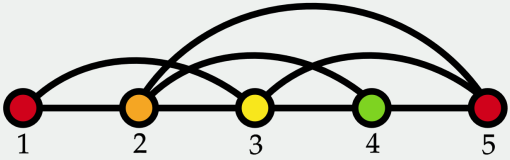
Higher Specht Bases, Tableaux Bijections, and Hessenberg Varieties

Kyle Salois, Colorado State University



Overview

- **Motivation:**
  - Stanley-Stembridge conjecture in symmetric function theory.
  - Shareshian-Wachs generalization for quasisymmetric functions.
- **Tools:**
  - Chromatic (quasi-)symmetric functions.
  - Cohomology rings of Hessenberg varieties.
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- **Tools:**
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  - Cohomology rings of Hessenberg varieties.
  - Tableaux and $\mathfrak{S}_n$ representation theory.
- **Results:**
  - New approaches and proof methods for a specific case.
  - Bijections between certain tableaux and basis elements of the cohomology ring.
  - Drawing new connections between the combinatorics and geometry of the problem.

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Symmetric Function Bases

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- Elementary basis:
  - $e_k(\mathbf{x}) = \sum_{i_1 < i_2 < \ldots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$.
  - For a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of n, we have $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_\ell}$.

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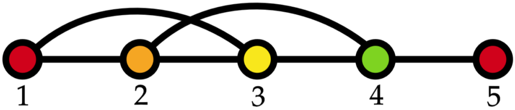
- Schur basis:
  - $$s_\lambda(\mathbf{x}) = \sum_{T \in SSYT(\lambda)} x^T = \sum_{T \in SSYT(\lambda)} x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k}$$

    where $SSYT(\lambda)$ is the set of semistandard Young tableaux of shape λ, and a_i is the number of times i was used in T.

4	5			
2	3	4		
1	2	2	2	3

$$\Leftrightarrow \quad x_1 x_2^4 x_3^2 x_4^2 x_5$$

- Let $G = (V, E)$ be a finite simple graph with $V = [n] = \{1, \dots, n\}$. A **proper coloring** of G is a function $\kappa : V \to \mathbb{N}$ such that if $vw \in E$, then $\kappa(v) \neq \kappa(w)$.
- An **ascent** in a coloring of G is a pair of vertices $v < w$ with $\kappa(v) < \kappa(w)$.



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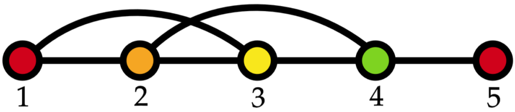


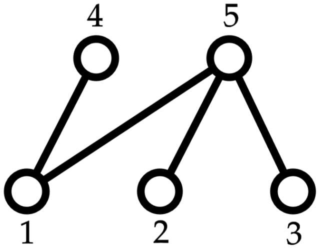
- The **chromatic quasisymmetric function** of G is:

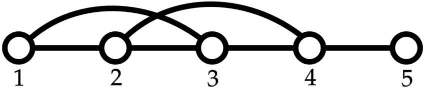
$$X_G(\mathbf{x}; q) = \sum_{\kappa : V \to \mathbb{N}} q^{\mathrm{asc}(\kappa)} x_{\kappa(1)} x_{\kappa(2)} \cdots x_{\kappa(n)}$$

- Let P be a poset on $[n]$, and $\mathrm{inc}(P)$ be its incomparability graph.

Conjecture ([SS93, SW16])

If P is a $(3+1)$-free poset, then $X_{\mathrm{inc}(P)}(\mathbf{x};q)$ is e-positive.





A Formula Using P-Tableaux

Proposition ([Gas96, SW16])

$$X_{\mathrm{inc}(P)}(\mathbf{x}; q) = \sum_{\lambda \vdash n} \left(\sum_{T \in \mathrm{PT}(\lambda)} q^{\mathrm{inv}_P(T)} \right) s_\lambda$$

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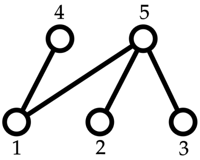
- A ***P*-tableau of shape** λ satisfies:
  - Each element of P is used at most once.
  - Rows are P-increasing.
  - Adjacent entries in columns are P-nondecreasing.



2	
1	4
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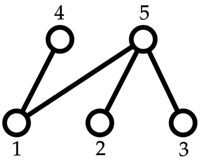

2	
1	4
3	5

- A ***P*-inversion** is a pair of entries (i, j) with $i < j$ which are incomparable in P, and i is in a higher row than j in T.

Representations of $\mathfrak{S}_n$

- Irreducible representations of the symmetric group $\mathfrak{S}_n$ are indexed by partitions λ of n.

