Two Analogues of Pascal's Triangle

Richard P. Stanley
U. Miami & M.I.T.

February 3, 2021

Pascal's triangle

rows 0–4:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

kth entry in row n, beginning with $k = 0$: $\binom{n}{k}$

Pascal's triangle

rows 0–4:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

kth entry in row n, beginning with $k = 0$: $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$

Pascal's triangle

rows 0–4:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

kth entry in row n, beginning with $k = 0$: $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$

$$\sum_k \binom{n}{k} x^k = (1+x)^n$$

Sums of powers

$$\sum_k \binom{n}{k}^2 = \binom{2n}{n}$$

Sums of powers

$$\sum_k \binom{n}{k}^2 = \binom{2n}{n}$$

$$\sum_{n\geq 0} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}},$$

not a rational function (quotient of two polynomials)

Sums of powers

$$\sum_k \binom{n}{k}^2 = \binom{2n}{n}$$

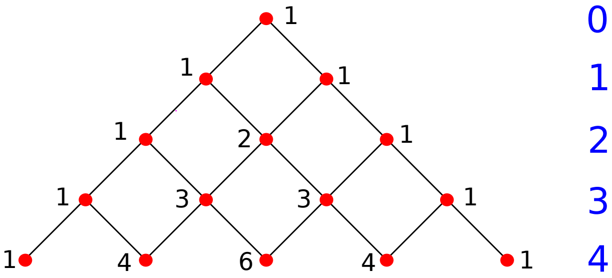
$$\sum_{n\geq 0} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}},$$

not a rational function (quotient of two polynomials)

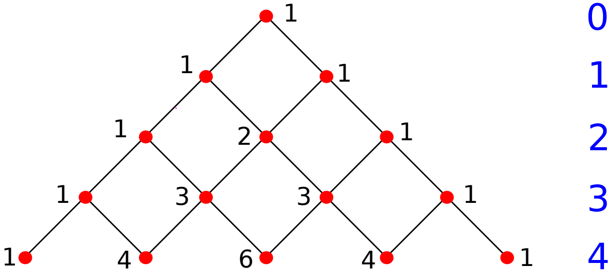
$$\sum_k \binom{n}{k}^3 = ??$$

Even worse! Generating function is not algebraic.

A diagram (poset) associated with Pascal's triangle

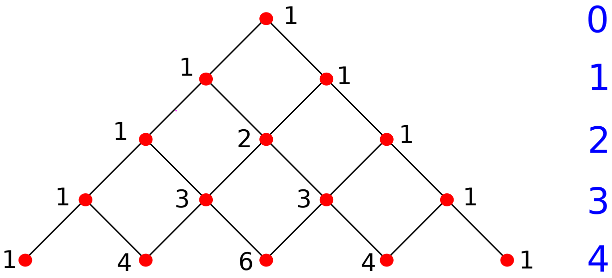


A diagram (poset) associated with Pascal's triangle



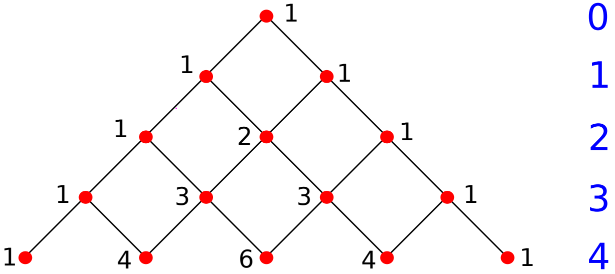
- Each point lies directly above two points.

A diagram (poset) associated with Pascal's triangle



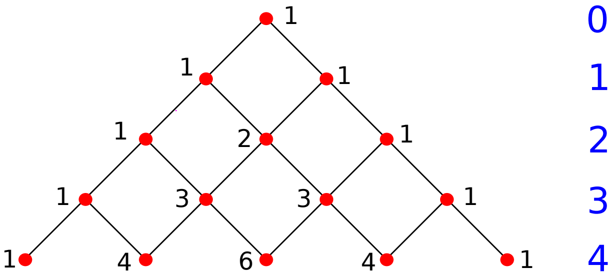
- Each point lies directly above two points.
- The diagram is planar.

A diagram (poset) associated with Pascal's triangle



- Each point lies directly above two points.
- The diagram is planar.
- Every  extends to 

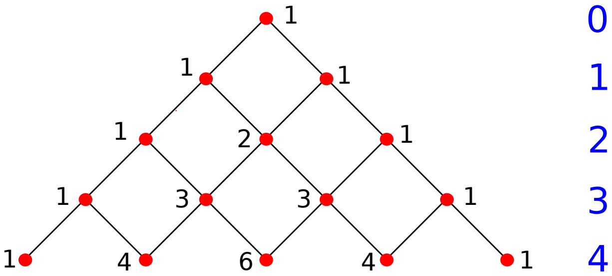
A diagram (poset) associated with Pascal's triangle



- Each point lies directly above two points.
- The diagram is planar.
- Every  extends to 

These properties **characterize** the diagram.

Two further properties



- Each label is the sum of those on the level above connected by an edge
- Each label is the number of paths from that label to the top.

Stern's triangle

Similar to Pascal's triangle, but we also "bring down" (copy) each number from one row to the next.

Stern's triangle

Similar to Pascal's triangle, but we also "bring down" (copy) each number from one row to the next.

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & & & 1 & & \\
 & 1 & & & & & & 1 & \\
 & & & & \vdots & & & &
\end{array}
$$

Stern's triangle

Similar to Pascal's triangle, but we also "bring down" (copy) each number from one row to the next.

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & \color{blue}{1} & 1 & & & \\
 & 1 & & & & & & 1 & \\
1 & & & & & & & & 1 \\
 & & & & \vdots & & & &
\end{array}
$$

Stern's triangle

Similar to Pascal's triangle, but we also "bring down" (copy) each number from one row to the next.

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & 1 & & 1 & & 1 & & \\
 & 1 & & {\color{red}2} & & {\color{red}2} & & 1 & \\
1 & & & & & & & & 1 \\
 & & & & \vdots & & & &
\end{array}
$$

Stern's triangle

Similar to Pascal's triangle, but we also "bring down" (copy) each number from one row to the next.

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & 1 & & 1 & & 1 & & \\
 & 1 & \color{blue}{1} & 2 & \color{blue}{1} & 2 & \color{blue}{1} & 1 & \\
1 & & & & & & & & 1 \\
 & & & & \vdots & & & &
\end{array}
$$

Stern's triangle

Similar to Pascal's triangle, but we also "bring down" (copy) each number from one row to the next.

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & 1 & & & & 1 & & & & 1 & & & \\
 & 1 & & 1 & & 2 & & 1 & & 2 & & 1 & & 1 & \\
1 & & {\color{red}2} & & {\color{red}3} & & {\color{red}3} & & {\color{red}3} & & {\color{red}3} & & {\color{red}2} & & 1 \\
 & & & & & & & \vdots & & & & & & &
\end{array}
$$

Stern's triangle

Similar to Pascal's triangle, but we also "bring down" (copy) each number from one row to the next.

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & 1 & & & & 1 & & & & 1 & & & \\
 & 1 & & 1 & & 2 & & 1 & & 2 & & 1 & & 1 & \\
1 & {\color{blue}1} & 2 & {\color{blue}1} & 3 & {\color{blue}2} & 3 & {\color{blue}1} & 3 & {\color{blue}2} & 3 & {\color{blue}1} & 2 & {\color{blue}1} & 1 \\
 & & & & & & & \vdots & & & & & & &
\end{array}
$$

Stern's triangle

Similar to Pascal's triangle, but we also "bring down" (copy) each number from one row to the next.

