Left Regular Bands of Groups
&
Mantaci–Reutenauer Algebras

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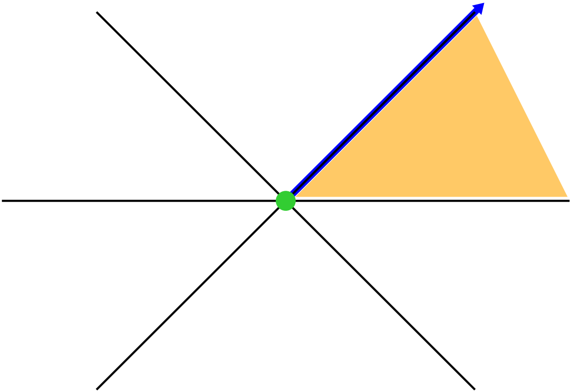
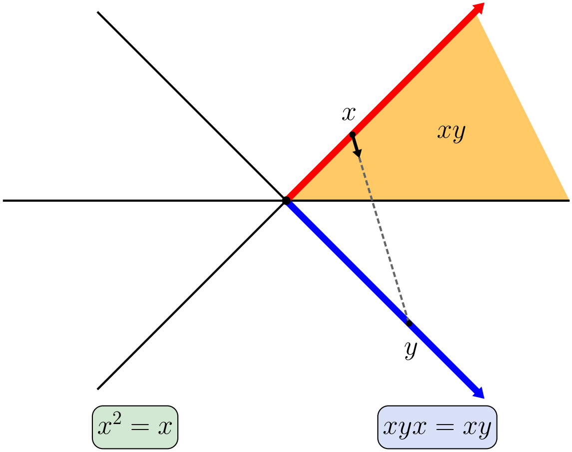
hyperplanes partition $\mathbb{R}^n$ into faces



chambers cut out by the hyperplanes
rays emanating from the origin
the origin

$$xy = \begin{cases} \text{the face first encountered after a small} \\ \text{movement along a line from } x \text{ toward } y \end{cases}$$



Left Regular Bands (LRBs)

A semigroup $\mathcal{B}$ is a **left regular band (LRB)** if, for all $x, y \in \mathcal{B}$,

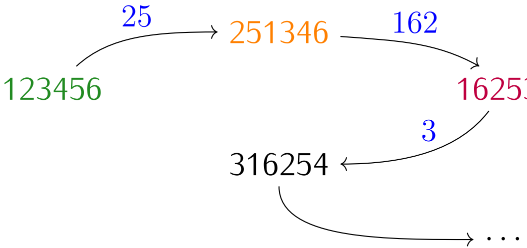
$$xx = x \qquad \text{and} \qquad xyx = xy.$$

- Every element of a LRB is **idempotent**: $x^2 = x$.
- Informally: identities say ignore "repetitions".
- Original motivation: unified framework for certain random walks (riffle shuffle, random-to-top shuffle, $\ldots$.)

Free LRB: repetition-free words;
concatenate and remove repeats

$$\mathbf{1} \cdot 427\mathbf{1}6 = \mathbf{1}427\cancel{\mathbf{1}}6 = \mathbf{1}4276$$

Step in the random walk: starting at an element $\boldsymbol{c}$, pick an element $\boldsymbol{y}$ at random, and move to $\boldsymbol{y} \cdot \boldsymbol{c}$.



Random Walks on LRBs

Introduced by Bidigare–Hanlon–Rockmore (1999):

- on monoid of faces of a hyperplane arrangement
- unified several known walks; eigenvalue formulas!

Further developed by Brown–Diaconis (1998):

- stationary distribution and diagonalizability

Brown (2000): extension to LRBs; and then *bands*

Steinberg (2006): extension to largest class of semigroups for which these results hold (pseudovariety **DA**)

LRBs are everywhere: Aguiar, Athanasiadis, Ayyer, Bastidas, Bidigare-Hanlon-Rockmore, Billera, Björner, Brauner, Brown, Chung, Commins, Denton, Diaconis, Fulman, Graham, Hivert, Hsiao, Lagr Lawvere, Mahajan, Margolis, Novelli, Petersen, Pike, Reiner, Schilling, Schützenberger, Steinberg, Thibon, Thiery, ...

