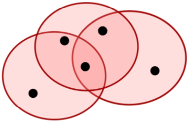

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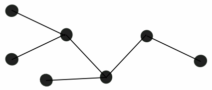

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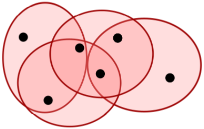

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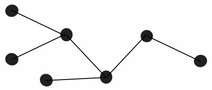

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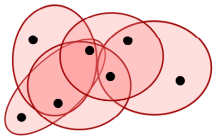

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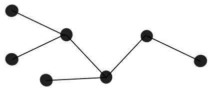

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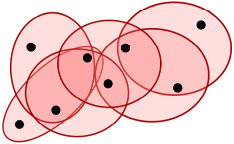

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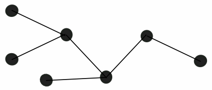

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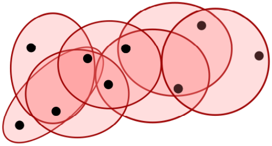

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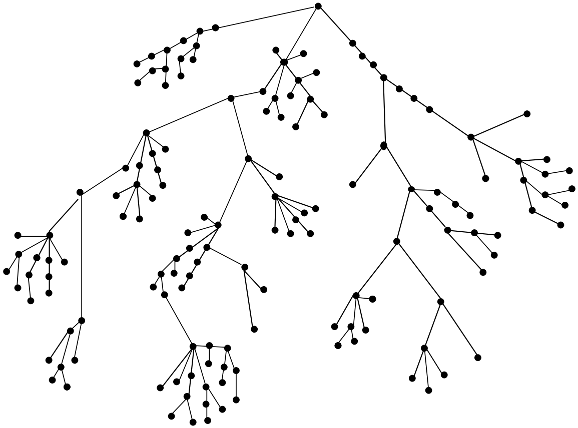

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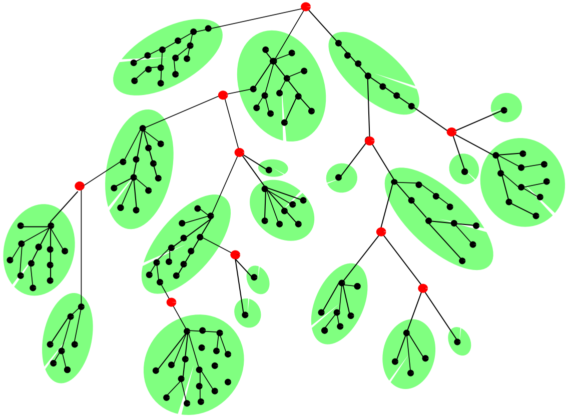

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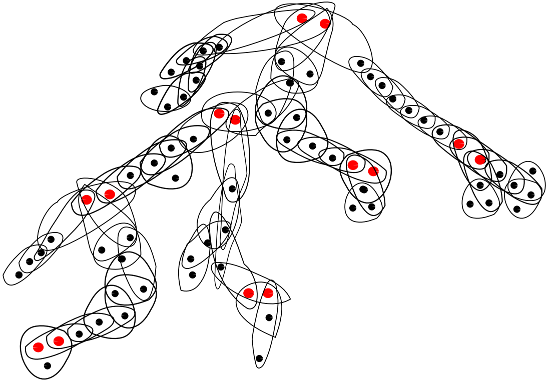

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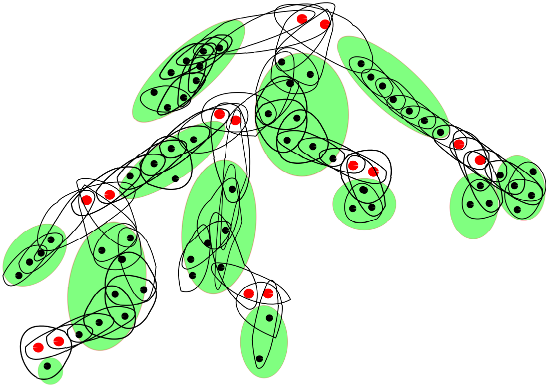

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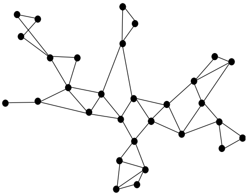

