

Log-concavity and -convexity via distributive lattices

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The Order Ideal Lemma (OIL)

Catalan numbers

Order polynomials

Generalized Lucas sequences

Comments and an open question

Consider a sequence of real numbers

$$(a_n) = (a_n)_{n \geq 0} = a_0, a_1, a_2, \dots$$

Say that (a_n) is *log-concave* if, for all $n \geq 1$, we have

$$a_n^2 \geq a_{n-1}a_{n+1}.$$

Say that (a_n) is *log-convex* if, for all $n \geq 1$, we have

$$a_n^2 \leq a_{n-1}a_{n+1}.$$

Log-concavity and -convexity have been intensely studied in combinatorics, algebra, and geometry.

A *lower order ideal* of a finite poset (P, \preceq) is $I \subseteq P$ such that

$$x \in I \text{ and } y \preceq x \text{ implies } y \in I.$$

An *upper order ideal* of P is $J \subseteq P$ such that

$$x \in J \text{ and } y \succeq x \text{ implies } y \in J.$$

Poset L is a *lattice* if every $x, y \in L$ have a *meet* $x \wedge y$ (greatest lower bound), and a *join* $x \vee y$ (least upper bound). Say L is *distributive* if it satisfies one of the two equivalent distributive laws

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \quad \forall x, y, z \in L,$$

or

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \quad \forall x, y, z \in L.$$

The following follows easily from the FKG (Fortuin, Kasteleyn, Ginibre) inequality.

Lemma (The Order Ideal Lemma, OIL, Liang-S)

Let L be a distributive lattice and $I, J \subseteq L$ be ideals.

(a) If I, J are both lower ideals or both upper ideals then

$$|I| \cdot |J| \leq |I \cap J| \cdot |L|$$

where $|\cdot|$ denotes cardinality.

(b) If one of I, J is a lower order ideal and the other is upper then

$$|I| \cdot |J| \geq |I \cap J| \cdot |L|.$$

The Order Ideal Lemma can be used to prove log-concavity and log-convexity results about

1. Lattice paths including Dyck, Motzkin, Schröder
2. Integer partitions
3. Order polynomials and Schur polynomials
4. Generalized Lucas sequences including the Fibonacci sequence
5. Descent and peak polynomials
6. Pattern avoidance classes
7. Stirling numbers of the second kind
8. Noncrossing partitions and Narayana numbers
9. Alternating sign matrices
10. q -analogues

The *Catalan numbers* are

$$C_n = \frac{1}{n+1} \binom{2n}{n}.$$

So

$$(C_n) = 1, 1, 2, 5, 14, 42, \dots$$

A *Dyck path of semilength n* is a lattice path P satisfying:

1. P starts at $(0, 0)$ and ends at $(2n, 0)$,
2. P uses up steps U parallel to the vector $[1, 1]$ and down steps D parallel to the vector $[1, -1]$,
3. never goes below the x-axis.