From random walks to algebra

Algebraic approach: encode the random walk as an operator acting on the semigroup algebra $\mathbb{C}\mathcal{B}$

random walk with probabilities $\boldsymbol{\rho}_x$	$\longleftrightarrow$	left mult. by $\sum_{x \in \mathcal{B}} \boldsymbol{\rho}_x \, x$

Left Regular Bands of Groups (LRBGs)

$\mathcal{S}$ is a **LRB of groups** if there is an $N \in \mathbb{N}$ s.t., for all $x, y \in \mathcal{S}$,

$$xx^N = x \qquad \text{and} \qquad xyx^N = xy$$

- x^N is an **idempotent**: $x^N x^N = x^N$
- $E(\mathcal{S}) = \{e \in \mathcal{S} : e^2 = e\}$ is a LRB
- if $e^2 = e$, then $G_e = \{x \in \mathcal{S} : x^N = e\}$ is a group and

$$\mathcal{S} = \bigsqcup_{e \in E(\mathcal{S})} G_e$$

- if $\mathcal{S}e\mathcal{S} = \mathcal{S}f\mathcal{S}$, then $G_e \cong G_f$
- $\mathcal{S}$ is **strict** if $G_e G_f \subseteq G_{ef}$ (equivalently, $(xy)^N = x^N y^N$)
  (*presheaf of groups*)

Block-labelled ordered set partitions

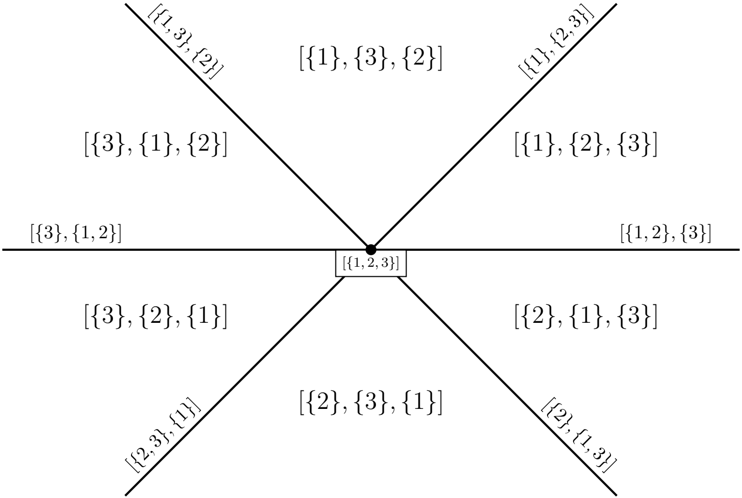
composition of $[n]$: ordered set partition of $[n]$

$$[\{2,5\},\{1,3,4,6\}] \qquad [\{1,3,4,6\},\{2,5\}] \qquad [\{3,6\},\{1,4,5\},\{2\}]$$

G-composition of $[n]$: label blocks by elements of G

$$[(\{2,5\},-1),(\{1,3,4,6\},1)] \qquad [(\{3,6\},-1),(\{1,4,5\},1),(\{2\},-1)]$$

$\Sigma_n[G]$: set of G-compositions of $[n]$



[{1,3},{2}]
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[{1},{2,3}]
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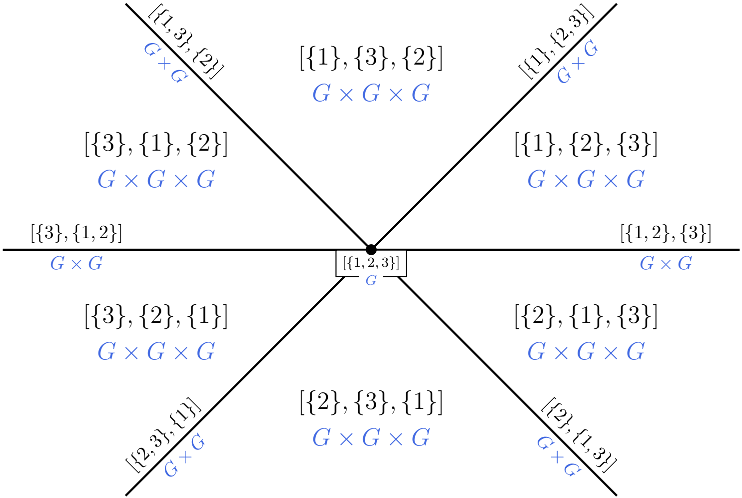
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[{2},{1,3}]
G × G

Product of G-compositions

$$\Big[(B_1,g_1),\ (B_2,g_2),\ (B_3,g_3),\ (B_4,g_4)\Big]\Big[(C_1,h_1),\ (C_2,h_2),\ (C_3,h_3)\Big]$$

$$\begin{aligned}= \Big[&(B_1\cap C_1, h_1g_1), \quad (B_1\cap C_2, h_2g_1), \quad (B_1\cap C_3, h_3g_1)\\ &(B_2\cap C_1, h_1g_2), \quad (B_2\cap C_2, h_2g_2), \quad (B_2\cap C_3, h_3g_2)\\ &(B_3\cap C_1, h_1g_3), \quad (B_3\cap C_2, h_2g_3), \quad (B_3\cap C_3, h_3g_3)\\ &(B_4\cap C_1, h_1g_4), \quad (B_4\cap C_2, h_2g_4), \quad (B_4\cap C_3, h_3g_4)\Big]^{\dagger}\end{aligned}$$