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & 1 & & & & 1 & & & & 1 & & & \\
 & 1 & & 1 & & 2 & & 1 & & 2 & & 1 & & 1 & \\
1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 \\
 & & & & & & & \vdots & & & & & & &
\end{array}
$$

Stern's triangle

Some properties

- Number of entries in row n (beginning with row 0): $2^{n+1} - 1$

Some properties

- Number of entries in row n (beginning with row 0): $2^{n+1} - 1$
- Sum of entries in row n: 3^n

Some properties

- Number of entries in row n (beginning with row 0): $2^{n+1} - 1$
- Sum of entries in row n: 3^n
- Largest entry in row n: F_{n+1} (Fibonacci number)

Some properties

- Number of entries in row n (beginning with row 0): $2^{n+1} - 1$
- Sum of entries in row n: 3^n
- Largest entry in row n: F_{n+1} (Fibonacci number)
- Let $\left\langle {n \atop k} \right\rangle$ be the kth entry (beginning with $k = 0$) in row n. Write

$$P_n(x) = \sum_{k \geq 0} \left\langle {n \atop k} \right\rangle x^k.$$

Then $\boxed{P_{n+1}(x) = (1 + x + x^2)P_n(x^2)}$, since $x\,P_n(x^2)$ corresponds to bringing down the previous row, and $(1 + x^2)P_n(x^2)$ to summing two consecutive entries.

Stern analogue of binomial theorem

Corollary. $P_n(x) = \prod_{i=0}^{n-1} \left(1 + x^{2^i} + x^{2 \cdot 2^i}\right)$

Historical note

An essentially equivalent array is due to **Moritz Abraham Stern** around 1858 and is known as **Stern's diatomic array**:

$$
\begin{array}{ccccccccccccccccc}
1 & & & & & & & & & & & & & & & & 1 \\
1 & & & & & & & & 2 & & & & & & & & 1 \\
1 & & & & 3 & & & & 2 & & & & 3 & & & & 1 \\
1 & & 4 & & 3 & & 5 & & 2 & & 5 & & 3 & & 4 & & 1 \\
1 & 5 & 4 & 7 & 3 & 8 & 5 & 7 & 2 & 7 & 5 & 8 & 3 & 7 & 4 & 5 & 1 \\
 & & & & & & & & \vdots & & & & & & & &
\end{array}
$$

Sums of squares

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & 1 & & & & 1 & & & & 1 & & & \\
 & 1 & & 1 & & 2 & & 1 & & 2 & & 1 & & 1 & \\
1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 \\
 & & & & & & & \vdots & & & & & & &
\end{array}$$

$$u_2(n) := \sum_k \left\langle {n \atop k} \right\rangle^2 = 1,\ 3,\ 13,\ 59,\ 269,\ 1227,\ \ldots$$

Sums of squares

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & 1 & & & & 1 & & & & 1 & & & \\
 & 1 & & 1 & & 2 & & 1 & & 2 & & 1 & & 1 & \\
1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 \\
 & & & & & & & \vdots & & & & & & &
\end{array}
$$

$$\boldsymbol{u_2(n)} := \sum_k \left\langle {n \atop k} \right\rangle^2 = 1,\ 3,\ 13,\ 59,\ 269,\ 1227,\ \ldots$$

$$u_2(n+1) = 5u_2(n) - 2u_2(n-1), \quad n \geq 1$$

Sums of squares

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & 1 & & & & 1 & & & & 1 & & & \\
 & 1 & & 1 & & 2 & & 1 & & 2 & & 1 & & 1 & \\
1 & 1 & 2 & 1 & 3 & 2 & 3 & 1 & 3 & 2 & 3 & 1 & 2 & 1 & 1 \\
 & & & & & & & \vdots & & & & & & &
\end{array}
$$

$$\boldsymbol{u_2(n)} \coloneqq \sum_k \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle^2 = 1,\ 3,\ 13,\ 59,\ 269,\ 1227,\ \ldots$$

$$u_2(n+1) = 5u_2(n) - 2u_2(n-1), \quad n \geq 1$$

$$\sum_{n\geq 0} u_2(n)x^n = \frac{1-2x}{1-5x+2x^2}$$

Proof

$$
\begin{aligned}
u_2(n+1) &= \cdots + \left\langle {n \atop k} \right\rangle^2 + \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle \right)^2 + \left\langle {n \atop k+1} \right\rangle^2 + \cdots \\
&= 3u_2(n) + 2\sum_k \left\langle {n \atop k} \right\rangle \left\langle {n \atop k+1} \right\rangle.
\end{aligned}
$$

Proof

$$
\begin{aligned}
u_2(n+1) &= \cdots + \left\langle {n \atop k} \right\rangle^2 + \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle \right)^2 + \left\langle {n \atop k+1} \right\rangle^2 + \cdots \\
&= 3u_2(n) + 2 \sum_k \left\langle {n \atop k} \right\rangle \left\langle {n \atop k+1} \right\rangle .
\end{aligned}
$$

Thus define $\boldsymbol{u_{1,1}(n)} \coloneqq \sum_k \left\langle {n \atop k} \right\rangle \left\langle {n \atop k+1} \right\rangle$, so

$$
u_2(n+1) = 3u_2(n) + 2u_{1,1}(n).
$$

What about $u_{1,1}(n)$?

$$\begin{aligned}
u_{1,1}(n+1) &= \cdots + \left(\left\langle {n \atop k-1} \right\rangle + \left\langle {n \atop k} \right\rangle\right)\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k} \right\rangle\left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle\right) \\
&\quad + \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle\right)\left\langle {n \atop k+1} \right\rangle + \cdots \\
&= 2u_2(n) + 2u_{1,1}(n)
\end{aligned}$$

What about $u_{1,1}(n)$?

$$
\begin{aligned}
u_{1,1}(n+1) &= \cdots + \left(\left\langle {n \atop k-1} \right\rangle + \left\langle {n \atop k} \right\rangle\right)\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k} \right\rangle\left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle\right) \\
&\quad + \left(\left\langle {n \atop k} \right\rangle + \left\langle {n \atop k+1} \right\rangle\right)\left\langle {n \atop k+1} \right\rangle + \cdots \\
&= 2u_2(n) + 2u_{1,1}(n)
\end{aligned}
$$

Recall also $u_2(n+1) = 3u_2(n) + 2u_{1,1}(n)$.

Two recurrences in two unknowns

Let $\boldsymbol{A} := \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$. Then

$$A \begin{bmatrix} u_2(n) \\ u_{1,1}(n) \end{bmatrix} = \begin{bmatrix} u_2(n+1) \\ u_{1,1}(n+1) \end{bmatrix}.$$

Two recurrences in two unknowns

Let $\boldsymbol{A} := \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$. Then

$$A \begin{bmatrix} u_2(n) \\ u_{1,1}(n) \end{bmatrix} = \begin{bmatrix} u_2(n+1) \\ u_{1,1}(n+1) \end{bmatrix}.$$

$$\Rightarrow A^n \begin{bmatrix} u_2(1) \\ u_{1,1}(1) \end{bmatrix} = \begin{bmatrix} u_2(n) \\ u_{1,1}(n) \end{bmatrix}$$

Two recurrences in two unknowns

Let $\boldsymbol{A} := \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$. Then

$$A \begin{bmatrix} u_2(n) \\ u_{1,1}(n) \end{bmatrix} = \begin{bmatrix} u_2(n+1) \\ u_{1,1}(n+1) \end{bmatrix}.$$

$$\Rightarrow A^n \begin{bmatrix} u_2(1) \\ u_{1,1}(1) \end{bmatrix} = \begin{bmatrix} u_2(n) \\ u_{1,1}(n) \end{bmatrix}$$

Characteristic (or minimum) polynomial of A: $x^2 - 5x + 2$

Two recurrences in two unknowns

Let $\boldsymbol{A} := \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$. Then

$$A \begin{bmatrix} u_2(n) \\ u_{1,1}(n) \end{bmatrix} = \begin{bmatrix} u_2(n+1) \\ u_{1,1}(n+1) \end{bmatrix}.$$

$$\Rightarrow A^n \begin{bmatrix} u_2(1) \\ u_{1,1}(1) \end{bmatrix} = \begin{bmatrix} u_2(n) \\ u_{1,1}(n) \end{bmatrix}$$

