Coefficientwise Hankel-total positivity in enumerative combinatorics

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A big project in collaboration with
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Overview

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  - Connection with multiple orthogonal polynomials (MOPs)
- Many examples of *conjectured* coefficientwise Hankel-TP
- (Tentative) conclusion

Some references

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- A bizarre concept: grossly basis-dependent.
- (Contrast with positive semidefiniteness.)
- But . . . In many areas of mathematics, there is a preferred basis.

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Applications of total positivity:

- Mechanics of oscillatory systems
- Zeros of polynomials and entire functions
- Numerical linear algebra
- Approximation theory
- Stochastic processes
- Lie theory and cluster algebras
- Representation theory of the infinite symmetric group
- Planar discrete potential theory and the planar Ising model
- Stieltjes moment problem
- Enumerative combinatorics

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$$H_\infty(\boldsymbol{a}) \;=\; (a_{i+j})_{i,j \ge 0} \;=\; \begin{pmatrix} a_0 & a_1 & a_2 & \cdots \\ a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & a_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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- We say that the sequence $\boldsymbol{a}$ is *Hankel-totally positive* if its Hankel matrix $H_\infty(\boldsymbol{a})$ is totally positive.
- This implies that the sequence is *log-convex*, but is much stronger.

Hankel-total positivity and the moment problem

Main Characterization (Stieltjes 1894, Gantmakher–Krein 1937)

For a sequence $\boldsymbol{a} = (a_n)_{n \geq 0}$ of real numbers, the following are equivalent:

(a) $\boldsymbol{a}$ is Hankel-totally positive.

(b) There exists a positive measure μ on $[0, \infty)$ such that $a_n = \int x^n \, d\mu(x)$ for all $n \geq 0$.

[That is, $\boldsymbol{a}$ is a **Stieltjes moment sequence**.]

(c) There exist numbers $\alpha_0, \alpha_1, \ldots \geq 0$ such that

$$\sum_{n=0}^{\infty} a_n t^n = \cfrac{\alpha_0}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cdots}}}$$

in the sense of formal power series.

[**Stieltjes-type continued fraction** with nonnegative coefficients]

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[or, From counting to counting-with-weights]

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   Counting permutations of $[n]$ by number of descents:

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An industry in combinatorics: cf. Sokal–Zeng 2020 and Deb–Sokal 2022

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Coefficientwise Hankel-TP *implies* that $(P_n(\mathbf{x}))_{n\geq 0}$ is a Stieltjes moment sequence for all $\mathbf{x} \geq 0$, but it is *stronger*.

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- More general approach: *production matrices* — still *sufficient but far from necessary*.

Classical continued fractions

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- Stieltjes-type continued fractions (S-fractions):

$$\sum_{n=0}^{\infty} \underbrace{S_n(\boldsymbol{\alpha})}_{\text{Stieltjes–Rogers polynomial}} t^n = \cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cdots}}}$$

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- This is combinatorialists' notation. Analysts take $t^n \to \dfrac{1}{z^{n+1}}$

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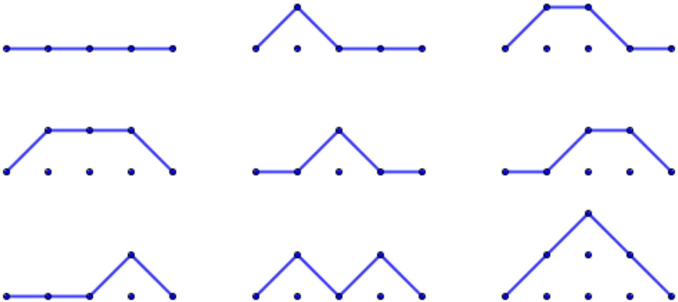

The nine Motzkin paths of length $n = 4$.

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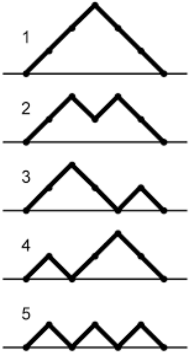

The five Dyck paths of length $2n = 6$.

