# Schur generating functions and the asymptotics of some structural constants from combinatorial representation theory.

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## I— Objects of study: Some structural constants appearing in the representation theory of the general linear group.

A representation of the general lineal group $GL(n,\mathbb{C})$ is a group homomorphism

Polynomial $\quad GL(n,\mathbb{C}) \to GL(m,\mathbb{C})$

A linear action of $GL(n,\mathbb{C})$ on the vector space $\mathbb{C}^m$

$GL(2,\mathbb{C}) \to GL(3,\mathbb{C})$.

$$x^2 \mapsto (a_{11}x + a_{21}y)^2 = a_{11}^2 x^2 + 2a_{11}a_{21}xy + a_{21}^2 y^2$$

$$\begin{pmatrix} a_{11}^2 & \cdot & \cdot \\ 2a_{11}a_{21} & \cdot & \cdot \\ a_{21}^2 & \cdot & \cdot \end{pmatrix}$$

# II— Background on polynomial representations of the general linear group

$$\rho : GL(n,\mathbb{C}) \to GL(m,\mathbb{C})$$

Irreducible representations of $GL(n,\mathbb{C})$

are indexed by conjugacy classes of $GL(m,\mathbb{C})$

Class representatives: Jordan Canonical Forms

Diagonalizable matrices are dense.

The trace of $\rho(A)$ is a symmetric polynomial in the eigenvalues of $A$

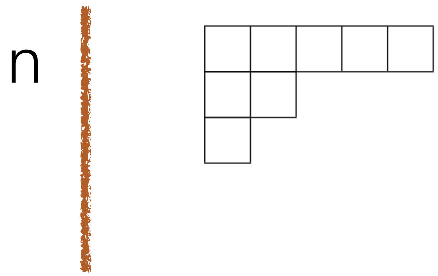
In the example the trace is $\alpha_1^2 + \alpha_1\alpha_2 + \alpha_2^2$

# i— Irreducible representations of the general linear group

Irreducible representations $W^\lambda$ of $GL(n,\mathbb{C})$

$$\boxed{GL(n,\mathbb{C}) \to GL(W^\lambda)}$$

are indexed by partitions of length $\leq n$.




(5,2,1) a partition of weight 8 of length 3

The traces of the irreducible representations are Schur polynomials.

In the example, the Schur polynomial $s_{(2)} = \alpha_1^2 + \alpha_1\alpha_2 + \alpha_2^2$

# ii— Structural constants for the general lineal group. The tensor product

The **Littlewood-Richardson coefficients** are the structure constants for the decomposition into irreducible of the tensor product of two irreducible representations of the general linear group

The Littlewood-Richardson coefficients

$$W^{\mu} \otimes W^{\nu} = \bigoplus_{\lambda} c^{\lambda}_{\mu,\nu} W^{\lambda}$$

with $|\mu| + |\nu| = |\lambda|$ three partitions of length $\leq \dim V$.

## iii— Structural constants for the general lineal group. The Kronecker product

The **Kronecker coefficients** are the structure constants for the **restriction** of irreducible representations of the general linear group

$$GL(nm)$$

into irreducibles for the subgroup

$$GL(n)\times GL(m)$$

via the tensor product of matrices.

# III— Motivation
## Stability results for the Kronecker coefficients

i. Murhagham's stability

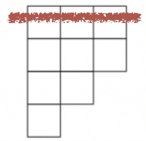
ii. A recurrent question

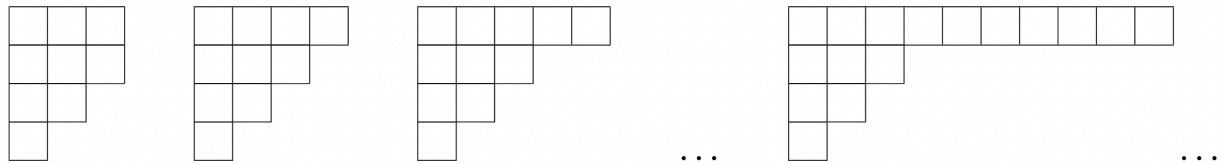
iii. Stembridge Conjecture (proved by Sam-Snowden)

