Double orthodontia formulas and Lascoux positivity

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Outline

- Schubert polynomials and flagged Weyl modules
- Orthodontia formula for flagged Weyl modules
  - and key positivity of their dual characters
- Orthodontia formula for double Grothendieck polynomials
  - and a curious Lascoux positivity result

Goal: Analogue of flagged Weyl module for Grothendieck polynomials.

Schubert polynomials

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Definition

The *i-th divided difference operator* is

$$\partial_i(f) := \frac{f - s_i \cdot f}{x_i - x_{i+1}},$$

for $i \in [n-1]$. $(s_i \cdot f := f(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n))$

Definition

For $w \in S_n$, recursively define *Schubert polynomials*:

$$\mathfrak{S}_w(\mathbf{x}) = \begin{cases} x_1^{n-1} x_2^{n-2} \ldots x_{n-1} & \text{if } w = w_0 \\ \partial_i(\mathfrak{S}_{ws_i}(\mathbf{x})) & \text{if } \ell(w) < \ell(ws_i). \end{cases}$$

Schur polynomials

Example

Schur polynomials $s_\lambda := \mathrm{ch}(V_\lambda)$ are $\mathfrak{S}_w$ for "Grassmannian w".

(The GL_n-irreps V_λ are "representation-theoretic avatars" of Grassmannian $\mathfrak{S}_w$.)

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$$[X_u] \cdot [X_v] = \sum_w c^w_{uv} [X_w] \quad \leftrightsquigarrow \quad V_\lambda \otimes V_\mu = \bigoplus_\nu V_\nu^{\oplus c^\nu_{\lambda\mu}}$$

intersection nos. $\leftrightsquigarrow$ multiplicities of irreps

c^w_{uv}: "Littlewood–Richardson coefficients"

Central problem: Combinatorial formula for c^w_{uv}?

Rothe diagrams

(Towards representation-theoretic avatars of general $\mathfrak{S}_w$)

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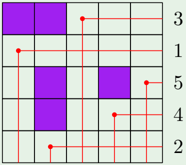
Definition

- Draw $n \times n$ grid with dots in i-th row and $w(i)$-th column
- Draw “death rays” emanating east and south of each dot
- Remaining squares are $D(w)$.

Running Example

D(31542): 

Flagged Weyl modules

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Theorem (Kraśkiewicz–Pragacz ’87)

The dual character $\mathrm{ch}^*(\mathcal{M}_{D(w)})$ *is the Schubert polynomial* $\mathfrak{S}_w$.

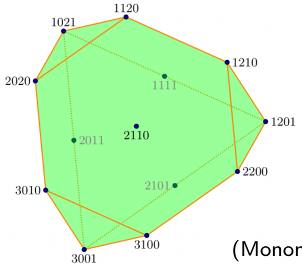
(*Dual character* of V is $\mathrm{ch}^*(V)(x_1, \ldots, x_n) = \mathrm{tr}(\mathrm{diag}(x_1^{-1}, \ldots, x_n^{-1}) \colon V \to V)$.)

Monomials in $\mathfrak{S}_w$

(What does $\mathcal{M}_D$ buy us?)

Question

Assume that $\mathbf{x}^{\alpha-\beta}$ *and* $\mathbf{x}^{\alpha+\beta}$ *appear in* $\mathfrak{S}_w$. *Does* $\mathbf{x}^{\alpha}$ *appear?*



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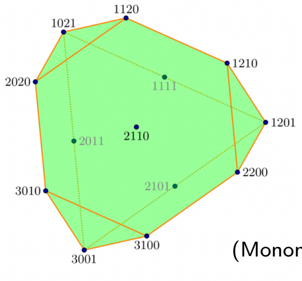

(Monomials in $\mathfrak{S}_{21543}$)

Conjecture (Monical–Tokcan–Yong '19)

$\mathcal{N}(w) := \{\mathrm{wt}(C) : \mathbf{x}^{\mathrm{wt}(C)}$ appears in $\mathfrak{S}_w\}$ *is saturated.*

(*Saturated*: $S = \mathrm{conv}(S) \cap \mathbb{Z}^n$.)