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Theorem (Flajolet 1980)

- The Jacobi–Rogers polynomial $J_n(\boldsymbol{\beta}, \boldsymbol{\gamma})$ is the generating polynomial for Motzkin paths of length n, in which each rise gets weight 1, each level step at height i gets weight γ_i, and each fall from height i gets weight β_i.
- The Stieltjes–Rogers polynomial $S_n(\boldsymbol{\alpha})$ is the generating polynomial for Dyck paths of length $2n$, in which each rise gets weight 1 and each fall from height i gets weight α_i.

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Theorem (A.S. 2014, based on Viennot 1983)

The sequence $(S_n(\alpha))_{n \geq 0}$ of Stieltjes–Rogers polynomials is coefficientwise Hankel-totally positive in the polynomial ring $\mathbb{Z}[\alpha]$.

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Can now specialize α to *nonnegative* elements in any partially ordered commutative ring, and get Hankel-TP.

I will show some examples . . .

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But the story is more complicated than for S-type fractions, because:

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- It is not even totally positive in $\mathbb{R}$ for all $\boldsymbol{\beta},\boldsymbol{\gamma} \ge 0$.
- Rather, the total positivity of $H_\infty(\boldsymbol{J})$ holds only when $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ satisfy suitable *inequalities*.

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Proof uses the method of *production matrices.*

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- When applied to tridiagonal matrices, this handles J-fractions.
- But the method is much more general.
- Big computational problem: Given a Hankel-TP sequence $\boldsymbol{a} = (a_n)_{n \ge 0}$, find a TP production matrix that generates $\boldsymbol{a}$ as the zeroth column of its output matrix. Does one even necessarily exist?

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- Elementary "renewal" argument on Dyck paths implies

$$\mathcal{N} \;=\; \frac{1}{1 - tx - t(\mathcal{N} - 1)}$$

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- **Conclusion:** The sequence $(N_n(x))_{n\geq 0}$ of Narayana polynomials is coefficientwise Hankel-totally positive.

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- See Sokal–Zeng 2020 for extensions to even more variables.

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- There is no S-type continued fraction *in the ring of polynomials*: we have

$$\alpha_1, \alpha_2, \ldots \;=\; 1+x,\; \frac{2x}{1+x},\; \frac{1+x^2}{1+x},\; \frac{x+x^2}{1+x^2},\; \frac{1+x^3}{1+x^2},\; \frac{x+x^3}{1+x^3},\; \ldots$$

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- **Conclusion** (A.S. unpublished 2014, Wang–Zhu 2016):
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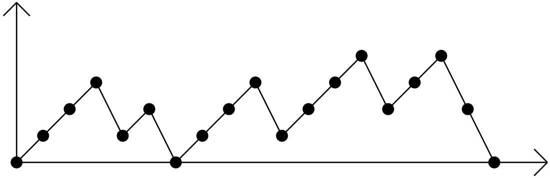
- A 2-Dyck path of length 18:



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Proof is essentially *identical* to the one for $m = 1$!

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But we hardly ever use these formulae.

We use (a) the graphical description, and/or (b) recurrences.

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- The α are indeed positive, but what the hell are they???

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- $(n!)^2$ has a nice m-branched continued fraction with $m = 2$:

$$\boldsymbol{\alpha} \;=\; 1, 1, 2, 4, 4, 6, 9, 9, 12, \ldots$$

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- But these are special cases of something vastly more general . . .

BCFs for ratios of contiguous hypergeometric functions

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- Euler also showed in 1746 that

$$\sum_{n=0}^{\infty} a(a+1)\cdots(a+n-1)\, t^n \;=\; \cfrac{1}{1 - \cfrac{at}{1 - \cfrac{1t}{1 - \cfrac{(a+1)t}{1 - \cfrac{2t}{1 - \cdots}}}}}$$

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- And this is the $b=1$ special case of

$$\frac{{}_2F_0\!\left(\begin{array}{c|}a,b\\ \text{—}\end{array}\; t\right)}{{}_2F_0\!\left(\begin{array}{c|}a,b-1\\ \text{—}\end{array}\; t\right)} \;=\; \cfrac{1}{1-\cfrac{at}{1-\cfrac{bt}{1-\cfrac{(a+1)t}{1-\cfrac{(b+1)t}{1-\cdots}}}}}$$

(${}_2F_0$ limiting case of Gauss continued fraction for ${}_2F_1$)