# i. Murhagham´s stability

## Sequences of Kronecker coefficients

**Example** $g^{(3,3,2,1)+(i)}_{(3,3,2,1)+(i),(3,3,2,1)+(i)}$





```
[(s[3+i,3,2,1].itensor(s[3+i,3,2,1])).scalar(s[3+i,3,2,1]) for i  in range(0,10) ]
```

```
[11, 117, 312, 429, 449, 449, 449, 449, 449, 449]
```

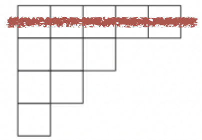
**Compare with** $g^{(i)}_{(i),(i)} = 1$ a constant

# Reduced Kronecker coefficients

The value of the stable Kronecker coefficients

$$c^{\lambda+(i)}_{\mu+(i),\nu+(i)}$$

(with $i >> 0$) only depends on the partitions $\overline{\mu}, \overline{\nu}, \overline{\lambda}$



$$g^{(3,3,2,1)+(i)}_{(3,3,2,1)+(i),(3,3,2,1)+(i)}$$

# A result of Littlewood

If $|\overline{\mu}| + |\overline{\nu}| = |\overline{\lambda}|$

then the stable Kronecker coefficients are

the Littlewood-Richardson coefficients $c^{\overline{\lambda}}_{\overline{\mu},\overline{\nu}}$

# ii. The recurrent question

# iii. Stembridge's conjecture (proved by Sam-Snowden)

The sequence

$$g(\alpha^0 + k\alpha, \beta^0 + k\beta, \gamma^0 + k\gamma)$$

Stabilizes if and only if

$$g(k\alpha, k\beta, k\gamma) = 1 \quad \text{For all} \quad k \geq 1.$$

the rate of growth of both sequences are equal, up to a constant.

$$g^{(i)}_{(i),(i)} = 1$$

There are further results on this topic due to Briand-R-Orellana, Pak-Panova, Manivel, Vallejo, …

# IV— Our question

Fix a triple of partitions $\omega^0 = (\alpha^0, \beta^0, \gamma^0)$

**What can we say about the rate of grow of sequences of the form**

$$m_F(\omega^0 + k\omega) \quad ?$$

where $m_F$ is a structural constant for a general lineal group.

(Think of the Kronecker or the Littlewood-Richardson coefficients,
Indexed by partitions of bounded length.)

# i— What is the nature of multiplicity functions

$$m_F(\omega^0 + k\omega)$$

$$Q(k) = g_{(3,3,2,1)+(k),(3,3,2,1)+(k)}^{(3,3,2,1)+(k)}$$

$$m_F(\omega)$$

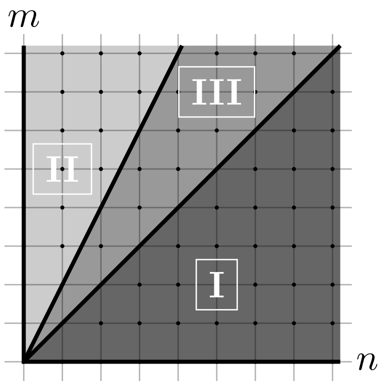
If $\omega^0$ is zero,
A result of Meinrenken and Sjamaar
implies that the multiplicity function
Is a piecewise quasi-polynomial.

# ii—Multiplicity functions $m_F(\omega)$ look like vector partition functions

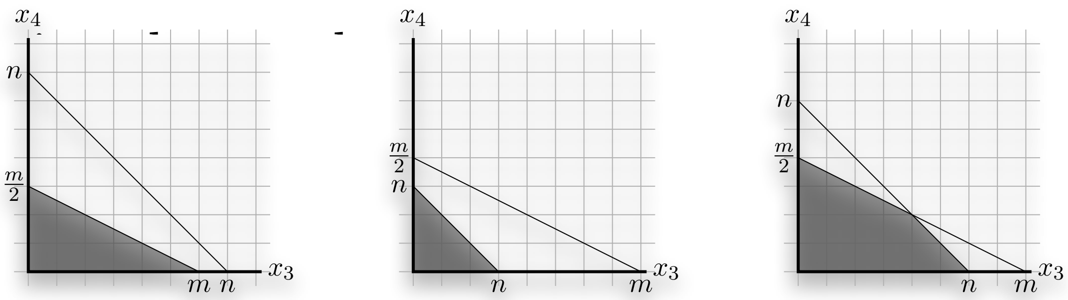
$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} n \\ m \end{bmatrix}$$