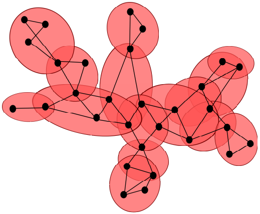

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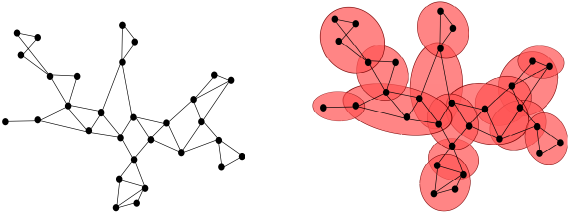

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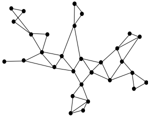
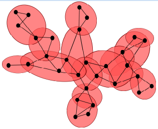
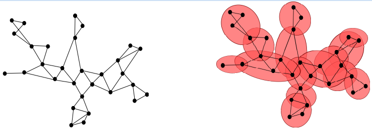




Hypertree-decompositions



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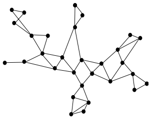

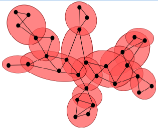

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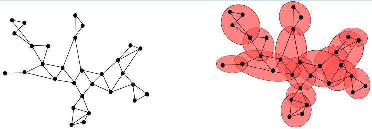

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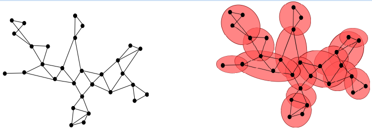

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If $\Delta_1(T) = \Delta$, we say H has a tree-decomposition of width r and max degree Δ.

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$\forall r, k, \Delta\varepsilon\ \exists n_0$ s.t. each k-graph H on $n \geq n_0$ vertices with $\delta_{k-1}(H) \geq (1 - \frac{1}{\binom{r-1}{k-1}+\binom{r-2}{k-2}} + \varepsilon)n$ contains every k-graph that admits a tree-decomposition of width $\leq r$ and max degree $\leq \Delta$.

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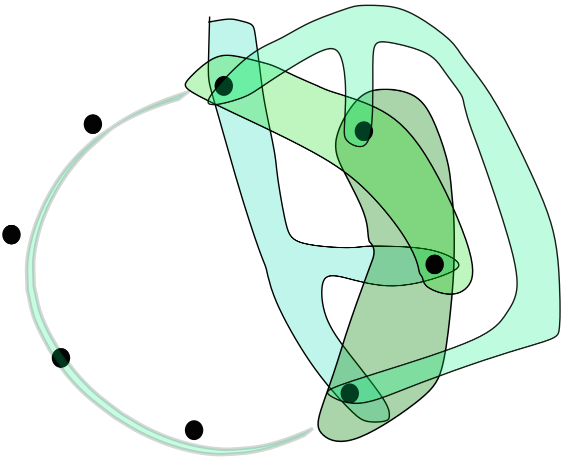

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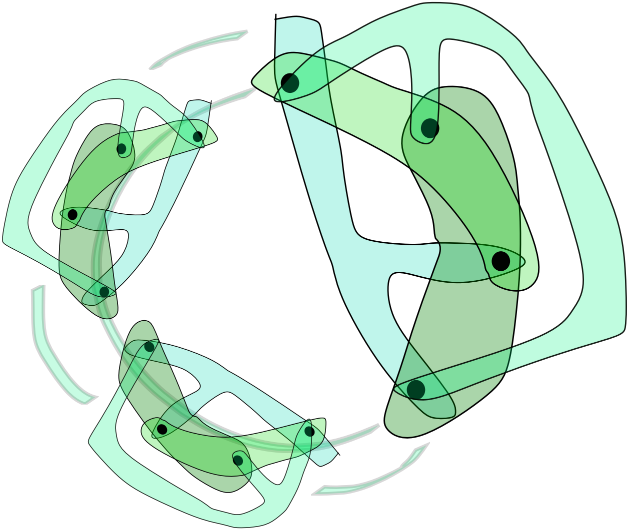

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