Characteristic (or minimum) polynomial of A: $x^2 - 5x + 2$

$$\begin{aligned} (A^2 - 5A + 2I)A^{n-1} &= 0_{2\times 2} \\ \Rightarrow u_2(n+1) &= 5u_2(n) - 2u_2(n-1) \end{aligned}$$

Two recurrences in two unknowns

Let $\boldsymbol{A} := \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$. Then

$$A \begin{bmatrix} u_2(n) \\ u_{1,1}(n) \end{bmatrix} = \begin{bmatrix} u_2(n+1) \\ u_{1,1}(n+1) \end{bmatrix}.$$

$$\Rightarrow A^n \begin{bmatrix} u_2(1) \\ u_{1,1}(1) \end{bmatrix} = \begin{bmatrix} u_2(n) \\ u_{1,1}(n) \end{bmatrix}$$

Characteristic (or minimum) polynomial of A: $x^2 - 5x + 2$

$$\begin{aligned} (A^2 - 5A + 2I)A^{n-1} &= 0_{2\times 2} \\ \Rightarrow u_2(n+1) &= 5u_2(n) - 2u_2(n-1) \end{aligned}$$

Also $u_{1,1}(n+1) = 5u_{1,1}(n) - 2u_{1,1}(n-1)$.

Sums of cubes

$$u_3(n) \coloneqq \sum_k \binom{n}{k}^3 = 1,\ 3,\ 21,\ 147,\ 1029,\ 7203,\ \ldots$$

Sums of cubes

$$u_3(n) \coloneqq \sum_k \left\langle {n \atop k} \right\rangle^3 = 1,\ 3,\ 21,\ 147,\ 1029,\ 7203,\ \ldots$$

$$u_3(n) = 3 \cdot 7^{n-1}, \quad n \geq 1$$

Sums of cubes

$$u_3(n) \coloneqq \sum_k \left\langle {n \atop k} \right\rangle^3 = 1,\ 3,\ 21,\ 147,\ 1029,\ 7203,\ \ldots$$

$$u_3(n) = 3 \cdot 7^{n-1}, \quad n \geq 1$$

Equivalently, if $\prod_{i=0}^{n-1} \left(1 + x^{2^i} + x^{2 \cdot 2^i}\right) = \sum a_j x^j$, then

$$\sum a_j^3 = 3 \cdot 7^{n-1}.$$

Why so simple?

Same method gives the matrix $\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$.

Why so simple?

Same method gives the matrix $\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$.

Characteristic polynomial: $x(x-7)$ (zero eigenvalue!)

Why so simple?

Same method gives the matrix $\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$.

Characteristic polynomial: $x(x-7)$ (zero eigenvalue!)

Thus $u_3(n+1) = 7u_3(n)$ and $u_{2,1}(n+1) = 7u_{2,1}(n)$ $(n \geq 1)$.

Why so simple?

Same method gives the matrix $\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$.

Characteristic polynomial: $x(x-7)$ (zero eigenvalue!)

Thus $u_3(n+1) = 7u_3(n)$ and $u_{2,1}(n+1) = 7u_{2,1}(n)$ $(n \geq 1)$.

In fact,

$$\begin{aligned} u_3(n) &= 3 \cdot 7^{n-1} \\ u_{2,1}(n) &= 2 \cdot 7^{n-1}. \end{aligned}$$

Why so simple?

Same method gives the matrix $\begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$.

Characteristic polynomial: $x(x-7)$ (zero eigenvalue!)

Thus $u_3(n+1) = 7u_3(n)$ and $u_{2,1}(n+1) = 7u_{2,1}(n)$ $(n \geq 1)$.

In fact,

$$\begin{aligned} u_3(n) &= 3 \cdot 7^{n-1} \\ u_{2,1}(n) &= 2 \cdot 7^{n-1}. \end{aligned}$$

Much nicer than $\sum_k \binom{n}{k}^3$

What about $u_r(n)$ for general $r \geq 1$?

By the same technique, can show that

$$\sum_{n \geq 0} u_r(n) x^n$$

is rational.

What about $u_r(n)$ for general $r \geq 1$?

By the same technique, can show that

$$\sum_{n \geq 0} u_r(n) x^n$$

is rational.

Example. $$\sum_{n \geq 0} u_4(n) x^n = \frac{1 - 7x - 2x^2}{1 - 10x - 9x^2 + 2x^3}$$

What about $u_r(n)$ for general $r \geq 1$?

By the same technique, can show that

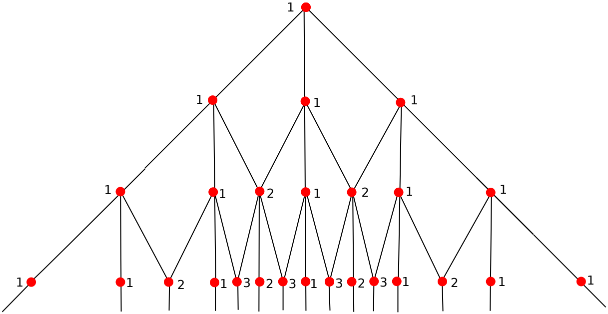
$$\sum_{n \geq 0} u_r(n) x^n$$

is rational.

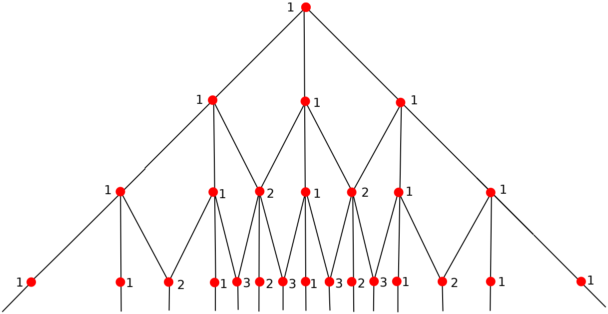
Example. $$\sum_{n \geq 0} u_4(n) x^n = \frac{1 - 7x - 2x^2}{1 - 10x - 9x^2 + 2x^3}$$

Much more can be said!

The Stern poset

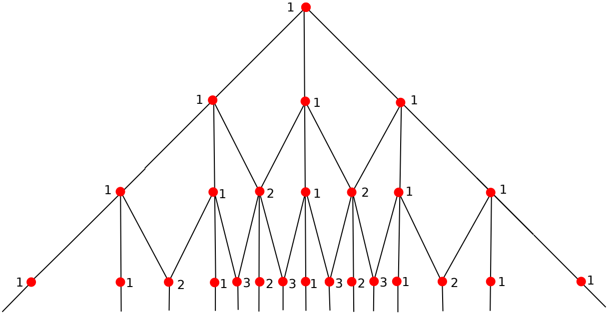


The Stern poset



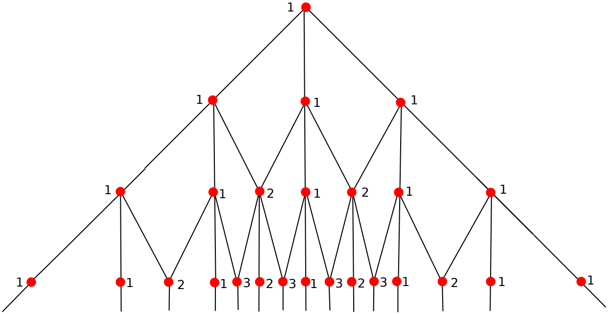
- Each point lies directly above **three** points.