*Piecewise quasipolynomial*

Fan/Chamber complex




Rational polyhedral cone




| Region | | $p_S(n,m)$ |
|---|---|---|
| I | $m \leq n$ | $\frac{m^2}{4} + m + \frac{7}{8} + \frac{(-1)^m}{8}$ |
| II | $2n \leq m$ | $\frac{n^2}{2} + \frac{3n}{2} + 1$ |
| III | $n \leq m \leq 2n$ | $nm - \frac{n^2}{2} - \frac{m^2}{4} + \frac{n+m}{2} + \frac{7}{8} + \frac{(-1)^m}{8}$ |

(Taken from Mishna-R-Sundaram)

# iii— An example: The Littlewood-Richardson cone

Cone generated by all nonzero Littlewood-Richardson coefficients.

**Rassart :** $m_F(\lambda, \mu, \nu)$

**is a piecewise polynomial**

| Chamber | Generators | Formula for $C$ |
|---|---|---|
| $\kappa_1$ | $b, c, d_1, e_2, d_2, e_1$ | $1 - \lambda_2 - \mu_2 + \nu_1$ |
| $\kappa_2$ | $b, c, d_1, g_1, d_2, g_2$ | $1 + \nu_2 - \nu_3$ |
| $\kappa_3$ | $b, c, e_2, g_1, e_1, g_2$ | $1 + \lambda_1 + \mu_1 - \nu_1$ |
| $\kappa_4$ | $b, f, d_1, e_2, d_2, e_1$ | $1 + \nu_1 - \nu_2$ |
| $\kappa_5$ | $b, f, d_1, g_1, d_2, g_2$ | $1 + \lambda_2 + \mu_2 - \nu_3$ |
| $\kappa_6$ | $b, f, e_2, g_1, e_1, g_2$ | $1 - \lambda_3 - \mu_3 + \nu_3$ |
| $\kappa_7$ | $b, c, d_1, g_1, d_2, e_1$ | $1 + \lambda_3 + \mu_1 - \nu_3$ |
| $\kappa_8$ | $b, c, d_1, e_2, d_2, g_2$ | $1 + \lambda_1 + \mu_3 - \nu_3$ |
| $\kappa_9$ | $b, c, d_1, e_2, e_1, g_2$ | $1 + \lambda_1 - \lambda_2$ |
| $\kappa_{10}$ | $b, c, e_2, g_1, d_2, e_1$ | $1 + \mu_1 - \mu_2$ |
| $\kappa_{11}$ | $b, c, d_1, g_1, e_1, g_2$ | $1 - \lambda_2 - \mu_3 + \nu_2$ |
| $\kappa_{12}$ | $b, c, e_2, g_1, d_2, g_2$ | $1 - \lambda_3 - \mu_2 + \nu_2$ |
| $\kappa_{13}$ | $b, f, d_1, g_1, d_2, e_1$ | $1 - \lambda_1 - \mu_3 + \nu_1$ |
| $\kappa_{14}$ | $b, f, d_1, e_2, d_2, g_2$ | $1 - \lambda_3 - \mu_1 + \nu_1$ |
| $\kappa_{15}$ | $b, f, d_1, g_1, e_1, g_2$ | $1 + \mu_2 - \mu_3$ |
| $\kappa_{16}$ | $b, f, e_2, g_1, d_2, g_2$ | $1 + \lambda_2 - \lambda_3$ |
| $\kappa_{17}$ | $b, f, d_1, e_2, e_1, g_2$ | $1 + \lambda_1 + \mu_2 - \nu_2$ |
| $\kappa_{18}$ | $b, f, e_2, g_1, d_2, e_1$ | $1 + \lambda_2 + \mu_1 - \nu_2$ |

**Knutson and Tao : There are no holes on the Littlewood-Richardson Cone.**

*lengths bounded by 2,2,3*

# iv— The Kronecker cone

There are interesting results on the **Kronecker cone and the Kronecker function** due to

Baldoni-Vergnes-Walter,
Briand-R-Orellana,
Christandl-Doran-Walter,
Mishna-R-Sundaram,
Pak-Panova,
Trandafir…

It is a more complicated object.

The **Kronecker cone** contains the **Littlewood-Richardson cone** on one of its walls.

