The Kromatic Symmetric Function

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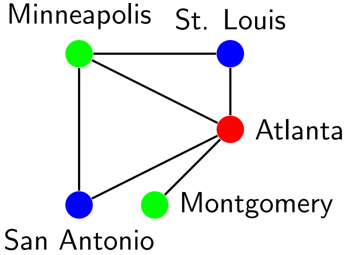
Graphs and colourings

A **graph** G consists of a set of vertices $V(G)$ and a set of edges $E(G)$.

A **colouring** of a graph G is a function $\kappa : V(G) \to \mathbb{N}_{>0}$. We say κ is a k-**colouring** if $\kappa(v) \leq k$ for all $v \in V(G)$.

A **proper colouring** is a colouring with $\kappa(u) \neq \kappa(v)$ for every edge $uv \in E(G)$.



The Chromatic Symmetric Function

Let $G = (V, E)$ be a graph.

Definition (Stanley (1995))

$$X_G(x_1, \dots) = \sum_{\kappa \text{ proper colouring}} \prod_{v \in V(G)} x_{\kappa(v)}.$$

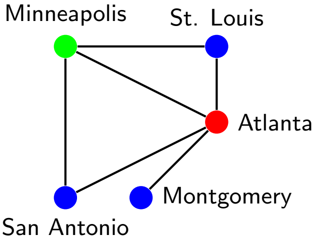
This function is a power series in $\mathbb{C}[[x_1, x_2, \dots]]$. It is called a **symmetric function** because for every permutation π of $\mathbb{N}$,

$$f(x_1, x_2, \dots) = f(x_{\pi(1)}, x_{\pi(2)}, \dots).$$

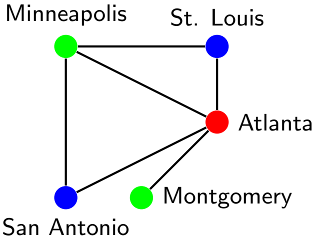
Example: X_Δ

Let blue $= 1$, green $= 2$, red $= 3$.

$$X_\Delta = x_1^3 x_2 x_3 + \cdots + x_1^2 x_2^2 x_3 + \ldots$$



$x_1^3 x_2 x_3$



$x_1^2 x_2^2 x_3$

Vertex-Weighted Graphs

A **vertex-weighted graph** (G, w) consists of a graph G and a weight function $w : V(G) \to \mathbb{Z}^+$. Extend the chromatic symmetric function as

$$X_{(G,w)}(x_1, x_2, \dots) = \sum_{\text{prop. } \kappa} \prod_{v \in V} x_{\kappa(v)}^{w(v)}$$

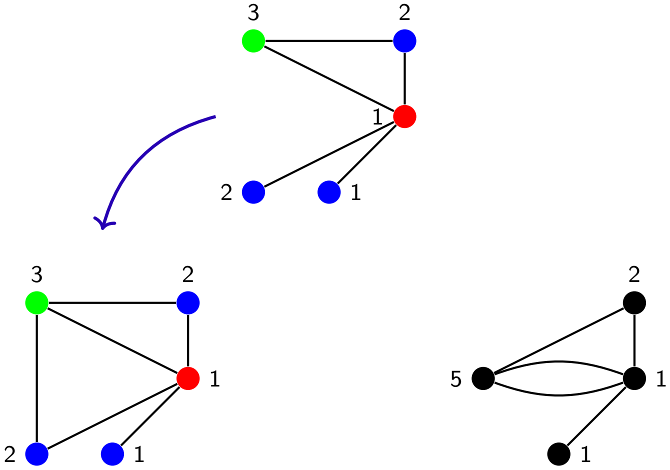
Theorem

Let $e = uv$. Then

$$X_{(G,w)} = X_{(G-e,w)} - X_{(G/e,w/e)}$$

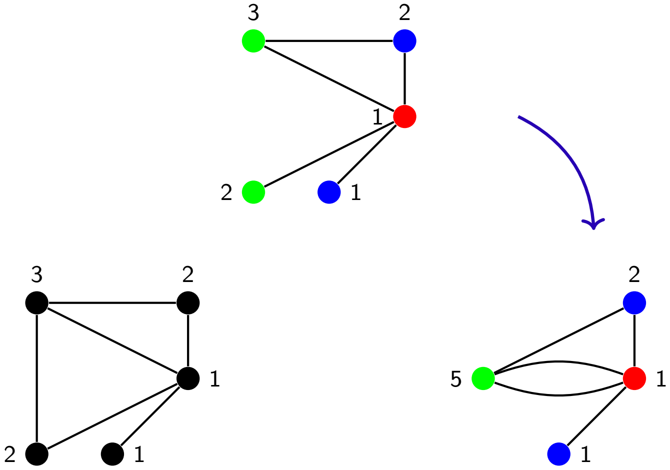
where $w/e(z) = w(u) + w(v)$ for the vertex z arising from contraction.