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- Irreducible representations of the symmetric group $\mathfrak{S}_n$ are indexed by partitions λ of n.
- Irreducible $\mathfrak{S}_n$-modules are isomorphic to the Specht modules:

$$V_\lambda = \langle F_T \mid T \in SYT(\lambda) \rangle \subseteq \mathbb{Q}[x_1, \ldots, x_n]$$

5		
2	6	
1	3	4

$$\Leftrightarrow \quad F_T = (x_5 - x_2)(x_5 - x_1)(x_2 - x_1)(x_6 - x_3)$$

Frobenius Map [Sag10]

- Given an $\mathfrak{S}_n$-module V, with character $\chi : \mathfrak{S}_n \to \mathbb{Z}$, define the Frobenius character map by

$$\mathrm{Frob}(V) = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} \chi(\pi) p_{c(\pi)}$$

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- **Fact:** If V decomposes as $V = \bigoplus_{i=1}^{m} V_{\lambda^{(i)}}$, then

$$\mathrm{Frob}(V) = \sum_{i=1}^{m} s_{\lambda^{(i)}}$$

Hessenberg Varieties [MPS92]

- $[n] = \{1, 2, \ldots, n\}$.
- A **Hessenberg function** $h : [n] \to [n]$ is a function such that:
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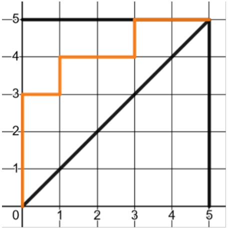
- Example: $h = (3, 4, 4, 5, 5)$



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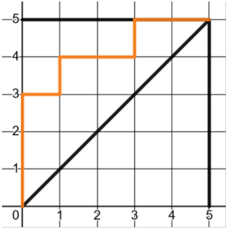

- Given a Hessenberg function h on $[n]$ and a matrix $X : \mathbb{C}^n \to \mathbb{C}^n$, the **Hessenberg variety** is the subvariety of $\mathrm{Fl}(\mathbb{C}^n)$ defined by

$$\mathrm{Hess}(X, h) = \left\{ F_\bullet := F_1 \subset F_2 \subset \cdots \subset F_n \mid X(F_i) \subset F_{h(i)} \right\}$$

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- The cohomology ring $H^*(\mathrm{Hess}(X, h))$ vanishes in the odd degree:

$$\mathrm{Poin}(\mathrm{Hess}(X, h), q) = \sum_{i=0}^{d} \dim(H^{2i}(\mathrm{Hess}(X, h); \mathbb{Q}))q^i$$

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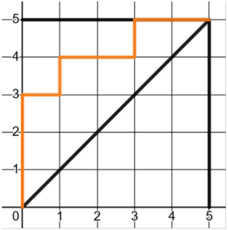
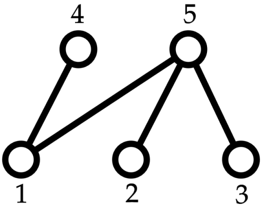
- $H^*(\mathrm{Hess}(S,h))$ is an $\mathfrak{S}_n$-module, under the dot action defined by Tymoczko.

- If h is a Hessenberg function, define the poset P_h by $i <_{P_h} j$ if and only if $h(i) < j$.





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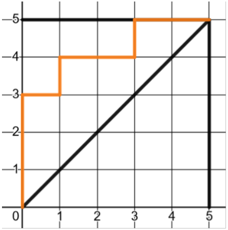

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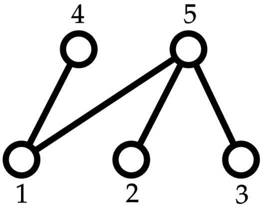

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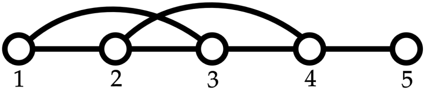

- Define $G_h = \mathrm{inc}(P_h)$.

Proposition ([BC18, GP16])

Let h be a Hessenberg function, $G_h = \mathrm{inc}(P_h)$, and S be a regular semisimple matrix. Then

$$\omega X_{G_h}(\mathbf{x}; q) = \sum_{i=1}^{|E|} \mathrm{Frob}(H^{2i}(\mathrm{Hess}(S, h)))q^i$$