Monomials in $\mathfrak{S}_w$, II

Theorem (Fink–Mészáros–St. Dizier '18)

$\mathcal{N}(D) := \{\mathrm{wt}(C) \colon \mathbf{x}^{\mathrm{wt}(C)} \text{ appears in } \mathrm{ch}^*(\mathcal{M}_D)\}$ *is saturated.*

Idea: use rep theory description of monomials in $\mathrm{ch}^*(\mathcal{M}_D)$.

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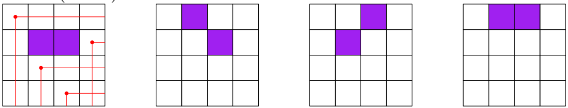

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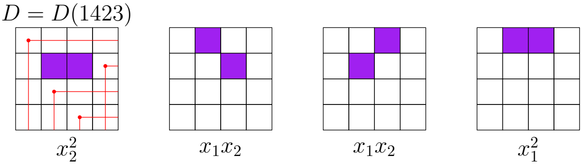

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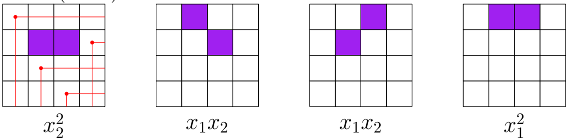


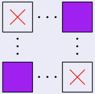
Also in Fink–Mészáros–St. Dizier: conv($\mathcal{N}(D)$) is a *generalized permutahedron*.

%-avoiding diagrams

(Towards the *orthodontia formula* computing $\mathrm{ch}^*(\mathcal{M}_D)$)

Definition (Reiner–Shimozono '98)

D is *%-avoiding* if it does not have any instance of:



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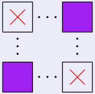
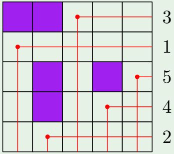


Proposition

The Rothe diagram $D(w)$ is %-avoiding for all $w \in S_n$.

Running Example



Orthodontic sequence

$$D_j := j\text{-th column of a diagram } D$$

Proposition (Reiner–Shimozono '98)

If D is %-avoiding, it can be reduced to the empty diagram via:

- *Remove columns: $D \mapsto D \setminus D_j$ when $D_j = [i]$*
- *Swap rows i and $i+1$: $D \mapsto s_i D$ when $i \in D_k \implies i+1 \in D_k$ for all k.*

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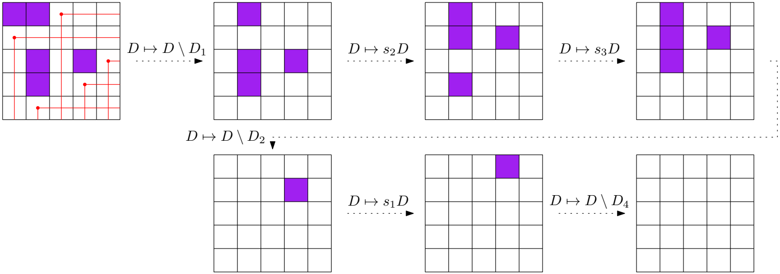

Orthodontia for flagged Weyl modules

$$\pi_i(f) := \partial_i(x_i f).$$

Theorem (Magyar '98, "orthodontia formula")

Let D be a %-avoiding diagram. Then:

- $\mathrm{ch}^*(\mathcal{M}_D) = x_1 \dots x_i \cdot \mathrm{ch}^*(\mathcal{M}_{D \setminus D_j})$ *if* $D_j = [i]$.
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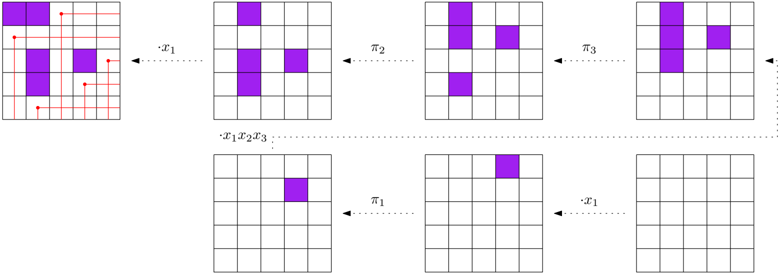

Proof involves: $\mathcal{M}_D \cong$ {sections of a line bundle on a variety}.

Uses comb. of chamber sets (Leclerc–Zelevinsky), geom. of Frobenius splitting (Van der Kallen).

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Corollary (Magyar '98)

For any %-avoiding diagram D, the dual character $\mathrm{ch}^*(\mathcal{M}_D)$ can be obtained from $1 \in \mathbb{C}[\mathbf{x}]$ by applying various $\cdot x_1 \dots x_i$ and π_i.

Key polynomials

Key polynomials κ_α were first defined as characters of *Demazure modules*.

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Definition

For $\alpha \in \mathbb{Z}^n_{\geq 0}$, recursively define *key polynomials*:

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Lemma (Reiner–Shimozono '98)

For any k and α, the polynomial $x_1 \dots x_k \cdot \kappa_\alpha$ is a $\mathbb{Z}_{\geq 0}$-linear combination of key polynomials.

What does orthodontia buy us?

Proposition

For %-avoiding D, the dual character $\mathrm{ch}^*(\mathcal{M}_D)$ *is a* $\mathbb{Z}_{\geq 0}$*-linear combination of key polynomials.*

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Since $\pi_i(\kappa_\alpha) = \kappa_{\alpha'}$ for some α', the operator π_i preserves key positivity.

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Since $x_1 \ldots x_i \cdot \kappa_\alpha$ is key positive, the operator $\cdot x_1 \ldots x_i$ preserves key positivity. □

Double Grothendieck polynomials

Double Grothendieck polynomials $\mathfrak{G}_w(\mathbf{x};\mathbf{y})$ are lifts of structure sheaves of Schubert varieties $[\mathcal{O}_{X_w}] \in K_T^*(\mathcal{F}\ell_n)$.

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Definition

For $w \in S_n$, recursively define *double Grothendieck polynomials*:

$$\mathfrak{G}_w(\mathbf{x};\mathbf{y}) = \begin{cases} \prod_{i+j\leq n}(x_i + y_j - x_i y_j) & \text{if } w = w_0 \\ \overline{\partial}_i(\mathfrak{G}_{ws_i}(\mathbf{x};\mathbf{y})) & \text{if } \ell(w) < \ell(ws_i), \end{cases}$$

where $\overline{\partial}_i(f) := \partial_i((1 - x_{i+1})f)$.

Lowest degree part of $\mathfrak{G}_w(\mathbf{x};\mathbf{0})$ is $\mathfrak{S}_w$.

Flagged Weyl modules in K-theory

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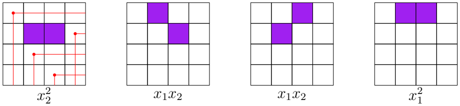

Pechenik–Speyer–Weigandt '24:

- $\deg(\mathfrak{G}_w) = \mathrm{raj}(w)$
- $\mathfrak{G}_w^{\mathrm{top}}(\mathbf{x}; \mathbf{y}) = f(\mathbf{x})g(\mathbf{y})$

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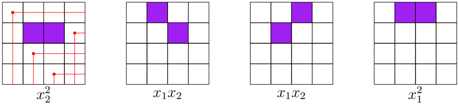
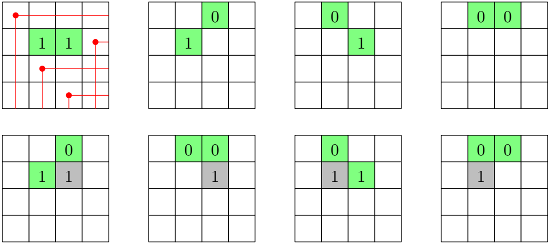


Hafner–Mészáros–S.–St. Dizier '24: {monomials in vexillary $\mathfrak{G}_w(\mathbf{x};\mathbf{0})$}.