A Deletion-Contraction Relation



$$X_{(G-e,w)} = X_{(G,w)} + X_{(G/e,w/e)}$$

A Deletion-Contraction Relation

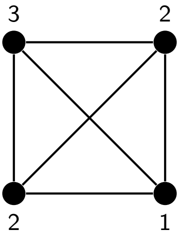


$$X_{(G-e,w)} = X_{(G,w)} + X_{(G/e,w/e)}$$

The Algebra of Symmetric Functions

- Monomial symmetric functions:

$$\widetilde{m}_{(\lambda_1,\ldots,\lambda_k)} = \sum_{i_1,\ldots,i_k \text{ distinct}} \prod_{j=1}^{k} x_{i_j}^{\lambda_j}$$

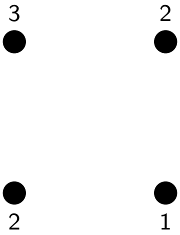


$\widetilde{m}_{(3,2,2,1)} = \sum_{\textbf{distinct } i_j} x_{i_1}^3 x_{i_2}^2 x_{i_3}^2 x_{i_4}$

The Algebra of Symmetric Functions

- Power-sum symmetric functions:

$$p_n = \sum_i x_i^n,\ p_\lambda = p_{\lambda_1} \dots p_{\lambda_k}$$

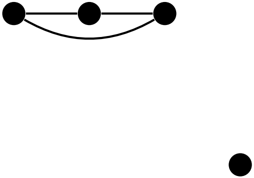


$$p_{(3,2,2,1)} = \sum x_i^3 \cdot \sum x_i^2 \cdot \sum x_i^2 \cdot \sum x_i$$

The Algebra of Symmetric Functions

- Elementary symmetric functions:

$$\widetilde{e}_n = \sum_{i_1 < \cdots < i_n} x_{i_1} \dots x_{i_n}, \widetilde{e}_\lambda = \widetilde{e}_{\lambda_1} \dots \widetilde{e}_{\lambda_k}$$



$\widetilde{e}_{(3,2,1)} = \sum_{i_1,i_2,i_3 \text{ distinct}} x_{i_1} x_{i_2} x_{i_3} \cdot \sum_{j_1 \neq j_2} x_{j_1} x_{j_2} \cdot \sum_{k_1} x_{k_1}$

Schur Functions

The **Young diagram of shape λ** consists of $l(\lambda)$ rows, top- and left-justified, where the i^{th} row from the top contains λ_i boxes.

A **semi-standard Young tableau (SSYT) of shape λ** is a filling of the Young diagram of shape λ with positive integers so that rows are weakly increasing left-to-right, and columns are strictly decreasing top-to-bottom.

1	1
2	3
3	

(2, 2, 1)

Schur Functions

The **Schur function of type λ** is given by

$$s_\lambda = \sum_{T \in SSYT(\lambda)} \prod_i x_i^{\#i \text{ in } T}.$$

So for example, one term of $s_{(2,2,1)}$ is $x_1^2 x_2 x_3^2$.

1	1
2	3
3	

Motivation from K-Theory: Grothendieck Functions

The **symmetric Grothendieck function** of type λ is a symmetric function of unbounded degree that is a **K-theoretic analogue of the Schur functions**.

A **multi-valued (semi-standard) Young tableau** (MVT) of shape λ fills each box with a non-empty set of positive integers such that choosing one integer from each box always gives a SSYT.

$1,2$	2
4	$5,7$
7	

Grothendieck Functions

Then

$$\overline{s}_\lambda = \sum_{T \in MVT(\lambda)} (-1)^{\#\text{numbers} - \#\text{boxes}} \prod_i x_i^{\#i \text{ in } T}.$$

For example, one term of $\overline{s}_{(2,2,1)}$ is $(-1)^{7-5} x_1 x_2^2 x_4 x_5 x_7^2$.

$$\begin{array}{|c|c|} \hline 1,2 & 2 \\ \hline 4 & 5,7 \\ \hline 7 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

The Kromatic Symmetric Function

K-theoretic analogues of objects in algebraic combinatorics generally have this form as a multiple-valued superposition.

Our idea: Apply this to the chromatic symmetric function. Instead of colouring a vertex with one color, give it a **set of colours**.