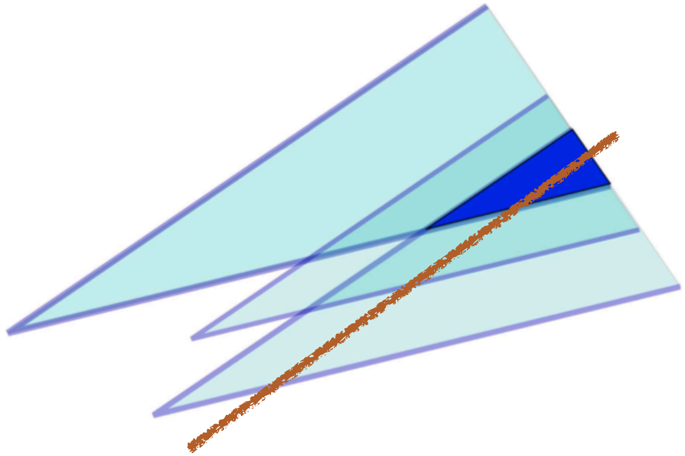
The **Kronecker function** is described by a **piecewise quasi polynomial**.

There are **holes** on the Kronecker cones

On the other hand,

$$m_F(\omega^0 + k\omega)$$

Is NOT a piecewise quasi polynomial.



As we start increasing $k$ on $m_F(\omega_0 + k\omega)$ we hit a translate of the original cone.

```
[(s[3+i,3,2,1].itensor(s[3+i,3,2,1])).scalar(s[3+i,3,2,1]) for i  in range(0,10) ]
```

```
[11, 117, 312, 429, 449, 449, 449, 449, 449, 449]
```

# V. Schur polynomials & Schur generating functions.

## The generating series for the complete homogeneous

$$\sigma[X] = \prod_{x \in X} \frac{1}{1-x} = \sum_{n \geq 0} h_n[X]$$

## R-S-K correspondence

$$\sigma[XY] = \prod_{x_i, y_j} \frac{1}{1 - x_i y_j} = \sum_{\lambda} s_\lambda[X] s_\lambda[Y]$$

# Our main tool.

We will study series of the form

Positive integers

$$\sigma[F(X,Y,Z)] = \sum_{\alpha,\beta,\gamma} m_f(\alpha,\beta,\gamma) s_\alpha[X] s_\beta[Y] s_\gamma[Z]$$

Where the lengths of $\alpha,\beta,\gamma$ are bounded.

# Some Schur generating series

## The Littlewood-Richardson coefficients

$$\sigma[XZ + YZ] = \sigma[XZ]\sigma[YZ] = \sum_{\lambda,\mu,\nu} c^{\lambda}_{\mu,\nu} s_{\mu}[X] s_{\nu}[Y] s_{\lambda}[Z]$$

## The Kronecker coefficients

$$\sigma[XYZ] = \prod_{x_i,y_j,z_k} \frac{1}{1 - x_i y_j z_k} = \sum_{\lambda} g_{\mu,\nu,\lambda} s_{\mu}[X] s_{\nu}[Y] s_{\lambda}[Z]$$

# Other examples

$$F(X,Y) = X s_\mu[Y]$$ plethysm

$$F(X,Y,Z) = XYZ + XZ + XY + YZ$$ reduced Kronecker

$$F(X,Y,Z) = XYZ + XZ + YZ$$ Heisenberg

# VI— GRAND generating series
## i. Extension of our family of coefficients

Define coefficients $m^*_F(\alpha,\beta,\gamma)$ indexed by integer vectors

$$m^*_F(\alpha,\beta,\gamma) = \langle \sigma[F(X,Y,Z)] | s_\alpha[X] s_\beta[Y] s_\gamma[Z] \rangle$$

Schur functions are defined using the Jacobi-Trudi determinant.

Compare with the original identity

$$\sigma[F(X,Y,Z)] = \sum_{\alpha,\beta,\gamma} m_f(\alpha,\beta,\gamma) s_\alpha[X] s_\beta[Y] s_\gamma[Z]$$

# VI— Grand generating series
## ii. Definition of the series

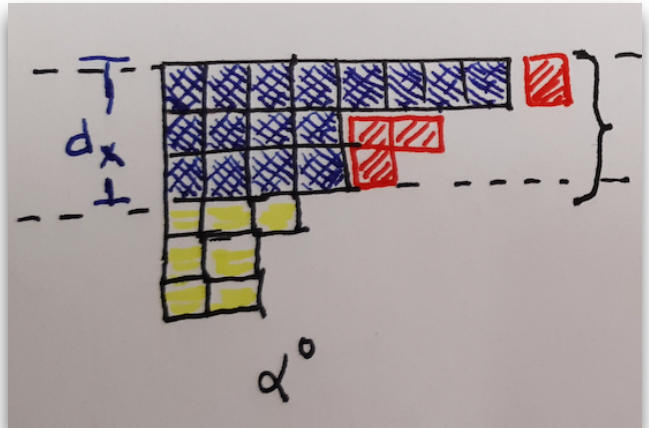
Fix $\omega^0 = (\alpha^0, \beta^0, \gamma^0)$, $d_X, d_Y, d_Z \geq 0$, $v = (v_{i,j})_{i,j}$.