Let

$$\mathcal{D}_n = \{P \mid P \text{ is a Dyck path of semilength } n\}.$$



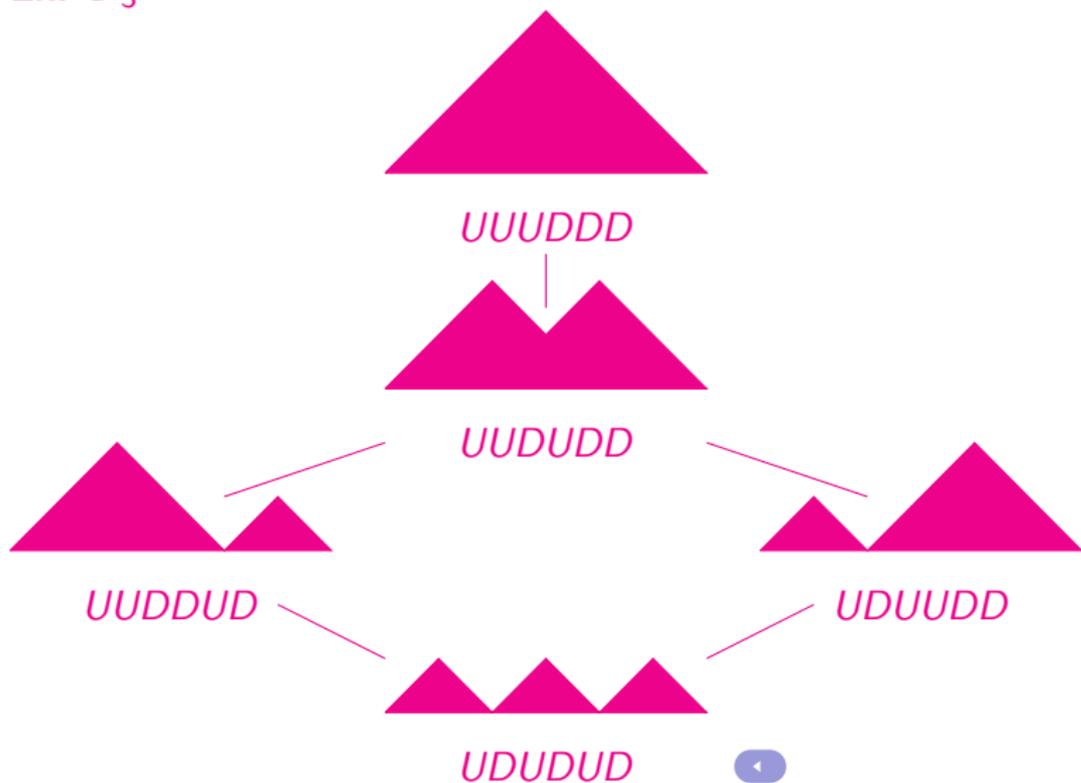
It is well known that

$$C_n = |\mathcal{D}_n|.$$

If $P \in \mathcal{D}_n$ then let $I(P)$ be the interior of the polygon enclosed by P and the x -axis. Turn \mathcal{D}_n into a poset by letting

$$P \preceq Q \iff I(P) \subseteq I(Q).$$

Ex. \mathcal{D}_3



Lemma (Ferrari, Pinzani)

For all $n \geq 0$ we have that \mathcal{D}_n is a distributive lattice. □

Theorem

The sequence (C_n) is log-convex.

Proof (Liang-S). Let $L = \mathcal{D}_{n+1}$ so that $|L| = C_{n+1}$. The maximum element in \mathcal{D}_n is the path $U^n D^n$. Construct the lower order ideals

$$I = \{P \in L \mid P \preceq U^n D^n U D\}$$

and

$$J = \{P \in L \mid P \preceq U D U^n D^n\}.$$

Therefore $I \cong J \cong \mathcal{D}_n$ so that $|I| = |J| = C_n$. Also

$$I \cap J = \{P \in L \mid P \preceq U D U^{n-1} D^{n-1} U D\}.$$

Therefore $I \cap J \cong \mathcal{D}_{n-1}$ so that $|I \cap J| = C_{n-1}$. Thus, by the Order Ideal Lemma

$$C_n^2 = |I| \cdot |J| \leq |I \cap J| \cdot |L| = C_{n-1} C_{n+1}$$

as desired. □

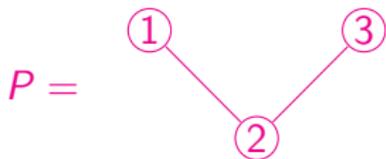
For a positive integer p we let $[p] = \{1, 2, \dots, p\}$. Let (P, \preceq) be a poset on $[p]$. A P -partition with range $[n]$ is a map $f : P \rightarrow [n]$ such that for all $x \prec y$:

1. $f(x) \geq f(y)$, i.e., f is order reversing, and
2. If $x > y$ then $f(x) > f(y)$.

Let

$$\mathcal{O}_P(n) = \{f \mid f \text{ is a } P\text{-partition with range } [n]\}.$$

Ex. Suppose $p = 3$ and



$$\therefore \mathcal{O}_P(n) = \{f : P \rightarrow [n] \mid f(2) > f(1) \text{ and } f(2) \geq f(3)\}.$$