(What is the rep-theoretic meaning of this?)

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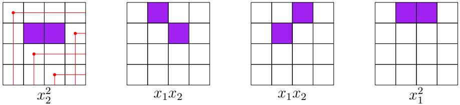

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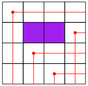
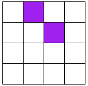


x_2^2



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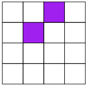

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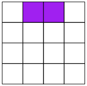

x_1^2

- Want to "access" $\mathfrak{G}_D$ for %-avoiding D:
  - To use for induction purposes
  - To collect certain $\mathfrak{G}_D$ together into generating functions

$$\sum_{\mathbf{m}} \mathfrak{G}_{D(\mathbf{m})} \cdot \mathbf{t}^{\mathbf{m}}$$

cf. generating function $\sum_\lambda s_\lambda(\mathbf{x})\mathbf{t}^\lambda = \mathrm{ch}(\mathbb{C}[G/U])$

Orthodontia for double Grothendieck polynomials

Schubert story:

Theorem (Magyar '98, "orthodontia formula")

Let D be a %-avoiding diagram. Then:

- $\mathrm{ch}^*(\mathcal{M}_D) = x_1 \dots x_i \cdot \mathrm{ch}^*(\mathcal{M}_{D \setminus D_j})$ *if* $D_j = [i]$.
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Goal

For %-avoiding D*, define* $\mathscr{G}_D \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$ *so that* $\mathscr{G}_{D(w)} = \mathfrak{G}_w(\mathbf{x}; \mathbf{y})$.

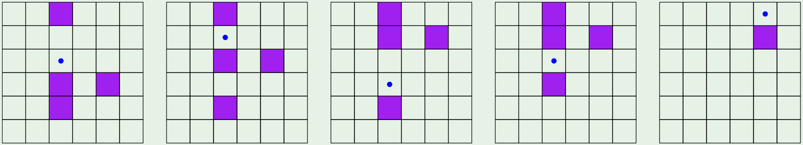
Easier goal: Define $\mathcal{G}_D \in \mathbb{C}[\mathbf{x}]$ so that $\mathcal{G}_{D(w)} = \mathfrak{G}_w(\mathbf{x}; \mathbf{0})$.

Orthodontia algorithm

Definition

Let C be the leftmost nonempty, non-up-aligned column of D. The *first missing tooth* is the minimal i so that $i \notin C$ and $i + 1 \in C$.

Example



$i := 3$ $i := 2$ $i := 4$ $i := 3$ $i := 1$

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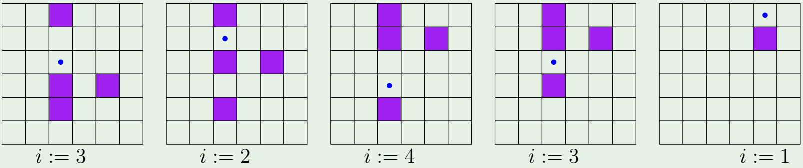
Example



Algorithm (Magyar '98, "orthodontia algorithm")

1. *Remove any columns* $D_j = [i]$
2. *Swap rows* i *and* $i+1$, *for* $i :=$ *first missing tooth*
3. *Repeat steps 1 & 2 until empty*

Orthodontia for ordinary Grothendieck polynomials

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$$\overline{\pi}_i := \pi_i((1 - x_{i+1})f)$$

Definition (Mészáros–S.–St. Dizier '22)

For %-avoiding D, define $\mathcal{G}_D \in \mathbb{C}[\mathbf{x}]$ recursively:

- $\mathcal{G}_D = x_1 \dots x_i \cdot \mathcal{G}_{D \setminus D_j}$ if some $D_j = [i]$,
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Theorem (Mészáros–S.–St. Dizier '22)

When $D = D(w)$ *is a Rothe diagram,* $\mathcal{G}_D = \mathfrak{G}_w(\mathbf{x}; \mathbf{0})$.