The grand Generating series

$$\Psi_F^{\omega^0} = \sum_{\omega \in \mathbb{Z}^{d_x} \times \mathbb{Z}^{d_y} \times \mathbb{Z}^{d_z}} m_f^*(\omega^0 + \omega) v^\omega$$

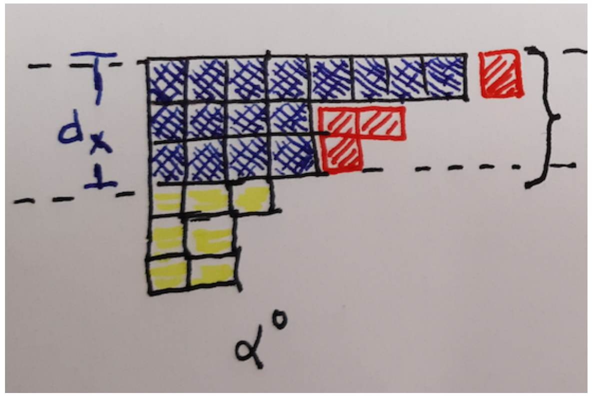
The grand generating series is always a **Laurent series.**

All partitions appear in this sum
Regardless of $\omega^0$



# VI— Grand generating series
## iii. The factorization lemma

For each fixed $\omega^0 = (\alpha^0, \beta^0, \gamma^0)$ we have a **factorization**:



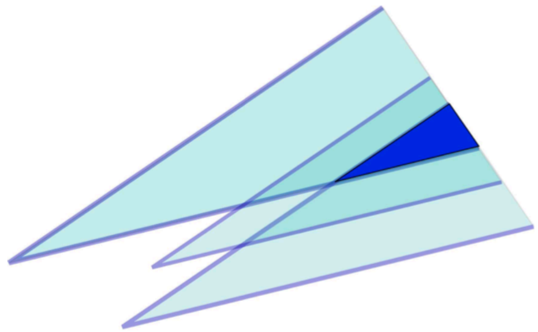

$$\Psi_F^{\omega_0} = P_F^{\omega^0} \ \Psi_F^0$$

**Laurent polynomial**

Depends on $\omega^0$

**Laurent Series**

A piecewise quasipolynomial
that only depends on
our fixed lengths



## How to compute the Laurent series?
## Use vertex operators!

When $\omega^0$ is a triple of empty partitions

$$\Psi_F^0 = V(X)V(Y)V(Z)\ \sigma[F(X,Y,Z)]$$

**Vertex operators**

$$V(X) = \prod_{j<k} (1 - x_k/x_j)$$

a shifted Vandermonde.

# VII— Main Theorem

Fix $\omega^0 = (\alpha^0, \beta^0, \gamma^0)$. and let $\omega = (\alpha, \beta, \gamma)$. There exist integers $k_o$ and $A(\omega^0)$ such that (*)

$$m_F(\omega^0 + k\omega) = A(\omega^0) \cdot m_F(k\omega)$$

Its degree in k

For all $k > k_0$

$+$ a quasipolynomial in $k$ of degree $<$ than d

$A(\omega^0)$ A particular coefficient in a Schur generating series.