Definition (Crew, Pechenik, S (2023+))

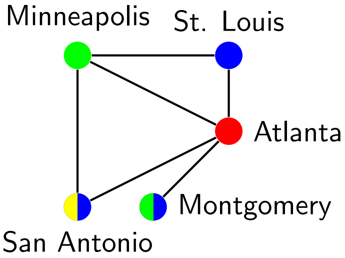
The **Kromatic symmetric function** $\overline{X}_G$ is given by

$$\overline{X}_G = \sum_{\kappa: V(G) \to \mathcal{P}(\mathbb{Z}^+) \setminus \{\emptyset\}} \prod_{v \in V(G)} \prod_{i \in \kappa(v)} x_i$$

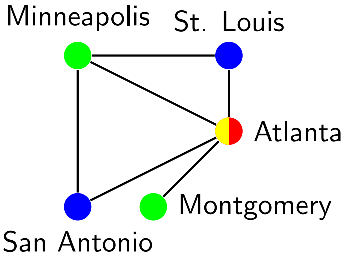
Example: $\overline{X}_\Delta$

Let blue $= 1$, green $= 2$, red $= 3$, yellow $= 4$.

$$\overline{X}_\Delta = x_1^3 x_2^2 x_3 x_4 + \cdots + x_1^2 x_2^2 x_3 x_4 + \ldots$$



$x_1^3 x_2^2 x_3 x_4$



$x_1^2 x_2^2 x_3 x_4$

Vertex-Weighted Kromatic Symmetric Function

Definition (Crew, Pechenik, S. (2023+))

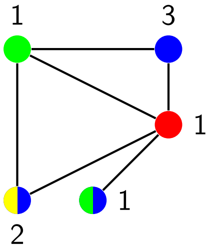
The **vertex-weighted Kromatic symmetric function** $\overline{X}_{(G,w)}$ is given by

$$\overline{X}_{(G,w)} = \sum_{\kappa : V(G) \to \mathcal{P}(\mathbb{Z}^+) \setminus \{\emptyset\}} \prod_{v \in V(G)} \prod_{i \in \kappa(v)} x_i^{w(v)}$$

Example: $\overline{X}_{(\Delta,w)}$

Let blue $= 1$, green $= 2$, red $= 3$, yellow $= 4$.

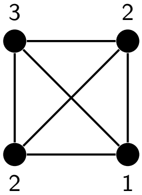
$$\overline{X}_{(\Delta,w)} = x_1^6 x_2^2 x_3 x_4^2 + \ldots$$



$x_1^6 x_2^2 x_3 x_4^2$

A K-Theoretic $\widetilde{m}$-basis Expansion

Define $\widetilde{m}_{(\lambda_1,\ldots,\lambda_k)}$ as the Kromatic symmetric function of a k-vertex complete graph with vertex weights $\lambda_1, \ldots, \lambda_k$:



$$\overline{\widetilde{m}}_{(3,2,2,1)} = \overline{X}_{K^{(3,2,2,1)}}$$

A K-Theoretic $\widetilde{m}$-basis Expansion

Lemma (Crew, Pechenik, S. (2023+))

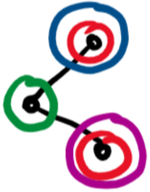
For any vertex-weighted graph (G, w), we have

$$\overline{X}_{(G,\omega)} = \sum_{C} \overline{\widetilde{m}}_{\lambda(C)}$$

where the sum ranges over all C that are covers of G by stable sets, and $\lambda(C)$ is the integer partition of the weights of the stable sets.

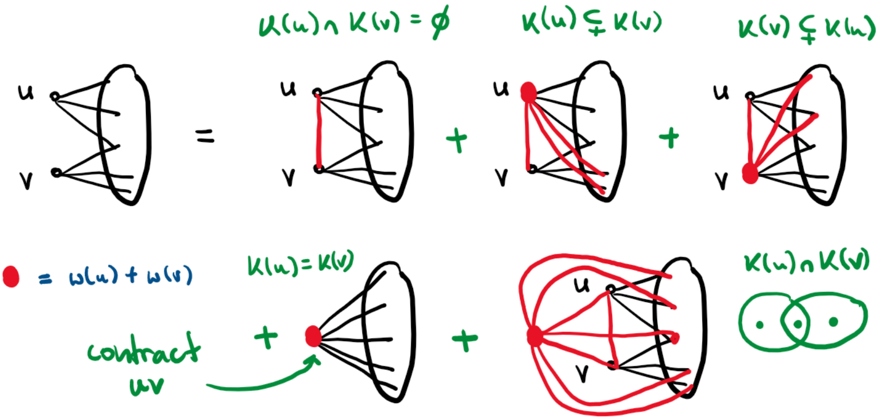
- For X_G, the same result holds if the stable sets are required to be disjoint [Stanley]