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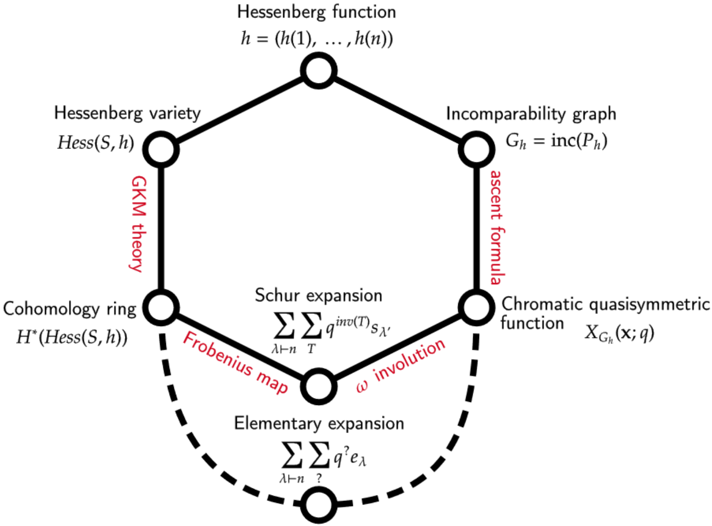
- Together with the P_h-tableaux expansion, we get:

$$\omega X_{G_h}(\mathbf{x}; q) = \sum_{\lambda \vdash n} \left(\sum_{T \in PT(\lambda)} q^{\mathrm{inv}_h(T)} \right) s_{\lambda'}$$

Hessenberg Connections



Proposition ([AHM17])

If $h = (h(1), n, \ldots, n)$, and S is regular semisimple, then the following types of monomials generate $H^*(\mathrm{Hess}(S, h)) \cong \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]/I$:

$$x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \quad \text{with no factor } \prod_{\ell=1}^{h(1)} x_\ell$$

$$x_n^{\ell_1} x_{n-1}^{\ell_2} \cdots x_2^{\ell_{n-1}} y_k \quad \text{with no factor } \prod_{\ell=h(1)+1}^{n} x_\ell$$

where $0 \le i_j \le n - j$ and $0 \le \ell_j \le n - j - 1$ and $1 \le k \le n - 1$.

- $\mathfrak{S}_n$ acts by fixing the x_i and permuting the y_i.

- A **higher Specht basis** of a polynomial ring is a basis acted on by $\mathfrak{S}_n$ in the same way as the Specht modules V_λ.
  - Higher Specht bases have been found for coinvariant rings R_n [ATY97] and generalized coinvariant rings $R_{n,k}$ [GR21].
- We want to find a higher Specht basis for $H^*(\mathrm{Hess}(S, h))$ so that the basis respects the decomposition into irreducible $\mathfrak{S}_n$-modules.

Proposition (S.)

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$$x_n^{\ell_1} x_{n-1}^{\ell_2} \cdots x_2^{\ell_{n-1}} (y_k - y_1) \quad \textit{with no factor} \prod_{\ell=h(1)+1}^{n} x_\ell$$

where $0 \leq i_j \leq n - j$ and $0 \leq \ell_j \leq n - j - 1$ and $2 \leq k \leq n$.

- $\mathfrak{S}_n$ acts by fixing the x_i and permuting the y_i.

Cohomology of Regular Semisimple Hessenberg Varieties

Corollary (S.)

We have a new proof of the following fact:
When $h = (h(1), n, \ldots, n)$*, the action of* $\mathfrak{S}_n$ *on* $H^*(\mathrm{Hess}(S,h))$ *decomposes into* $h(1)(n-1)!$ *copies of the trivial representation* $V_{(n)}$ *and* $(n - h(1))(n-2)!$ *copies of the standard representation* $V_{(n-1,1)}$*.*

1	2	3	4	5

- $V_{(n)}$, the trivial representation.

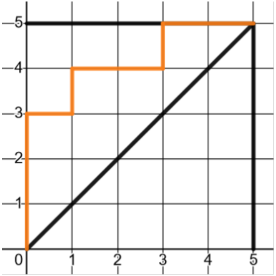
3			
1	2	4	5

- $V_{(n-1,1)}$, the standard representation.

Theorem ([HHM$^+$21])

If h is any Hessenberg function, and N is regular nilpotent, then the following type of monomials generate $H^*(\mathrm{Hess}(N, h))$:

$$x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \qquad 0 \leq i_j \leq h(j) - j.$$



- For $h = (3, 4, 4, 5, 5)$, the highest degree element is $x_1^2\, x_2^2\, x_3^1\, x_4^1$.

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Theorem ([AHHM16])

There exists an isomorphism of graded $\mathbb{Q}$-algebras

$$\mathcal{A} : H^*(\mathrm{Hess}(N, h)) \to H^*(\mathrm{Hess}(S, h))^{\mathfrak{S}_n}$$

Regular Nilpotent Hessenberg Varieties

- Let $\mathcal{N}_h$ be the set of monomials:

$$x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \qquad 0 \le i_j \le h(j) - j.$$

- Let $PT(h, \lambda)$ be the set of P_h-tableaux of shape λ.

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There exists a weight preserving bijection between the set $\mathcal{N}_h$ of monomials and the set $PT(h, \lambda)$ for $\lambda = (1^n)$.