$m_F(k\omega)$ A piecewise quasipolynomial on k.

$$m_F(\omega^0 + k\omega) = A(\omega^0) \cdot m_F(k\omega)$$ For all $k > k_0$

$$+ \text{ a quasipolynomial in } k \text{ of degree} < \text{than d}$$

---

# Corollary

When the multiplicities are **always one**

$$m_F(\omega^0 + k\omega) = A(\omega^0) \qquad \text{For all } k > k_0$$

where

$$A(\omega^0) = \langle \sigma[F'], s_{\bar{\alpha}}[X] s_{\bar{\beta}}[X] s_{\bar{\gamma}}[X] \rangle$$

$$F' = F(X+1, Y+1, Z+1) - F(1,1,1) - X - Y - Z$$

# VII— The Littlewood-Richardson coefficients. Increasing first rows.

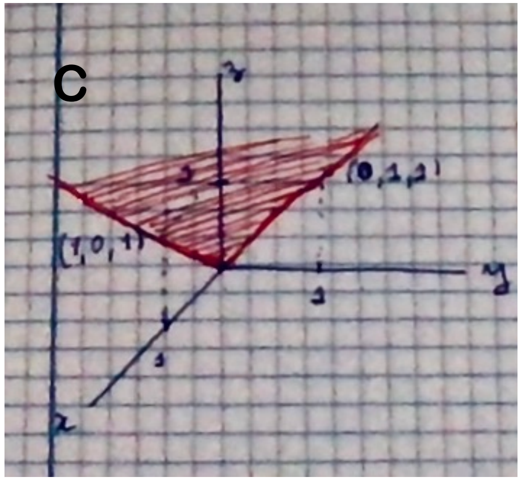
$$\sigma[XZ+YZ] = \sum_{\lambda,\mu,\nu} c^{\lambda}_{\mu,\nu} s_\mu[X] s_\nu[Y] s_\lambda[Z]$$

$$\Psi^0 = \sigma[xz+yz] = \frac{1}{(1-xz)(1-yz)}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

Let **C** be the 2-dim cone generated by (1,0,1) and (0,1,1)
The coefficients inside of **C** are one
Outside of **C** are zero

$$c^{(a+b)}_{(a),(b)} = 1$$



## Stability cone

$$c^{(a+b)}_{(a),(b)} = 1$$

```
[lrcalc.lrcoef([10+3*i,10,5,5],[5+2*i,5,5],[10+i,5]) for i  in range(0,15) ]
```

```
[1, 3, 5, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6]
```

```
[lrcalc.lrcoef([10+2*i,10,5,5],[5+i,5,5],[10+i,5]) for i  in range(0,15) ]
```

```
[1, 2, 3, 4, 5, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6]
```

```
[lrcalc.lrcoef([10+4*i,10,5,5],[5+3*i,5,5],[10+i,5]) for i  in range(0,15) ]
```

```
[1, 4, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6]
```

```
[lrcalc.lrcoef([10+5*i,10,5,5],[5+2*i,5,5],[10+3*i,5]) for i  in range(0,15) ]
```

```
[1, 3, 5, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6]
```

## A formula for the stable value?

$$F[x,y,z] = xy + xz$$

$$\sigma[F(X+1,Y+1,z+1) - F(1,1,1) - x - y - z]$$

$$= \sigma[xy + yz + z] = \frac{1}{(1-xy)(1-yz)(1-z)}$$

```
f(x+1,y+1,z+1)-f(1,1,1)-x-y-z
```

```
x*z + y*z + z
```

# VIII— The stable Littlewood-Richardson coefficients.

Stable value the coefficient of $s_{\bar{\alpha}}[X]s_{\bar{\beta}}[Y]s_{\bar{\gamma}}[Z]$ is

$$\bar{c}^{\,\overline{\gamma}}_{\overline{\alpha},\overline{\beta}} = [s_{\overline{\alpha}}[X]s_{\overline{\beta}}[Y]s_{\overline{\gamma}}[Z]]\,\sigma[XZ + YZ + Z] = \sum_{\lambda} c^{\lambda}_{\overline{\alpha},\overline{\beta}}$$

$$\sigma[xy + yz + z] = \frac{1}{(1 - xy)(1 - yz)(1 - z)}$$

What happens if I iterate this construction?

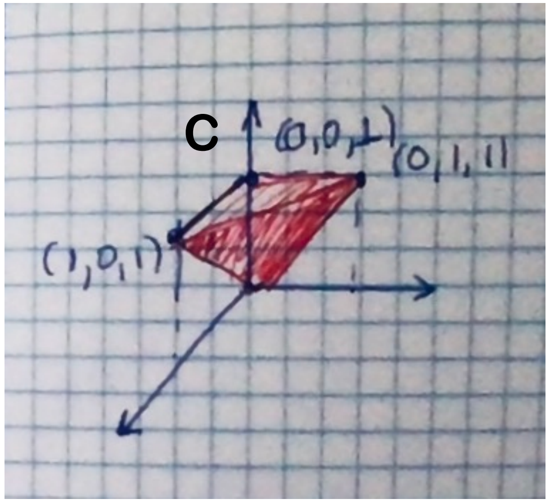
# The stable stable-Littlewood-Richardson coefficients.