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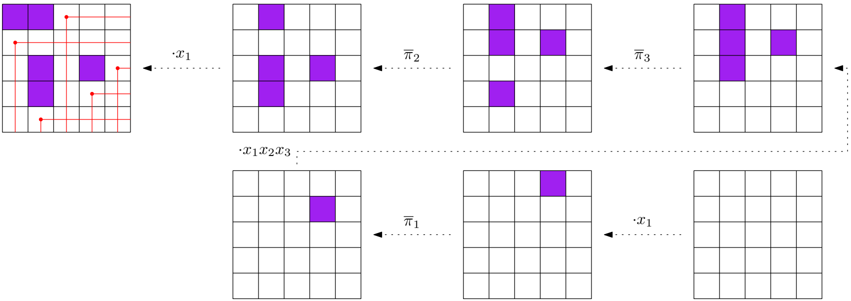
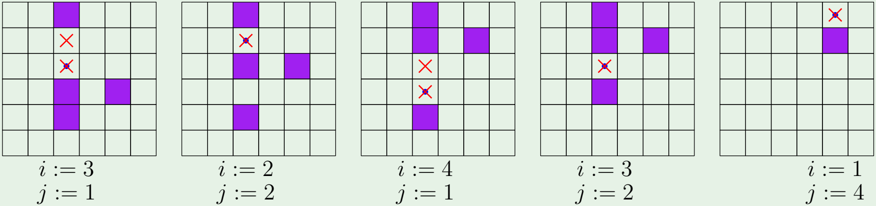


Orthodontia algorithm, II

Definition

Let D_k be the leftmost nonempty column of D. Let i be the first missing tooth and $j := k - \#\{a \leq i \colon a \notin D_k\}$. The *first missing double-tooth* is (i, j).

Example



Double orthodontic polynomials

Goal

For %-avoiding D, define $\mathscr{G}_D \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$ *so that* $\mathscr{G}_{D(w)} = \mathfrak{G}_w(\mathbf{x}; \mathbf{y})$.

$$\overline{\omega}_i^{\{j\}} := \prod_{k=1}^{i} (x_k + y_j - x_k y_j)$$

$$\overline{\pi}_{i,j} := \overline{\partial}_i((x_i + y_j - x_i y_j) f)$$

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Orthodontia for double Grothendieck polynomials

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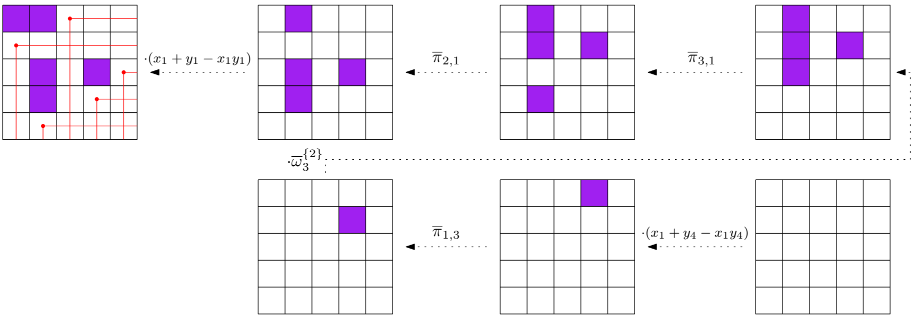

$$\left(\overline{\omega}_3^{\{2\}} := (x_1 + y_2 - x_1 y_2)(x_2 + y_2 - x_2 y_2)(x_3 + y_2 - x_3 y_2)\right)$$

Orthodontia for double Grothendieck polynomials, II

Theorem (S.–St. Dizier)

When $D = D(w)$ is a Rothe diagram, $\mathscr{G}_D = \mathfrak{G}_w(\mathbf{x}; \mathbf{y})$.