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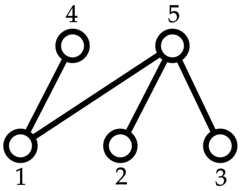
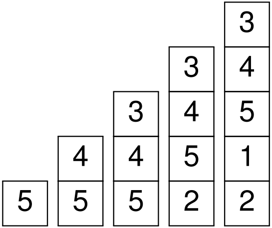
Example: Let $h = (3, 4, 4, 5, 5)$. Consider $x_1\, x_3\, x_4 \in \mathcal{N}_h$.





- Begin with 5. Insert 4 with one P_h-inversion.
  Insert 3 with one P_h-inversion. Insert 2 with no P_h-inversions.
  Insert 1 with one P_h-inversions.

- Let B_1 be the set of monomials:

$$x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \quad \text{with no factor } \prod_{\ell=1}^{h(1)} x_\ell \text{ such that } 0 \leq i_j \leq n - j$$

- Note that $\mathfrak{S}_n$ acts trivially on B_1.
- Let P_N be the poset for $h = (n, \ldots, n)$ - all elements incomparable.

Bijections for Fixed Monomials

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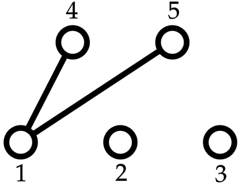

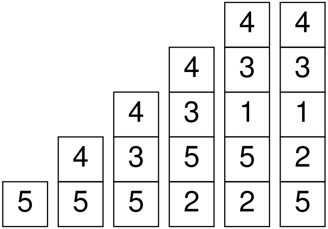

- Begin with 5. Insert 4 with one P_N-inversion. Insert 3 with one P_N-inversion. Insert 2 with no inversions. Insert 1 with two P_N-inversions. Then shift entry 2 to below the 1.

- Let B_3 be the set of monomials:

$$x_n^{\ell_1} x_{n-1}^{\ell_2} \cdots x_2^{\ell_{n-1}} (y_k - y_1) \text{ with no } \prod_{i=h(1)+1}^{n} x_i \text{ and } 0 \leq \ell_j \leq n - j - 1$$

- Let $PSPT(h, \lambda)$ be the set of pairs (S, T) where S is a standard tableau and T is a P_h-tableau, both of shape λ.

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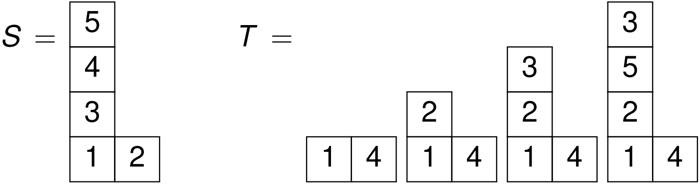
Example: Let $h = (3, 5, 5, 5, 5)$. Consider $M = x_5^2\, x_3\, (y_2 - y_1) \in B_3$.



- M has $(y_2 - y_1)$, so construct S with 1, 2 in the bottom row.
- Largest x-index not present in M is 4.
- Insert 2, then 3, then 5 with inversions determined by powers in M.

- Recall that we have the following expression:

$$\sum_{j=0}^{|E|} \mathrm{Frob}(H^{2j}(\mathrm{Hess}(S,h)))q^j = \omega X_G(\mathbf{x};q) = \sum_{\lambda \vdash n} \left(\sum_{T \in PT(h,\lambda)} q^{\mathrm{inv}_h(T)} \right) s_{\lambda'}$$

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- So, the Poincaré polynomials of $\mathrm{Hess}(S,h)$ is:

$$\mathrm{Poin}(\mathrm{Hess}(S,h),q) = \sum_{\lambda \vdash n} \left(\sum_{T \in \mathrm{PT}(h,\lambda)} q^{\mathrm{inv}_h(T)} \right) \#\mathrm{SYT}(\lambda')$$

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Proposition ([AHM17])

If $h = (h(1), n, \ldots, n)$, then the Poincaré polynomial of $\mathrm{Hess}(S,h)$ is given by:

$$\frac{1-q^{h(1)}}{1-q} \prod_{j=1}^{n-1} \frac{1-q^j}{1-q} + (n-1)q^{h(1)-1} \frac{1-q^{n-h(1)}}{1-q} \prod_{j=1}^{n-2} \frac{1-q^j}{1-q}$$

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- Alternative proof using P_h-tableaux:
- Each term in q is the generating function for number of P_h-inversions in P_h-tableaux of shape $\lambda = (1^n)$ and $\mu = (2, 1^{n-2})$.

- **Poincaré Polynomials**:
  - We can find more Poincaré polynomials for Hessenberg varieties by counting the number of inversions in P_h-tableaux for different functions h and partitions λ.
- **Cohomology Rings:**
  - We can extrapolate these bijections to come up with conjectured bases for the cohomology ring in other cases, together with GKM theory.

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