The *order polynomial* of P is

$$\Omega_P(n) = |\mathcal{O}_P(n)|.$$

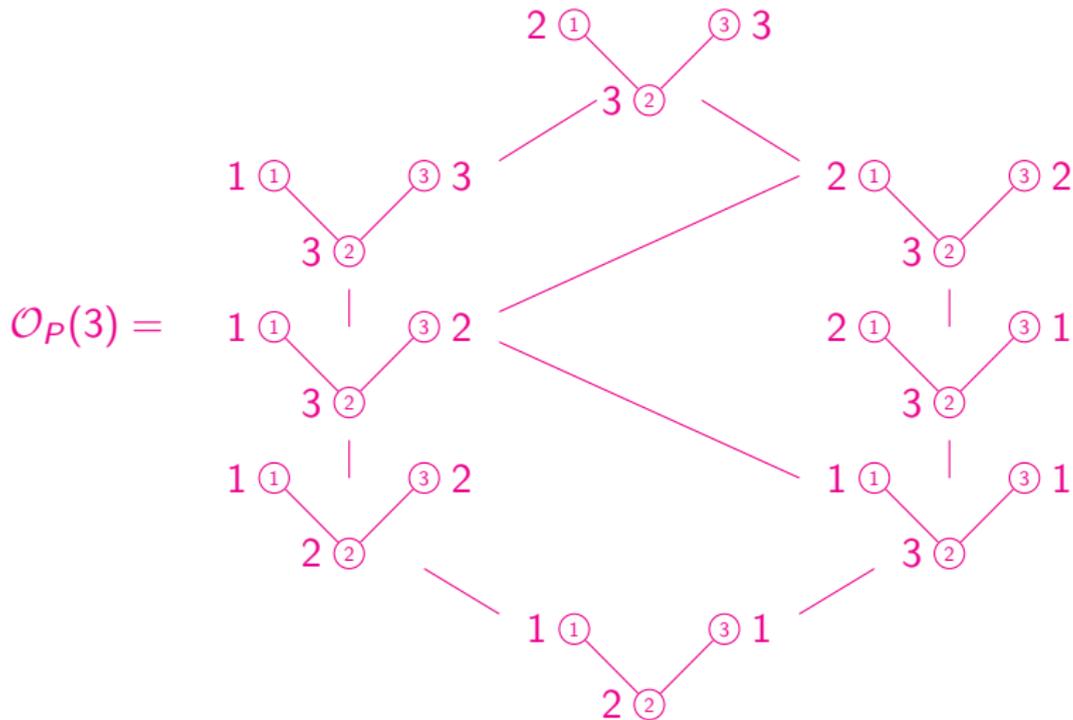
Theorem (Stanley)

For any P on $[p]$ we have that $\Omega_P(n)$ is a polynomial in n . □

Turn $\mathcal{O}_P(n)$ into a poset by letting

$$f \preceq g \iff f(x) \leq g(x) \text{ for all } x \in P.$$

Ex. With P as in the previous example we have



Theorem (Brenti)

For any P on $[p]$, the sequence $(\Omega_P(n))_{n \geq 1}$ is log-concave.

Proof (Liang-S). We have that $\mathcal{O}_P(n)$ is a lattice with

$$(f \wedge g)(x) = \min\{f(x), g(x)\} \text{ and } (f \vee g)(x) = \max\{f(x), g(x)\}.$$

Also $\mathcal{O}_P(n)$ is distributive since min distributes over max. Let $L = \mathcal{O}_P(n+1)$ so $|L| = \Omega_P(n+1)$. Define the lower order ideal

$$I = \{f \in L \mid f(x) \leq n \text{ for all } x \in P\}$$

and upper ideal

$$J = \{f \in L \mid f(x) \geq 2 \text{ for all } x \in P\}.$$

Therefore $I \cong J \cong \mathcal{O}_P(n)$ so that $|I| = |J| = \Omega_P(n)$. Similarly $|I \cap J| = \Omega_P(n-1)$. By the Order Ideal Lemma we are done. \square

Call a_0, a_1, a_2, \dots *log-concave at n* if

$$a_n^2 \geq a_{n-1}a_{n+1}.$$

Similarly define being *log-convex at n* . A *generalized Lucas sequence* $(l_n)_{n \geq 0}$ has initial values l_0, l_1 and

$$l_n = l_{n-1} + l_{n-2} \text{ for } n \geq 2.$$

Ex. The Fibonacci sequence $F_0 = F_1 = 1$ and

$$F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2.$$

Call a generalized Lucas sequence $(l_n)_{n \geq 0}$ *positive* if

$$l_0, l_1 > 0.$$

Theorem (Liang-S)

Any positive Lucas sequence has an index shift which is log-concave at odd indices and log-convex at even ones.