$\mathrm{ch}^*(\mathcal{M}_D)$ is invariant under reordering columns, but $\mathscr{G}_D$ is not.

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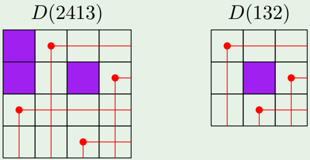
Example

$$\mathfrak{S}_{2413}(\mathbf{x}) = x_1 x_2 \mathfrak{S}_{132}(\mathbf{x})$$
$$\mathfrak{G}_{2413}(\mathbf{x};\mathbf{0}) = x_1 x_2 \mathfrak{G}_{132}(\mathbf{x};\mathbf{0})$$
$$\mathfrak{G}_{2413}(\mathbf{x};\mathbf{y}) \neq g(\mathbf{x},\mathbf{y}) \cdot \mathfrak{G}_{132}(\mathbf{x};\mathbf{y}) \quad \text{for any } g$$

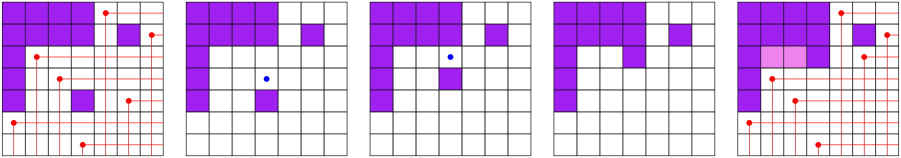


Orthodontia for double Grothendieck polynomials, III

Theorem (S.–St. Dizier)

When $D = D(w)$ is a Rothe diagram, $\mathscr{G}_D = \mathfrak{G}_w(\mathbf{x}; \mathbf{y})$.

Proof idea: "Find almost-Rothe-diagrams in reduction sequence for $D(w)$"



Orthodontia for double Grothendieck polynomials, III

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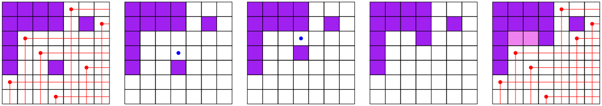
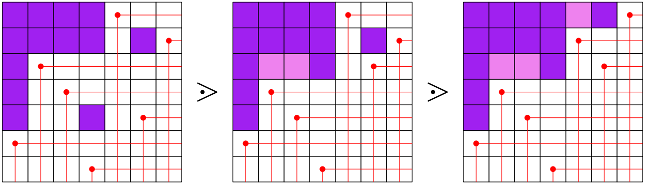


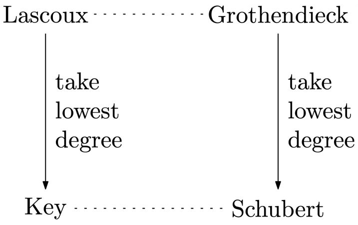
“orthodontic sort”:



(what’s the geometric meaning of this?)

Lascoux polynomials

Lascoux polynomials are "K-theoretic analogues" of key polynomials:



Definition

For $\alpha \in \mathbb{Z}^n_{\geq 0}$, recursively define *Lascoux polynomials*:

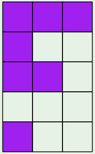
$$\mathfrak{L}_\alpha(\mathbf{x}) = \begin{cases} x_1^{\alpha_1} \dots x_n^{\alpha_n} & \text{if } \alpha_1 \geq \dots \geq \alpha_n \\ \overline{\pi}_i(\mathfrak{L}_{s_i\alpha}(\mathbf{x})) & \text{if } \alpha_i < \alpha_{i+1}, \end{cases}$$

where $\overline{\pi}_i(f) := \pi_i((1 - x_{i+1})f)$.