† – remove empty intersections

$$\Big[\big(\{2{,}5\},\text{-1}\big),\ \big(\{1{,}3{,}4{,}6\},1\big)\Big]\cdot\Big[\big(\{3{,}6\},\text{-1}\big),\ \big(\{1{,}4{,}5\},1\big),\ \big(\{2\},\text{-1}\big)\Big]$$

$$=\Big[\big(\{5\},\text{-1}\big),\big(\{2\},1\big),\big(\{3{,}6\},\text{-1}\big),\big(\{1{,}4\},1\big)\Big]$$

$$\Big[(B_1,g_1),(B_2,g_2),\dots,(B_l,g_l)\Big]^2=\Big[\dots,(B_i\cap B_j,g_jg_i),\dots\Big]^\dagger$$
$$=\Big[(B_1,g_1^2),(B_2,g_2^2),\dots,(B_l,g_l^2)\Big]$$

Theorem (S. Hsiao 2009)

$\Sigma_n[G]$ *is a strict LRBG:*

$$xx^{|G|}=x \qquad xyx^{|G|}=xy \qquad (xy)^{|G|}=x^{|G|}y^{|G|}$$

$\mathfrak{S}_n$-action on $\Sigma_n[G]$

For $\sigma \in \mathfrak{S}_n$ and $\big[(B_1, g_1), \ldots, (B_l, g_l)\big] \in \Sigma_n[G]$,

$$\sigma \cdot \big[(B_1, g_1), \ldots, (B_l, g_l)\big] = \big[(\sigma(B_1), g_1), \ldots, (\sigma(B_l), g_l)\big]$$

$$\begin{aligned}(1\,3)(2\,4\,6\,5) \cdot \Big[\big(\{2{,}5\}, \text{-}1\big), \big(\{1{,}3{,}4{,}6\}, 1\big)\Big] &= \Big[\big(\{4{,}2\}, \text{-}1\big), \big(\{3{,}1{,}6{,}5\}, 1\big)\Big] \\ &= \Big[\big(\{2{,}4\}, \text{-}1\big), \big(\{1{,}3{,}5{,}6\}, 1\big)\Big]\end{aligned}$$

In $\mathbb{Z}\Sigma_3[G]$, the following element is invariant for the $\mathfrak{S}_3$-action:

$$\big[(\{1\}, g), (\{2, 3\}, h)\big] + \big[(\{2\}, g), (\{1, 3\}, h)\big] + \big[(\{3\}, g), (\{1, 2\}, h)\big]$$

Theorem (J Tits 1976, TP Bidigare 1997, S Hsiao 2009)

Let G be a finite group and $\Sigma_n[G]$ the LRBG of G-compositions of $[n]$.

$$\big(\mathbb{Z}\Sigma_n[G]\big)^{\mathfrak{S}_n} = \big\{a \in \mathbb{Z}\Sigma_n[G] : \sigma \cdot a = a \text{ for all } \sigma \in \mathfrak{S}_n\big\}$$

is anti-isomorphic to a subalgebra of $\mathbb{Z}\mathfrak{S}_n[G]$, *where $\mathfrak{S}_n[G]$ is the wreath product.*

- *If G is trivial,* *[Tits 1976, Bidigare 1997]*

  $$\big(\mathbb{Z}\Sigma_n\big)^{\mathfrak{S}_n} \text{ is Solomon's descent algebra of } \mathfrak{S}_n.$$

- *For the cyclic group C_p,* *[Hsiao 2009]*

  $$\big(\mathbb{Z}\Sigma_n[C_p]\big)^{\mathfrak{S}_n} \text{ is the Mantaci–Reutenauer algebra of } \mathfrak{S}_n[C_p].$$

The descent algebra was introduced by Solomon (1976), and has connections with the representation theory of $\mathfrak{S}_n$, combinatorial Hopf algebras, probability; studied by Baumann, the Bergerons, Bishop, Blessenohl, Bonnafé, Douglass, Foissy, Garsia, Hohlweg, Howlett, Malvenuto, Novelli, Patras, Pfeiffer, Reutenauer, Schocker, Taylor, Thibon, ...

The Mantaci–Reutenauer (descent) algebras are generalizations of the descent algebra introduced by Mantaci–Reutenauer (1995) and studied by: Monica Vazirani; Aguiar–Bergeron–Nyman (2004); Baumann–Hohlweg (2008); Hsiao (2009); Novelli–Thibon (2010); Margolis–Steinberg (2011).