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- Recall the definition of the hypergeometric series:

$${}_rF_s\!\left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} \,\middle|\, t\right) \;=\; \sum_{n=0}^{\infty} \frac{a_1^{\overline{n}} \cdots a_r^{\overline{n}}}{b_1^{\overline{n}} \cdots b_s^{\overline{n}}} \, \frac{t^n}{n!}$$

where $a^{\overline{n}} = a(a+1)\cdots(a+n-1)$

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We generalize this to ratios of contiguous ${}_{m+1}F_0$:

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Theorem (Pétréolle–A.S.–Zhu 2018)

For each $m \ge 1$,

$$\frac{{}_{m+1}F_0\left(\begin{matrix} a_1, \ldots, a_{m+1} \\ - \end{matrix} \,\middle|\, t\right)}{{}_{m+1}F_0\left(\begin{matrix} a_1, \ldots, a_m, a_{m+1}-1 \\ - \end{matrix} \,\middle|\, t\right)} = \sum_{n=0}^{\infty} S_n^{(m)}(\boldsymbol{\alpha})\, t^n$$

where the $\boldsymbol{\alpha}$ *are very simple polynomials in* $a_1, \ldots, a_{m+1}$*:*

$$\boldsymbol{\alpha} = a_1 \cdots a_m,\ a_2 \cdots a_{m+1},\ a_3 \cdots a_{m+1}(a_1+1),\ a_4 \cdots a_{m+1}(a_1+1)(a_2+1),\ \ldots$$

which can be seen as products of m successive "pre-alphas":

$$\boldsymbol{\alpha}^{\rm pre} = a_1, \ldots, a_{m+1},\ a_1+1, \ldots, a_{m+1}+1,\ a_1+2, \ldots, a_{m+1}+2,\ \ldots$$

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We generalize this to ratios of contiguous ${}_{m+1}F_0$:

the result is an m-branched continued fraction ...

Corollary

The polynomials $P_n^{(m)}(a_1, \ldots, a_m; a_{m+1}) = S_n^{(m)}(\alpha)$ arising as the Taylor coefficients of this ratio are coefficientwise Hankel-TP jointly in $a_1, \ldots, a_{m+1}$.

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When $a_{m+1} = 1$, these polynomials are products of rising factorials (= products of Pochhammer symbols):

$$P_n^{(m)}(a_1, \ldots, a_m; 1) \;=\; a_1^{\overline{n}} \cdots a_m^{\overline{n}} \;=\; \frac{\Gamma(a_1 + n) \cdots \Gamma(a_m + n)}{\Gamma(a_1) \cdots \Gamma(a_2)}$$

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- Generalizes Gauss (1813) continued fraction for ${}_2F_1$.
- Can further generalize to q-hypergeometric functions ${}_r\phi_s$.
- But corollaries for Hankel-TP are more subtle than for $s = 0$. (Already this was the case for ${}_2F_1$ compared to ${}_2F_0$.)

Connection with multiple orthogonal polynomials
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- MOPs of type **II**: $P_{\mathbf{n}}(x)$ for $\mathbf{n} = (n_1, \ldots, n_r)$, of degree $|\mathbf{n}| = \sum n_i$
- Let $(\widetilde{P}_n(x))_{n \ge 0}$ be MOPs along the stepline (near-diagonal sequence) $\ldots \to (n, n, \ldots, n) \to (n+1, n, \ldots, n) \to (n+1, n+1, \ldots, n) \to \ldots$
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- Stepline MOPs satisfy $(r+2)$-term recurrence

$$\widetilde{P}_{n+1}(x) \;=\; (x - \pi_{nn})\, \widetilde{P}_n(x) \;-\; \sum_{k=n-r}^{n-1} \pi_{nk}\, \widetilde{P}_k(x)$$

or equivalently

$$x\widetilde{P}_n(x) \;=\; \sum_{k=n-r}^{n+1} \pi_{nk}\, \widetilde{P}_k(x)$$

where $\pi_{n,n+1} = 1$.

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- Hélder Lima and I are pursuing this connection

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    - Trees, forests etc (A.S. 2021, Chen–A.S. 2023)

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⋮

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Dedicated to the memory of Philippe Flajolet (1948–2011)