The Stern poset



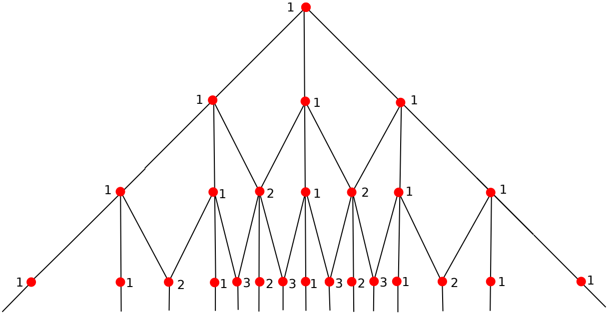
- Each point lies directly above **three** points.
- The diagram is planar.

The Stern poset



- Each point lies directly above **three** points.
- The diagram is planar.
- Every  extends to 

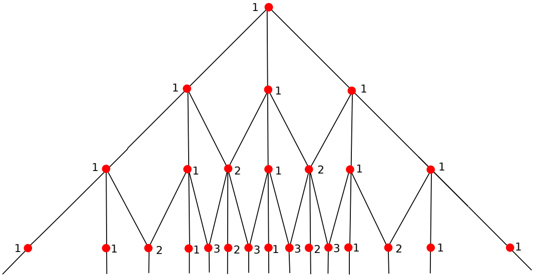
The Stern poset



- Each point lies directly above **three** points.
- The diagram is planar.
- Every  extends to 

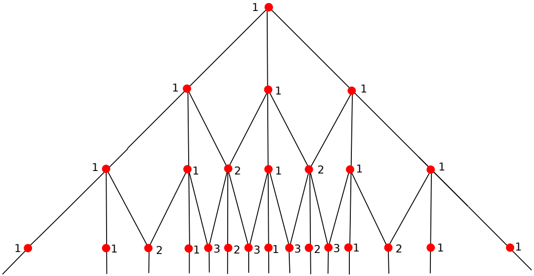
These properties **characterize** the diagram.

Two further properties



- Each label is the sum of those on the level above connected by an edge
- Each label is the number of paths from that label to the top.

Two further properties



- Each label is the sum of those on the level above connected by an edge
- Each label is the number of paths from that label to the top.

The kth label (beginning with $k = 0$) at rank n is $\left\langle {n \atop k} \right\rangle$:

$$\sum_k \left\langle {n \atop k} \right\rangle x^k = \prod_{i=0}^{n-1} \left(1 + x^{2^i} + x^{2 \cdot 2^i}\right).$$

A Fibonacci product

Fibonacci numbers: $F_1 = F_2 = 1, \; F_n = F_{n-1} + F_{n-2} \; (n \geq 3)$

A Fibonacci product

Fibonacci numbers: $F_1 = F_2 = 1, \; F_n = F_{n-1} + F_{n-2} \; (n \geq 3)$

$$I_n(x) = \prod_{i=1}^{n} \left(1 + x^{F_{i+1}}\right)$$

A Fibonacci product

Fibonacci numbers: $F_1 = F_2 = 1,\ F_n = F_{n-1} + F_{n-2}\ (n \geq 3)$

$$I_n(x) = \prod_{i=1}^{n} \left(1 + x^{F_{i+1}}\right)$$

$$\begin{aligned} I_4(x) &= (1+x)(1+x^2)(1+x^3)(1+x^5) \\ &= 1 + x + x^2 + 2x^3 + x^4 + 2x^5 + 2x^6 + x^7 + 2x^8 + x^9 + x^{10} + x^{11} \end{aligned}$$

A Fibonacci product

Fibonacci numbers: $F_1 = F_2 = 1,\ F_n = F_{n-1} + F_{n-2}\ (n \geq 3)$

$$I_n(x) = \prod_{i=1}^{n} \left(1 + x^{F_{i+1}}\right)$$

$$\begin{aligned} I_4(x) &= (1+x)(1+x^2)(1+x^3)(1+x^5) \\ &= 1 + x + x^2 + 2x^3 + x^4 + 2x^5 + 2x^6 + x^7 + 2x^8 + x^9 + x^{10} + x^{11} \end{aligned}$$

$v_2(n)$: sum of squares of coefficients of $I_n(x)$

A Fibonacci product

Fibonacci numbers: $F_1 = F_2 = 1,\ F_n = F_{n-1} + F_{n-2}\ (n \geq 3)$

$$I_n(x) = \prod_{i=1}^{n} \left(1 + x^{F_{i+1}}\right)$$

$$\begin{aligned} I_4(x) &= (1+x)(1+x^2)(1+x^3)(1+x^5) \\ &= 1 + x + x^2 + 2x^3 + x^4 + 2x^5 + 2x^6 + x^7 + 2x^8 + x^9 + x^{10} + x^{11} \end{aligned}$$

$v_2(n)$: sum of squares of coefficients of $I_n(x)$

Goal: $\displaystyle\sum_{n \geq 0} v_2(n) x^n = \frac{1 - 2x^2}{1 - 2x - 2x^2 + 2x^3}$

The Fibonacci triangle $\mathcal{F}$

$$
\begin{array}{ccccccccccccccccccccccccccccc}
 & & & & & & & & & & & & & & 1 & & & & & & & & & & & & & & \\
 & & & & & & & & & & & 1 & & & & & & 1 & & & & & & & & & & & \\
 & & & & & 1 & & & & & & 1 & & & \bullet & & & 1 & & & & & & 1 & & & & & \\
 & & & 1 & & 1 & & & \bullet & & & 1 & & & 2 & & & 1 & & & \bullet & & & 1 & & 1 & & & \\
 & 1 & & 1 & \bullet & 1 & & & 2 & & & 1 & \bullet & 2 & & 2 & \bullet & 1 & & & 2 & & & 1 & \bullet & 1 & & 1 & \\
1 & 1 & \bullet & 1 & 2 & 1 & \bullet & 2 & & 2 & \bullet & 1 & 3 & 2 & \bullet & 2 & 3 & 1 & \bullet & 2 & & 2 & \bullet & 1 & 2 & 1 & \bullet & 1 & 1
\end{array}
$$

The Fibonacci triangle $\mathcal{F}$

1

1 1

1 1 • 1 1

1 1 • 1 2 1 • 1 1

1 1 • 1 2 1 • 2 2 • 1 2 1 • 1 1

1 1 • 1 2 1 • 2 2 • 1 3 2 • 2 3 1 • 2 2 • 1 2 1 • 1 1

- Copy each entry of row $n \geq 1$ to the next row.
- Add two entries if separated by at bullet (and form group of 3)
- Copy once more the middle entry of a group of three (group of 2)
- Adjoin 1 at beginning and end of each row after row 0.

“Binomial theorem” for $\mathcal{F}$

$\begin{bmatrix} n \\ k \end{bmatrix}$: kth entry (beginning with $k = 0$) in row n (beginning with $n = 0$) in $\mathcal{F}$

“Binomial theorem” for $\mathcal{F}$

$\left[{n \atop k} \right]$: kth entry (beginning with $k=0$) in row n (beginning with $n=0$) in $\mathcal{F}$

Theorem. $\displaystyle\sum_k \left[{n \atop k} \right] x^k = I_n(x) := \prod_{i=1}^{n} \left(1 + x^{F_{i+1}}\right)$

“Binomial theorem” for $\mathcal{F}$

$\left[{n \atop k}\right]$: kth entry (beginning with $k = 0$) in row n (beginning with $n = 0$) in $\mathcal{F}$

Theorem. $\sum_k \left[{n \atop k}\right] x^k = I_n(x) := \prod_{i=1}^{n} \left(1 + x^{F_{i+1}}\right)$

Proof omitted.

$\sum_k \begin{bmatrix} n \\ k \end{bmatrix}^2$

Now can obtain a system of recurrences analogous to

$$\begin{aligned} u_2(n+1) &= 3u_2(n) + 2u_{1,1}(n) \\ u_{1,1}(n+1) &= 2u_2(n) + 2u_{1,1}(n) \end{aligned}$$

for Stern's triangle.