Idempotents in algebras

Let e be an idempotent in an algebra A, and set $f = 1_A - e$. Then:

$f^2 = f$	$ef = 0 = fe$	$e + f = 1_A$
(f is also idempotent)	(e, f are orthogonal)	(e, f decompose 1_A)

$\{e_1, \dots, e_l\}$ is a decomposition of 1_A into orthogonal idempotents if

$$e_i^2 = e_i, \qquad e_i e_j = 0 \text{ for } i \neq j, \qquad e_1 + \cdots + e_l = 1_A.$$

decompositions of 1_A into orthogonal idempotents $\{e_1, e_2, \dots, e_l\}$	$\longleftrightarrow$	direct sum decompositions of A into left A-modules $A = Ae_1 \oplus Ae_2 \oplus \cdots \oplus Ae_l$

A complete system of primitive orthogonal idempotents (CSPOI) is a decomp. of 1_A into orthogonal idempotents $\{e_1, \dots, e_l\}$ with l maximal.

Idempotents in group algebras $\mathbb{C}G$

- **isotypic projector**: if ψ is an irreducible character of G, then

$$E_\psi = \frac{\psi(1_G)}{|G|} \sum_{g \in G} \psi(g^{-1})\, g$$

- $\{E_\psi\}_{\psi \in \mathrm{Irr}(G)}$ is a decomposition of 1_G into orthogonal idempotents

$$E_\psi E_\phi = \begin{cases} E_\psi, & \text{if } \psi = \phi \\ 0, & \text{if } \psi \neq \phi \end{cases} \qquad \text{and} \qquad \sum_\psi E_\psi = 1_G$$

- $\{E_\psi\}_{\psi \in \mathrm{Irr}(G)}$ is a CSPOI iff G is abelian.

Idempotents in LRB algebras $\mathbb{C}\mathcal{B}$

$\mathscr{J}$-equivalence on $\mathcal{B}$: $\quad x \sim_{\mathscr{J}} y \quad$ iff $\quad \mathcal{B}x\mathcal{B} = \mathcal{B}y\mathcal{B}$

For Σ_n: $\quad [B_1,\ldots,B_l] \sim_{\mathscr{J}} [C_1,\ldots,C_m] \quad$ iff $\quad \{B_1,\ldots,B_l\} = \{C_1,\ldots,C_m\}$

In a CSPOI for $\mathbb{C}\mathcal{B}$, there is one idempotent e_X° for each $\mathscr{J}$-class X:

$$u_X = \frac{1}{|X|}\sum_{x\in X} x \qquad \text{and} \qquad e_X^\circ = u_X - \sum_{Y >_{\mathscr{J}} X} u_X e_Y^\circ$$

CSPOI for LRBG algebras $\mathbb{C}\mathcal{S}$

$$\mathcal{S} = \bigsqcup_{f \in E(\mathcal{S})} G_f \qquad \text{and} \qquad G_f \cong G_{f'} \text{ if } f \sim_{\mathscr{J}} f'$$

Therefore, for each $\mathscr{J}$-class X in $E(\mathcal{S})$, we have

- e_X°, a primitive idempotent from the CSPOI for $\mathbb{C}E(\mathcal{S})$
- G_X, a subgroup of the form G_f with $f \in X$ (unique up to isomorphism)

Theorem (J Bastidas, S Brauner, FS, 2023)

Let $\mathcal{S}$ be a strict LRBG, and $E(\mathcal{S})$ its LRB of idempotents. Let

- $\{e^\circ_X\}_X$ *be a CSPOI for* $\mathbb{C}E(\mathcal{S})$ *(one for each $\mathscr{J}$-class X in $E(\mathcal{S})$)*
- $\{E^{(i)}_X\}_{i\in I_X}$ *be a CSPOI for* $\mathbb{C}G_X$ *(one family for each $\mathscr{J}$-class X)*

Then the elements $e_{(X,i)} := e^\circ_X E^{(i)}_X e^\circ_X$ *form a CSPOI for* $\mathbb{C}\mathcal{S}$.

Furthermore:

- *when a group W acts on $\mathcal{S}$, we can construct a CSPOI for* $(\mathbb{C}\mathcal{S})^W$
- $(\mathbb{C}\Sigma_n[\pm 1])^{\mathfrak{S}_n}$*: we recover the CSPOI constructed by M Vazirani under the anti-isomorphism with the Mantaci–Reutenauer algebra*
- *when $\mathcal{S}$ is a LRB of abelian groups, we get new bases of* $\mathbb{C}\mathcal{S}$

Takeaways

- LRBs are to the representation theory of $\mathfrak{S}_n$ as
LRBGs are to the representation theory of $\mathfrak{S}_n[G]$

- Aguiar–Mahajan (Topics in Hyperplane Arrangements) developed a vast theory for LRBs associated with hyperplane arrangements; the idempotent theory of LRB algebras play a central role.

- Our results merge the idempotent theories of LRBs and group algebras, allowing us to initiate a study of LRBGs that parallels the Aguiar–Mahajan theory.