$$m_F(\omega^0 + k\omega) = \quad A(\omega^0) \cdot m_F(k\omega)$$

Its degree

$$+ \text{ a quasipolynomial in } k \text{ of degree} < \text{than d}$$

Stable value the stable LR coefficient of $s_{\bar{\alpha}}[X]s_{\bar{\beta}}[Y]s_{\bar{\gamma}}[Z]$ in




$$\sigma[XZ + YZ + Z]$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Let **C** be the 3 dim cone generated by (1,0,1), (0,1,1), (0,0,1)

They stabilize.

The coefficients inside **C** are one
Outside **C** are zero.

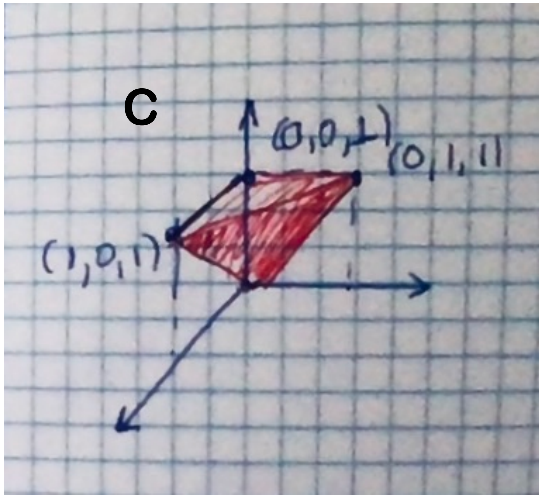
# The Littlewood-Richardson coefficients
# Increasing the second row.

$$m_F(\omega^0 + k\omega) = \quad A(\omega^0) \cdot m_F(k\omega)$$

Its degree

$$+ \text{ a quasipolynomial in } k \text{ of degree } < \text{ than d}$$

Increase the first row of the stable Littlewood-Richardson coefficients the resulting Schur generating series is :




$$\sigma[XZ + YZ + 2Z]$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

**dim of the nullspace 1**

$$m_F(\omega^0 + k\omega)$$

Is a **linear** quasipolynomial

# The Littlewood-Richardson coefficients

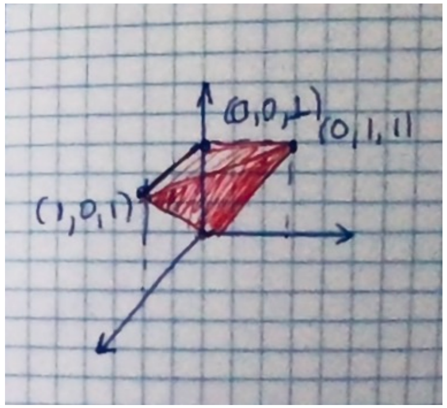
## Keep iterating this construction

In the (k+1)-step the Schur series for the asymptotic coefficients is

$$\sigma[XZ + YZ + kZ]$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

k times




Same chamber complex

**dim of the nullspace k-1**

The LR grow like a polynomial of degree k when the first k+1 rows are really long.

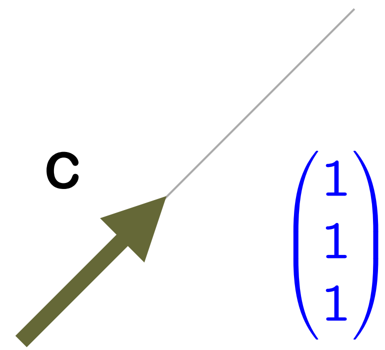
# XIX—The Kronecker coefficients
# Increasing first rows.

$$\sigma[XYZ] = \sum_{\lambda} g_{\mu,\nu,\lambda} s_{\mu}[X] s_{\nu}[Y] s_{\lambda}[Z]$$

$$\Psi^0 = \sigma[xyz] = \frac{1}{(1-xyz)}$$

$$\sigma[XYZ + XY + XZ + YZ]$$

$$\sigma[F(X+1, Y+1, z+1) - F(1,1,1) - x - y - z]$$




The cone **C** is generated by (1,1,1). In **C** the coefficients are one, and zero outside.