$\sum_k \left[{n \atop k}\right]^2$

Now can obtain a system of recurrences analogous to

$$\begin{aligned} u_2(n+1) &= 3u_2(n) + 2u_{1,1}(n) \\ u_{1,1}(n+1) &= 2u_2(n) + 2u_{1,1}(n) \end{aligned}$$

for Stern's triangle.

Need such sums as $\sum_k \left[{n \atop k}\right]^2$, where k ranges over all integers for which the kth entry in row n is the last in its group of two or three.

$\sum_k \left[{n \atop k} \right]^2$

Now can obtain a system of recurrences analogous to

$$\begin{aligned} u_2(n+1) &= 3u_2(n) + 2u_{1,1}(n) \\ u_{1,1}(n+1) &= 2u_2(n) + 2u_{1,1}(n) \end{aligned}$$

for Stern's triangle.

Need such sums as $\sum_k \left[{n \atop k} \right]^2$, where k ranges over all integers for which the kth entry in row n is the last in its group of two or three.

Seven sums in all $\Rightarrow$ 7×7 matrix.

$\sum_k \left[{n \atop k} \right]^2$

Now can obtain a system of recurrences analogous to

$$\begin{aligned} u_2(n+1) &= 3u_2(n) + 2u_{1,1}(n) \\ u_{1,1}(n+1) &= 2u_2(n) + 2u_{1,1}(n) \end{aligned}$$

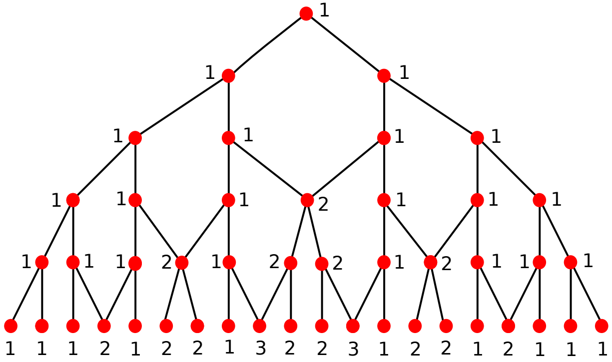
for Stern's triangle.

Need such sums as $\sum_k \left[{n \atop k} \right]^2$, where k ranges over all integers for which the kth entry in row n is the last in its group of two or three.

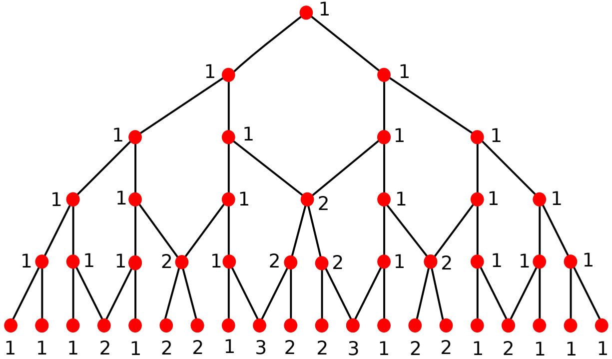
Seven sums in all $\Rightarrow 7 \times 7$ matrix.

Probably a simpler argument using this method.

A diagram (poset) associated with $\mathfrak{F}$

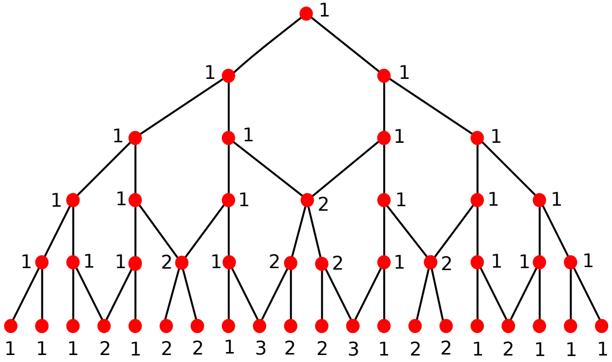


A diagram (poset) associated with $\mathfrak{F}$



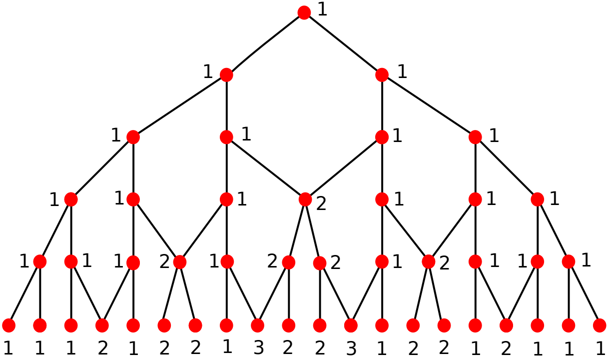
- Each point lies directly above **two** points.

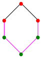
A diagram (poset) associated with $\mathfrak{F}$



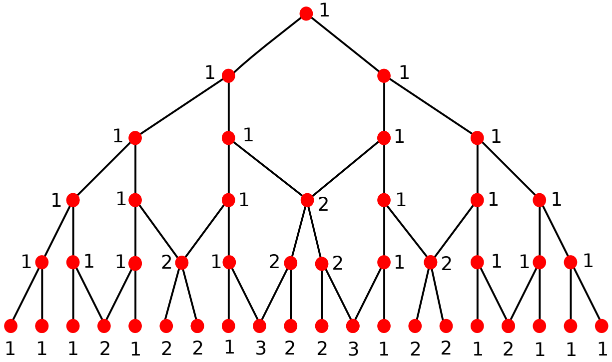
- Each point lies directly above **two** points.
- The diagram is planar.

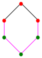
A diagram (poset) associated with $\mathfrak{F}$



- Each point lies directly above **two** points.
- The diagram is planar.
- Every  extends to 

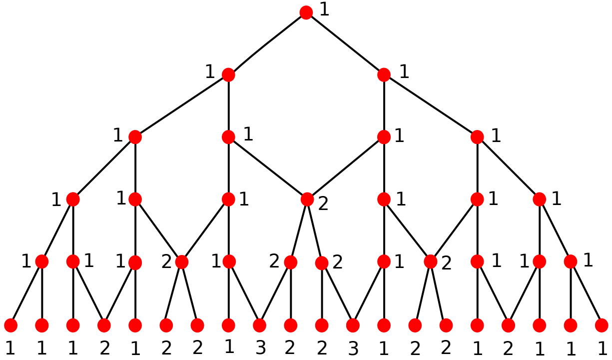
A diagram (poset) associated with $\mathfrak{F}$



- Each point lies directly above **two** points.
- The diagram is planar.
- Every  extends to 

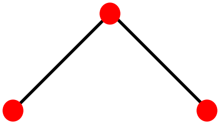
These properties **characterize** the diagram.

Two further properties

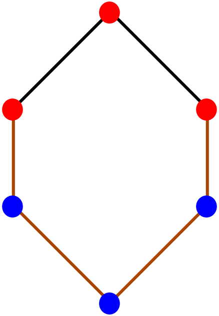


- Each label is the sum of those on the level above connected by an edge
- Each label is the number of paths from that label to the top.