A K-Theoretic $\widetilde{m}$-basis Expansion



$$\widetilde{m}_{2,1} + \widetilde{m}_{2,1,1} + \widetilde{m}_{2,1,1} + \widetilde{m}_{2,1,1,1} + \widetilde{m}_{1,1,1}$$

A Deletion-Contraction Relation for $\overline{X}_{(G,w)}$

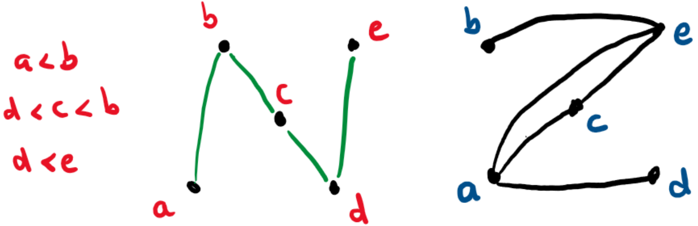


- Reduces number of stable sets of size more than 1.

Schur and Grothendieck Basis Expansions

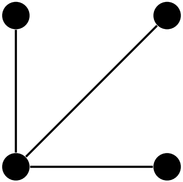
A **poset**, or partially-ordered set, consists of a pair $(P, <_P)$ of a set P and a relation $<_P$ on P that is antisymmetric and transitive.

The **incomparability graph of P**, denoted $inc(P)$, is the graph with vertex set P, and where two distinct elements of P have an edge between them if and only if they are incomparable with respect to $<_P$.



Schur and Grothendieck Basis Expansions

A graph is **claw-free** if it has no induced subgraph isomorphic to $K_{1,3}$.



Schur and Grothendieck Basis Expansions

Theorem (Gasharov (1996))

If G is a claw-free incomparability graph of a poset P, then the coefficient of s_λ in X_G enumerates fillings of the Young diagram of shape λ with elements of P such that rows are $<_P$-increasing left-to-right, and columns have no $<_P$-decreases among consecutive boxes top-to-bottom.

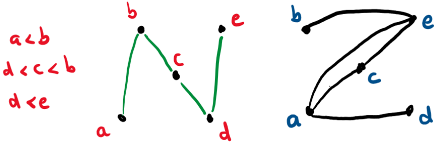
In particular, X_G is s-positive.

a	b
d	e
c	

d	c
a	b
e	

d	e
a	b
c	

d	e
c	b
a	



Schur and Grothendieck Basis Expansions

Theorem (Crew, Pechenik, S. (2023+))

If G is a claw-free incomparability graph, then $\overline{X}_G$ is $\overline{s}$-positive, and the coefficient of $\overline{s}_\lambda$ enumerates a weighted sum of the aforementioned fillings of Young tableaux with shape contained in λ.

- Does this imply a topological interpretation of Schur- and Grothendieck-positivity for these symmetric functions?

The Stanley-Stembridge Conjecture

Conjecture (Stanley-Stembridge (1993))

The chromatic symmetric function of a claw-free incomparability graph is e-positive.

- Stronger condition than s-positivity.
- Does the Stanley-Stembridge conjecture extend to the Kromatic symmetric function?
- What is the "correct" $\overline{e}$-basis?

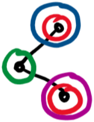
The Stanley-Stembridge Conjecture

There are two options for a multiplicative $\overline{e}$:

- $\overline{e}_n = \overline{s}_{1^n}$; or
- $\overline{e}_n = \frac{1}{n!}\overline{X}_{K_n}$.

These differ by a different constant factor for each fixed degree.

In both cases, even a three-vertex path is not $\overline{e}$-positive!



$\tilde{m}_{2,1} + \tilde{m}_{2,1,1} + \tilde{m}_{2,1,1} + \tilde{m}_{2,1,1,1} + \tilde{m}_{1,1,1}$

1	1, 2
2	

Further Directions

- Is there a good interpretation of the p-basis coefficients?

$$p_{1,1,1} - 2p_{1,2} + p_3 - 4p_{1,1,2} + 7p_{1,3} + p_{2,2} - 4p_4 - 2p_{1,1,1,2} + 11p_{1,1,3} + 5p_{1,2,2} - 20p_{1,4} - 6p_{2,3} + 12p_5 + \dots$$

- Do there exist graphs with equal Kromatic symmetric function?

The End

Thank You!