Double Lascoux polynomials...?

$\alpha \rightsquigarrow D(\alpha)$ *"skyline diagram"*

Example

$\alpha = (3, 1, 2, 0, 1) \quad \rightsquigarrow \quad D(\alpha) =$ 

Observation (Mészáros–S.–St. Dizier, '22)

When $D = D(\alpha)$ is a skyline diagram, $\mathcal{G}_D = \mathfrak{L}_\alpha(\mathbf{x})$.

Who is $\mathscr{G}_{D(\alpha)}(\mathbf{x}; \mathbf{y})$? And what about reordered-column $D(\alpha)$'s?

A curious Lascoux positivity conjecture

$$\mathscr{G}_D^{\mathrm{bot}} := \text{lowest degree part of } \mathscr{G}_D.$$

($\mathscr{G}_{D(w)}^{\mathrm{bot}}(\mathbf{x}; -\mathbf{y})$ is the *double Schubert polynomial*.)

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Conjecture (S.–St. Dizier)

If D is %-avoiding, $x_1^n \dots x_n^n \mathscr{G}_D^{\mathrm{bot}}(x_n^{-1}, \dots, x_1^{-1}; -1, \dots, -1)$ is a graded nonnegative sum of Lascoux polynomials.

Example

The polynomial $x_1^4 x_2^4 x_3^4 x_4^4 \mathscr{G}_{D(2143)}^{\mathrm{bot}}(x_4^{-1}, x_3^{-1}, x_2^{-2}, x_1^{-1}; -1, -1, -1, -1)$ is

$$x_1^4 x_2^3 x_3^4 x_4^3 + x_1^4 x_2^4 x_3^4 x_4^2 + x_1^4 x_2^4 x_3^3 x_4^3 - x_1^4 x_2^3 x_3^4 x_4^4 - x_1^4 x_2^4 x_3^3 x_4^4 - 4 x_1^4 x_2^4 x_3^4 x_4^3 + 3 x_1^4 x_2^4 x_3^4 x_4^4$$

which is

$$(\mathfrak{L}_{(4,3,4,3)} + \mathfrak{L}_{(4,4,4,2)}) - (\mathfrak{L}_{(4,3,4,4)} + 2\mathfrak{L}_{(4,4,4,3)}) + \mathfrak{L}_{(4,4,4,4)}$$

A curious Lascoux positivity conjecture, II

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Proof??

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- $f \mapsto \overline{\pi}_i(f)$,
- $f \mapsto x_1 \dots x_i (1 - x_{i+1}) \dots (1 - x_n) f$.

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Conjecture: The product $\mathfrak{L}_\alpha \cdot x_1 \dots x_i(1 - x_{i+1}) \dots (1 - x_n)$ is graded Lascoux positive. (cf. key positivity of $\kappa_\alpha \cdot x_1 \dots x_i$.) □

A curious Lascoux positivity result

Corollary (S.–St. Dizier)

When the columns of D can be ordered by inclusion, the polynomial $x_1^n \dots x_n^n \mathscr{G}_D^{\mathrm{bot}}(x_n^{-1}, \dots, x_1^{-1}; -1, \dots, -1)$ is a graded nonnegative sum of Lascoux polynomials.

($D(w)$ ordered by inclusion $\iff$ w vexillary.)

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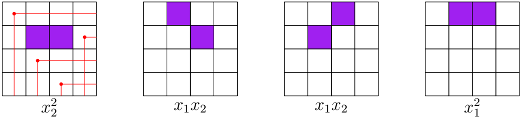
Follows from Orelowitz–Yu '23: $G_w \cdot \mathfrak{L}_\alpha$ is graded Lascoux positive.

($G_w :=$ stable Grothendieck) □

Thank you!

Goal

Find analogue of $\mathcal{M}_D$ for Grothendieck polynomials.



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