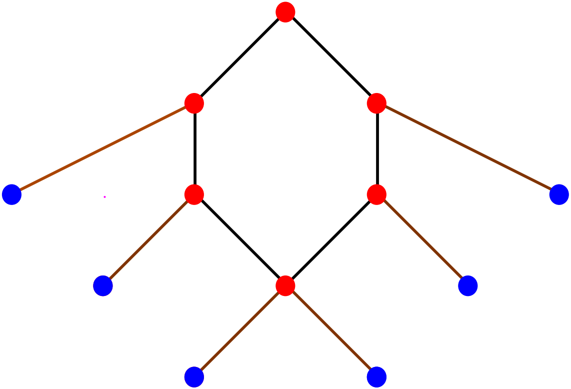
Constructing $\mathfrak{F}$



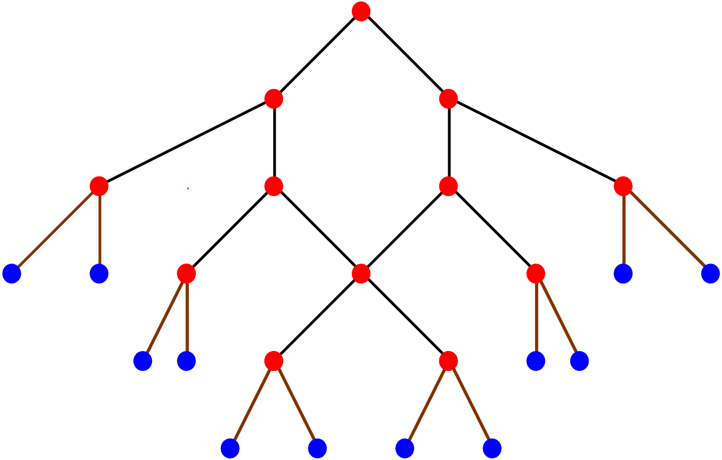
Constructing $\mathfrak{F}$



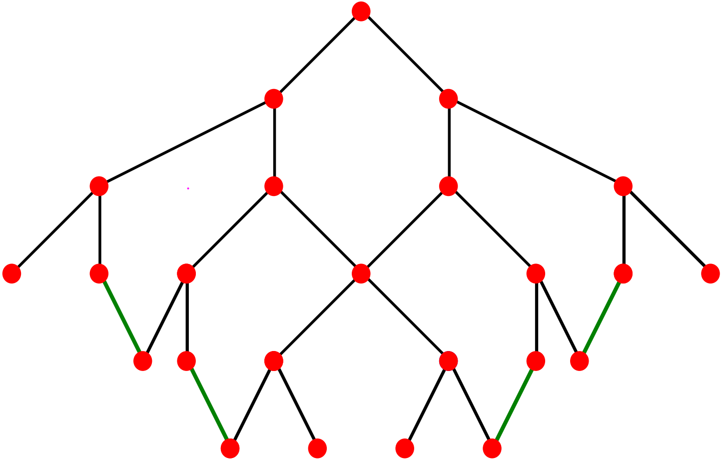
Constructing $\mathfrak{F}$



Constructing $\mathfrak{F}$



Constructing $\mathfrak{F}$



Number of elements at level n

p_n: number of elements of $\mathfrak{F}$ at level n

$$(p_0, p_1, \ldots) = (1, 2, 4, 7, 12, 20, \ldots)$$

Number of elements at level n

p_n: number of elements of $\mathfrak{F}$ at level n

$(p_0, p_1, \ldots) = (1, 2, 4, 7, 12, 20, \ldots)$

Each entry lies above two entries. Each entry at level $n \geq 3$ is the bottom element of a hexagon (with top at level $n-3$)

$$\Rightarrow p_n = 2p_{n-1} - p_{n-3}.$$

Number of elements at level n

p_n: number of elements of $\mathfrak{F}$ at level n

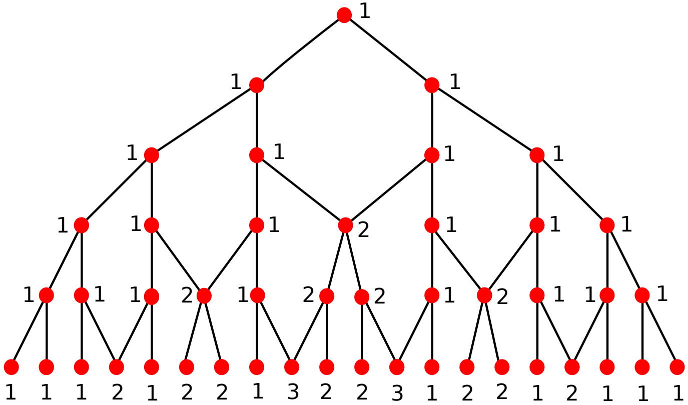
$(p_0, p_1, \dots) = (1, 2, 4, 7, 12, 20, \dots)$

Each entry lies above two entries. Each entry at level $n \geq 3$ is the bottom element of a hexagon (with top at level $n - 3$)

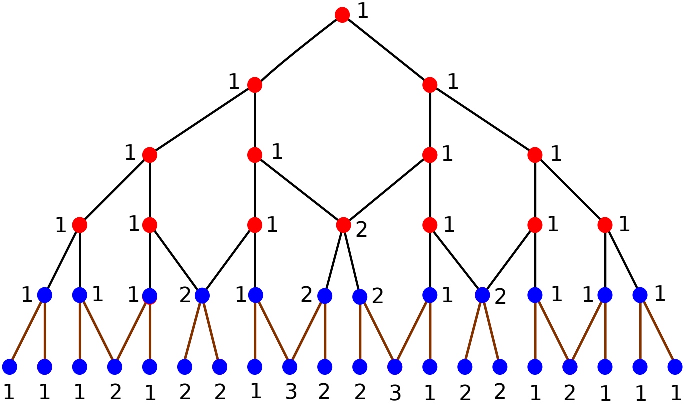
$$\Rightarrow p_n = 2p_{n-1} - p_{n-3}.$$

Solution with $p_0 = 1, p_1 = 2$ is $\boxed{p_n = F_{n+3} - 1}$

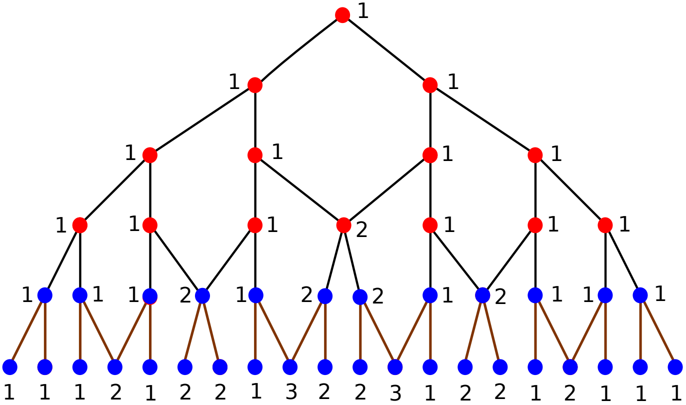
The groups of size two and three



The groups of size two and three



The groups of size two and three



What is the sequence of group sizes on each level? E.g., on level 5, the sequence $2, 3, 2, 3, 3, 2, 3, 2$.

The limiting sequence

As $n \to \infty$, we get a "limiting sequence"

$$2, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, \ldots .$$

The limiting sequence

As $n \to \infty$, we get a "limiting sequence"

$$2, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, \dots .$$

Let $\phi = (1 + \sqrt{5})/2$, the golden mean.

The limiting sequence

As $n \to \infty$, we get a "limiting sequence"

$$2, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, \dots .$$

Let $\phi = (1 + \sqrt{5})/2$, the golden mean.

Theorem. *The limiting sequence* $(c_1, c_2, \dots)$ *is given by*

$$c_n = 1 + \lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor.$$

Properties of $c_n = 1 + \lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor$

$$2, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, \ldots.$$

- $\gamma = (c_2, c_3, \ldots)$ characterized by invariance under $2 \to 3$, $3 \to 32$ (**Fibonacci word** in the letters 2,3).