In this case we recover stability Murnaghan result, and a symmetric version of Michel Brion's formula for the reduced Kronecker coefficients.

# The Kronecker coefficients

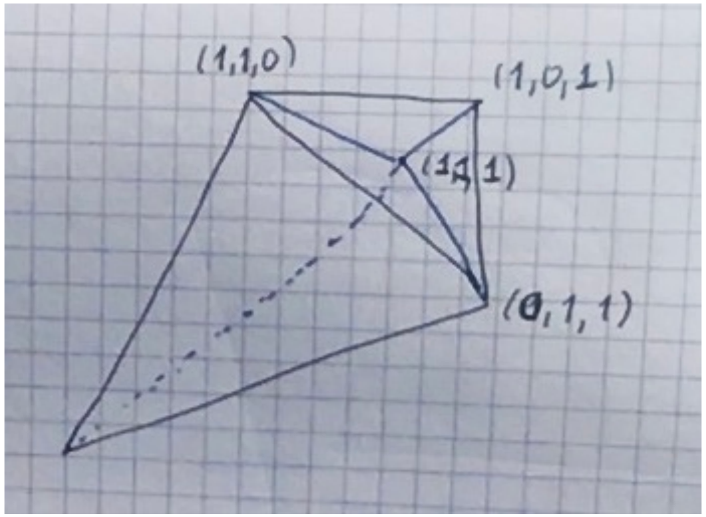
First iteration

Increasing the first rows of the reduced Kronecker coefficients.

$$\sigma[XYZ + XY + XZ + YZ]$$

$$\Psi^0 = \sigma[xyz + xy + xz + yz] = \frac{1}{(1-xyz)(1-xy)(1-xz)(1-yz)}$$

Briand-Rattan-R




The vector partition associated to vectors (1,1,1), (1,1,0), (1,0,1), (0,1,1).

$m_F(\omega^0 + k\omega)$

$=$ piecewise quasipolynomial of degree 1

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

# The Kronecker coefficients
# Increasing the 2nd rows of the partitions

The leading term is a linear polynomial

$$A\,\frac{j+k-i}{2} + \text{ a periodic function.}$$

Schur generating series for the constant A is given by

$$\sigma[XYZ + 2(XY + XZ + YZ + X + Y + Z)]$$

(The first two parts of the partitions should be removed)

# Kronecker coefficients
# Increasing the kth row of the partitions

## The n-reduced Kronecker coefficients

$$\sigma[XYZ + n(XY + XZ + YZ) + n(n-1)(X + Y + Z)]$$

Asymptotically behaves like a quasi-polynomial of degree $3n^2 - 2$

## Kronecker coefficients

$\sigma[xyz]$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

corank 0

## Reduced Kronecker coefficients

$\sigma[xyz + xy + xz + yz]$

$$\left(\begin{array}{c|ccc} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{array}\right)$$

corank 1

## 2-Reduced Kronecker coefficients

$\sigma[xyz + 2(xy + xz + yz) + 2(x + y + z)]$

corank 10

$$\left(\begin{array}{c|ccc|ccc|ccc|ccc} 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

# The some plethysm coefficients
# Increasing first rows.

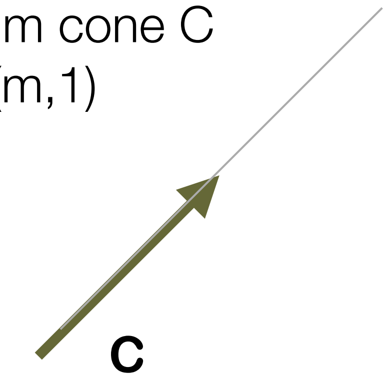
Manivel

Fix $\mu$

$$\sigma[Y s_\mu[X]] = \sum_{\lambda,\gamma} a^{\gamma}_{\lambda,\mu} s_\lambda[X] s_\gamma[Y]$$

We increase **the sizes of the first rows**.

A 1-dim cone C
(m,1)




$$\Psi^0 = \sigma[y x^m] = \frac{1}{(1 - y x^m)}$$

We get a formula the stable value.

$$Y \sum_{n=1}^{m} s_n[X] + \sum_{n=2}^{m} s_n[X]$$

$$\sigma[F(X+1, Y+1, z+1) - F(1,1,1) - x - y - z]$$

THANKS