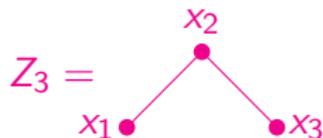


Outline of proof for F_n . For any poset P , let

$$\mathcal{J}(P) = \{I \mid I \text{ is a lower order ideal of } P\}$$

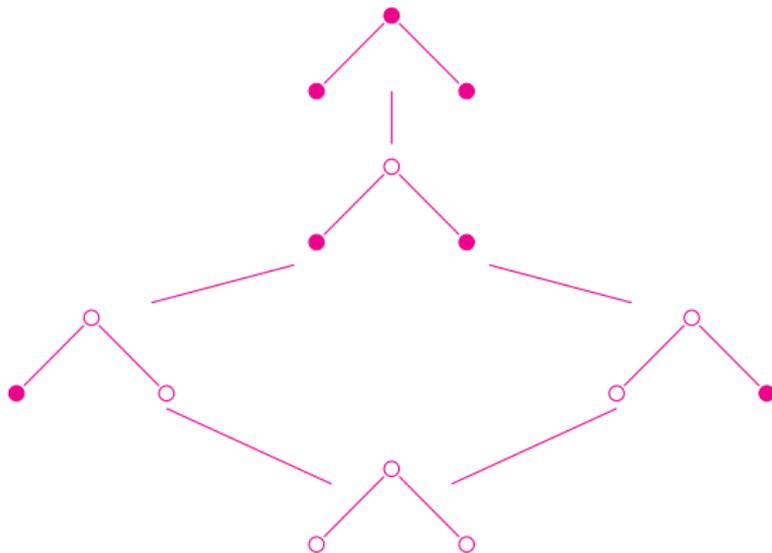
ordered by inclusion.

Ex. Suppose



If \bullet and \circ denote elements in and not in an ideal, respectively

$\mathcal{J}(Z_3) =$



Theorem (Fundamental Thm of Finite Distributive Lattices)

For any finite poset P : $\mathcal{J}(P)$ is a distributive lattice. For any finite distributive lattice L there is a poset P with $L \cong \mathcal{J}(P)$. \square

The *zigzag poset*, Z_n has elements x_1, \dots, x_n and relations

$$x_1 < x_2 > x_3 < x_4 > x_5 < \dots .$$

Claim: for $n \geq 0$ we have

$$\#\mathcal{J}(Z_n) = F_{n+1}. \quad \leftarrow$$

To see this, induct on n . Suppose n is even so Z_n ends $x_{n-1} < x_n$. Thus

$$\begin{aligned} \#\mathcal{J}(Z_n) &= \#\{I \in \mathcal{J}(Z_n) \mid x_n \notin I\} + \#\{I \in \mathcal{J}(Z_n) \mid x_n \in I\} \\ &= \#\{I \in \mathcal{J}(Z_n) \mid I \subseteq Z_{n-1}\} + \#\{I \in \mathcal{J}(Z_n) \mid x_n, x_{n-1} \in I\} \\ &= \#\mathcal{J}(Z_{n-1}) + \#\mathcal{J}(Z_{n-2}). \end{aligned}$$

To finish proving that F_n alternates log-concavity and -convexity, one now finds appropriate ideals in $\mathcal{J}(Z_{n+1})$ and applies OIL. \square

1. Peak polynomials. Let \mathfrak{S}_n be the symmetric group on $[n]$ and $\pi = \pi_1\pi_2 \dots \pi_n \in \mathfrak{S}_n$. The *peak set* of π is

$$\text{Pk } \pi = \{i \mid \pi_{i-1} < \pi_i > \pi_{i+1}\}.$$

Ex. If $\pi = 352164 = \pi_1 < \pi_2 > \pi_3 > \pi_4 < \pi_5 > \pi_6$ then

$$\text{Pk } \pi = \{2, 5\}.$$

For any index set I of positive integers, let

$$P_I(n) = \{\pi \in \mathfrak{S}_n \mid \text{Pk } \pi = I\}.$$

Ex. $P_{\{2\}}(4) = \{1324, 1423, 1432, 2314, 2413, 2431, 3412, 3421\}$.

Theorem (Billey-Burdzy-S)

For any I , there is a polynomial $p_I(n)$ such that for $n \geq 0$

$$|P_I(n)| = p_I(n) 2^{n-|I|-1}.$$

□

Theorem (Liang-S)

For any I , the sequence $(p_I(n))_{n \geq 0}$ is log-concave.

□

2. **Stirling numbers.** The *Stirling numbers of the second kind* are

$S(n, k)$ = number of partitions of $[n]$ into k subsets (blocks).

By defining a new poset on such partitions which is a distributive lattice we have used the OIL to prove the following.

Theorem (Neuman)

For fixed k the sequence $(S(n, k))_{n \geq 0}$ is log-concave. □

The (*signless*) *Stirling numbers of the first kind* are

$c(n, k)$ = number of permutations of $[n]$ having k cycles.

Computations support the following conjecture.

Conjecture

Given k , there is an integer N_k such that $(c(n, k))_{n \geq 0}$ is log-concave for $n < N_k$ and log-convex for $n \geq N_k$. □

Note that this conjecture is equivalent to $\left(\frac{c(n-1, k)}{c(n, k)}\right)_{n \geq 1}$ being unimodal with mode at N_k .

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THANKS FOR
LISTENING!