Properties of $c_n = 1 + \lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor$

$$2, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, \ldots.$$

- $\gamma = (c_2, c_3, \ldots)$ characterized by invariance under $2 \to 3$, $3 \to 32$ (**Fibonacci word** in the letters 2,3).
- $\gamma = z_1 z_2 \ldots$ (concatenation), where $z_1 = 3$, $z_2 = 23$, $z_k = z_{k-2} z_{k-1}$

$$3 \cdot 23 \cdot 323 \cdot 23323 \cdot 32323323 \cdots$$

Properties of $c_n = 1 + \lfloor n\phi \rfloor - \lfloor (n-1)\phi \rfloor$

$$2, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, 3, 2, 3, 2, 3, 3, 2, 3, \dots .$$

- $\gamma = (c_2, c_3, \dots)$ characterized by invariance under $2 \rightarrow 3$, $3 \rightarrow 32$ (**Fibonacci word** in the letters 2,3).
- $\gamma = z_1 z_2 \dots$ (concatenation), where $z_1 = 3$, $z_2 = 23$, $z_k = z_{k-2} z_{k-1}$

$$3 \cdot 23 \cdot 323 \cdot 23323 \cdot 32323323 \cdots$$

- Sequence of number of 3's between consecutive 2's is the original sequence with 1 subtracted from each term.

$$2 \underbrace{3}_{1} 2 \underbrace{33}_{2} 2 \underbrace{3}_{1} 2 \underbrace{33}_{2} 2 \underbrace{33}_{2} 2 \underbrace{3}_{1} 2 \underbrace{33}_{2} 2 \dots .$$

An edge labeling of $\mathfrak{F}$

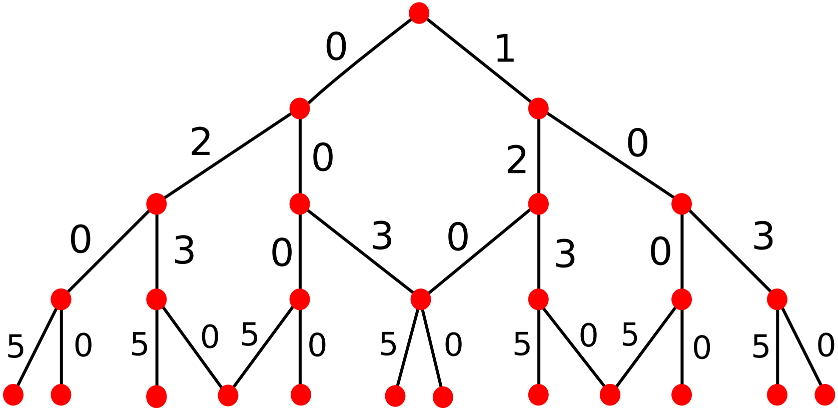
The edges between ranks $2k$ and $2k+1$ are labelled alternately $0, F_{2k+2}, 0, F_{2k+2}, \ldots$ from left to right.

An edge labeling of $\mathfrak{F}$

The edges between ranks $2k$ and $2k + 1$ are labelled alternately $0, F_{2k+2}, 0, F_{2k+2}, \ldots$ from left to right.

The edges between ranks $2k - 1$ and $2k$ are labelled alternately $F_{2k+1}, 0, F_{2k+1}, 0, \ldots$ from left to right.

Diagram of the edge labeling



Connection with sums of Fibonacci numbers

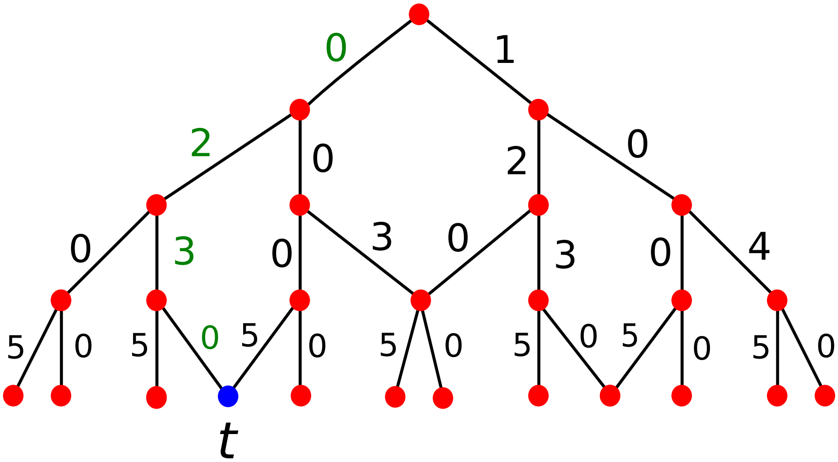
Let $t \in \mathfrak{F}$. All paths (saturated chains) from the top to t have the same sum of their elements $\sigma(t)$.

Connection with sums of Fibonacci numbers

Let $\boldsymbol{t} \in \mathfrak{F}$. All paths (saturated chains) from the top to t have the same sum of their elements $\boldsymbol{\sigma(t)}$.

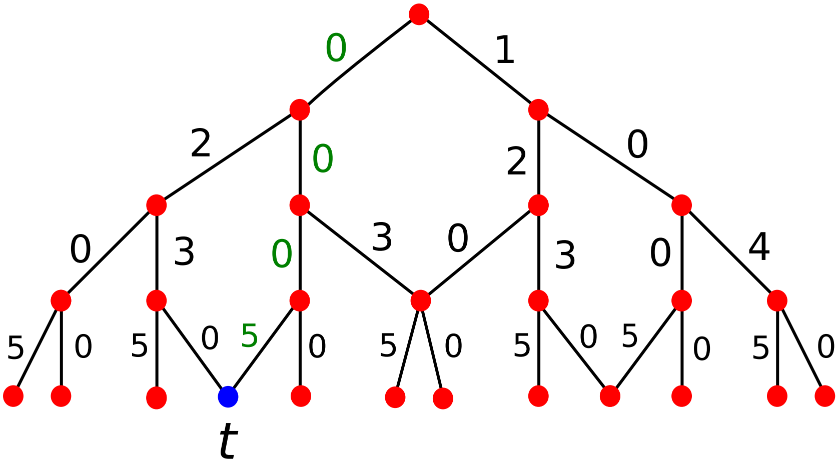
If $\mathrm{rank}(t) = n$, this gives all ways to write $\sigma(t)$ as a sum of distinct Fibonacci numbers from $F_2, F_3, \ldots, F_{n+1}$.

An example



$2 + 3 = F_3 + F_4$

An example



$5 = F_5$

Second proof: factorization in a free monoid

$$\begin{aligned} I_n(x) &:= \prod_{i=1}^{n} \left(1 + x^{F_{i+1}}\right) \\ &= \sum_k \begin{bmatrix} n \\ k \end{bmatrix} x^k \end{aligned}$$

Second proof: factorization in a free monoid

$$\begin{aligned} I_n(x) &:= \prod_{i=1}^{n} \left(1 + x^{F_{i+1}}\right) \\ &= \sum_k \begin{bmatrix} n \\ k \end{bmatrix} x^k \end{aligned}$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \# \left\{ (a_1, \ldots, a_n) \in \{0,1\}^n \, : \, \sum_i a_i F_{i+1} = k \right\}$$

Second proof: factorization in a free monoid

$$I_n(x) := \prod_{i=1}^{n} \left(1 + x^{F_{i+1}}\right) = \sum_k \begin{bmatrix} n \\ k \end{bmatrix} x^k$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \# \left\{ (a_1, \dots, a_n) \in \{0,1\}^n \; : \; \sum_i a_i F_{i+1} = k \right\}$$

$$v_2(n) := \sum_k \begin{bmatrix} n \\ k \end{bmatrix}^2 = \# \left\{ \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix} : \sum a_i F_{i+1} = \sum b_i F_{i+1} \right\}$$

A concatenation product

$$\mathcal{M}_n := \left\{ \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix} : \sum a_i F_{i+1} = \sum b_i F_{i+1} \right\}$$

A concatenation product

$$\mathcal{M}_n := \left\{ \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix} : \sum a_i F_{i+1} = \sum b_i F_{i+1} \right\}$$

Let

$$\alpha = \begin{pmatrix} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \end{pmatrix} \in \mathcal{M}_n, \quad \beta = \begin{pmatrix} c_1 & \cdots & c_m \\ d_1 & \cdots & d_m \end{pmatrix} \in \mathcal{M}_m.$$

Define

$$\alpha\beta = \begin{pmatrix} a_1 & \cdots & a_n & c_1 & \cdots & c_m \\ b_1 & \cdots & b_n & d_1 & \cdots & d_m \end{pmatrix},$$

A concatenation product

$$\mathcal{M}_n := \left\{ \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix} : \sum a_i F_{i+1} = \sum b_i F_{i+1} \right\}$$

Let

$$\alpha = \begin{pmatrix} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \end{pmatrix} \in \mathcal{M}_n, \quad \beta = \begin{pmatrix} c_1 & \cdots & c_m \\ d_1 & \cdots & d_m \end{pmatrix} \in \mathcal{M}_m.$$

Define

$$\alpha\beta = \begin{pmatrix} a_1 & \cdots & a_n & c_1 & \cdots & c_m \\ b_1 & \cdots & b_n & d_1 & \cdots & d_m \end{pmatrix},$$

Easy to check: $\alpha\beta \in \mathcal{M}_{n+m}$

The monoid $\mathcal{M}$

$$\mathcal{M} := \mathcal{M}_0 \cup \mathcal{M}_1 \cup \mathcal{M}_2 \cup \cdots,$$

a **monoid** (semigroup with identity) under concatenation. The identity element is $\varnothing \in \mathcal{M}_0$.

The monoid $\mathcal{M}$

$$\mathcal{M} := \mathcal{M}_0 \cup \mathcal{M}_1 \cup \mathcal{M}_2 \cup \cdots,$$

a **monoid** (semigroup with identity) under concatenation. The identity element is $\varnothing \in \mathcal{M}_0$.

Definition. A subset $\mathcal{G} \subset \mathcal{M}$ **freely generates** $\mathcal{M}$ if every $\alpha \in \mathcal{M}$ can be written uniquely as a product of elements of $\mathcal{G}$. (We then call $\mathcal{M}$ a **free** monoid.)

The monoid $\mathcal{M}$

$$\mathcal{M} := \mathcal{M}_0 \cup \mathcal{M}_1 \cup \mathcal{M}_2 \cup \cdots,$$

a **monoid** (semigroup with identity) under concatenation. The identity element is $\varnothing \in \mathcal{M}_0$.

Definition. A subset $\mathcal{G} \subset \mathcal{M}$ **freely generates** $\mathcal{M}$ if every $\alpha \in \mathcal{M}$ can be written uniquely as a product of elements of $\mathcal{G}$. (We then call $\mathcal{M}$ a **free** monoid.)

Suppose $\mathcal{G}$ freely generates $\mathcal{M}$, and let $\boldsymbol{G(x)} = \sum_{n\geq 1} \#(\mathcal{M}_n \cap \mathcal{G}) x^n$. Then

$$\begin{aligned}
\sum_n v_2(n) x^n &= \sum_n \#\mathcal{M}_n \cdot x^n \\
&= 1 + G(x) + G(x)^2 + \cdots \\
&= \frac{1}{1 - G(x)}.
\end{aligned}$$

Free generators of $\mathcal{M}$

Theorem. *$\mathcal{M}$ is freely generated by the following elements:*

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & * & 1 & * & 1 & * & 1 & * & \cdots & * & 1 & 0 \\ 00 & * & 0 & * & 0 & * & 0 & * & \cdots & * & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 00 & * & 0 & * & 0 & * & 0 & * & \cdots & * & 0 & 1 \\ 11 & * & 1 & * & 1 & * & 1 & * & \cdots & * & 1 & 0 \end{pmatrix},$$

where each $*$ can be 0 or 1, but two $*$'s in the same column must be equal.

Free generators of $\mathcal{M}$

Theorem. *$\mathcal{M}$ is freely generated by the following elements:*

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & * & 1 & * & 1 & * & 1 & * & \cdots & * & 1 & 0 \\ 00 & * & 0 & * & 0 & * & 0 & * & \cdots & * & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 00 & * & 0 & * & 0 & * & 0 & * & \cdots & * & 0 & 1 \\ 11 & * & 1 & * & 1 & * & 1 & * & \cdots & * & 1 & 0 \end{pmatrix},$$

where each $*$ can be 0 or 1, but two $*$'s in the same column must be equal.

Example. $\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$: $1 + 2 + 3 + 5 = 3 + 8$

$G(x)$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 11 & * & 1 & * & 1 & * & 1 & * & \cdots & * & 1 & 0 \\ 00 & * & 0 & * & 0 & * & 0 & * & \cdots & * & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 00 & * & 0 & * & 0 & * & 0 & * & \cdots & * & 0 & 1 \\ 11 & * & 1 & * & 1 & * & 1 & * & \cdots & * & 1 & 0 \end{pmatrix}$$

Two elements of length one: $G(x) = 2x + \cdots$

$G(x)$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 11 & * & 1 & * & 1 & * & 1 & * & \cdots & * & 1 & 0 \\ 00 & * & 0 & * & 0 & * & 0 & * & \cdots & * & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 00 & * & 0 & * & 0 & * & 0 & * & \cdots & * & 0 & 1 \\ 11 & * & 1 & * & 1 & * & 1 & * & \cdots & * & 1 & 0 \end{pmatrix}$$

Two elements of length one: $G(x) = 2x + \cdots$

Let $\boldsymbol{k}$ be the number of columns of $*$'s. Length is $2k+3$. Thus

$$\begin{aligned} G(x) &= 2x + 2\sum_{k\geq 0} 2^k x^{2k+3} \\ &= 2x + \frac{2x^3}{1-2x^2}. \end{aligned}$$

Completion of proof

$$\sum_n v_2(n)x^n = \frac{1}{1-G(x)} = \frac{1}{1-\left(2x+\frac{2x^3}{1-2x^2}\right)} = \frac{1-2x^2}{1-2x-2x^2+2x^3} \quad \square$$

Further vistas?

Let $i, j \geq 1$. Define the diagram (poset) $\boldsymbol{P_{ij}}$ by

- Each point lies directly above i points.

Further vistas?

Let $i, j \geq 1$. Define the diagram (poset) $\boldsymbol{P_{ij}}$ by

- Each point lies directly above i points.
- The diagram is planar.

Further vistas?

Let $i, j \geq 1$. Define the diagram (poset) $\boldsymbol{P_{ij}}$ by

- Each point lies directly above i points.
- The diagram is planar.
- Every  extends to a $2(j+1)$-gon ($j+1$ edges on each side)

Further vistas?

Let $i, j \geq 1$. Define the diagram (poset) $\boldsymbol{P_{ij}}$ by

- Each point lies directly above i points.
- The diagram is planar.
- Every  extends to a $2(j+1)$-gon ($j+1$ edges on each side)

Example. P_{11}: diagram for Pascal's triangle
P_{21}: diagram for Stern's triangle
P_{12}: diagram for the Fibonacci triangle

Further vistas?

Let $i, j \geq 1$. Define the diagram (poset) $\boldsymbol{P_{ij}}$ by

- Each point lies directly above i points.
- The diagram is planar.
- Every  extends to a $2(j+1)$-gon ($j+1$ edges on each side)

Example. P_{11}: diagram for Pascal's triangle
P_{21}: diagram for Stern's triangle
P_{12}: diagram for the Fibonacci triangle

What can be said about P_{ij}?

References

These slides:
www-math.mit.edu/~rstan/transparencies/msu.pdf

The Stern triangle: *Amer. Math. Monthly* **127** (2020), 99–111;
arXiv:1901.04647

The Fibonacci triangle (and much more): **arXiv:2101.02131**

Fibonacci word: **Wikipedia**

Factorization in free monoids: **EC1, second ed., §4.7.4**

The final slide

The final slide