Gallai and transitive colorings

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Based on joint work with
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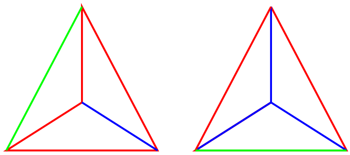
Jian-Rong Li and Avichai Marmor



MSU, Sep '23

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Hundreds of follow-ups.

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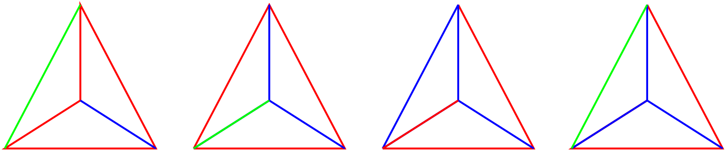

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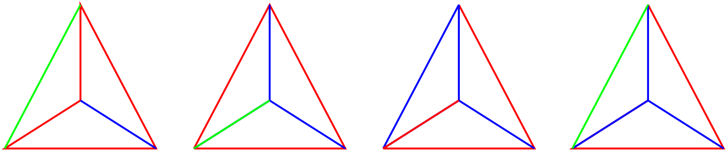

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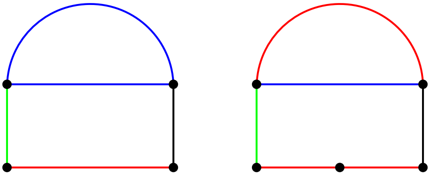

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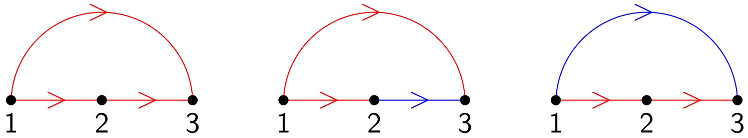

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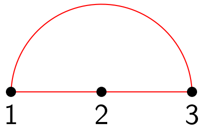

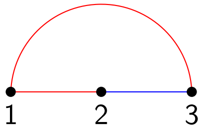

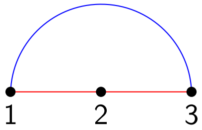

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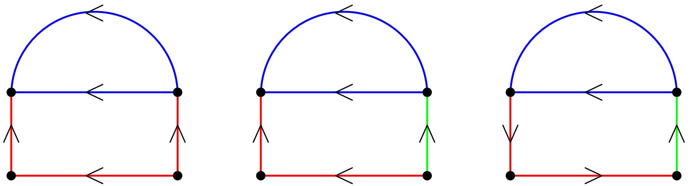

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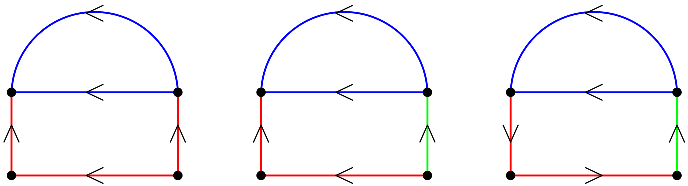

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Even more general?

Matroids, vectors and root systems

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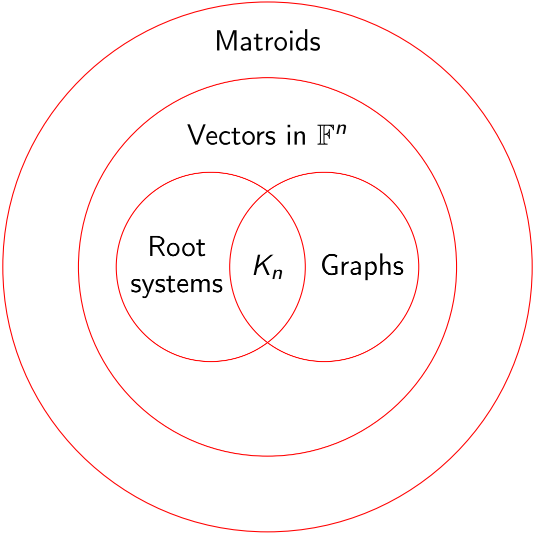
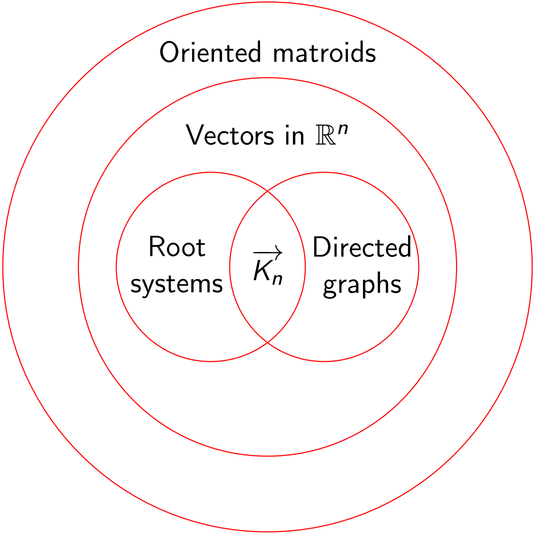


Oriented matroids, vectors and root systems



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Remark: Gallai / transitive colorings of the root system of type A,

$$\Phi^+(A_{n-1}) = \{e_i - e_j : \ 1 \leq i < j \leq n\},$$

are equivalent to Gallai / transitive colorings of K_n / $\overrightarrow{K_n}$, respectively.

Properties

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Remark: This generalizes the Erdős-Sinonovits-Sós result ($G = K_n$).

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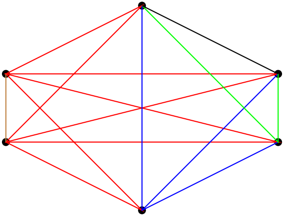

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Problem:

Find the exact number of Gallai (transitive) k-colorings of a matroid (oriented matroid) M, for any $1 \leq k \leq \text{rank}(M)$.

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2. For any finite Coxeter group W, the number of transitive 2-colorings of the set $\Phi^+(W)$ of positive roots is equal to $|W|$.

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Theorem: The number of maximal Gallai partitions of K_n is equal to $(2n-3)!! = 1 \cdot 3 \cdots (2n-3)$.

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Definition: A Gallai / transitive partition of K_n / $\overrightarrow{K_n}$ is maximal if it has the maximal possible number of parts, namely $n-1$.

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Theorem: The number of maximal transitive partitions of $\overrightarrow{K_n}$ is equal to the Catalan number $C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1}$.

Number of maximal partitions of root systems

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Corollary: The number of maximal transitive partitions of the set $\Phi^+(A_n) = \{e_i - e_j : \ 1 \le i < j \le n+1\}$ of positive roots of type A_n is equal to the n-th Catalan number C_n.

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Conjecture: The number of maximal transitive partitions of the set $\Phi^+(B_n) = \{e_i \ : \ 1 \le i \le n\} \cup \{e_i \pm e_j \ : \ 1 \le i < j \le n\}$ of positive roots of type B_n is equal to

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Remark:

C_n is equal to the number of pairs (α, β) of compositions of n, with the same number of parts, s.t. $\sum_{i=1}^{r} \alpha_i \ge \sum_{i=1}^{r} \beta_i \ (\forall r)$.

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Schur-positivity

Symmetric functions

A formal power series $f(x_1, x_2, \ldots) \in \mathbb{Q}[[x_1, x_2, \ldots]]$ is symmetric if it is invariant under permuting variables.

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Example

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The set of homogeneous symmetric functions of degree k forms a vector space over $\mathbb{Q}$, denoted by Λ_k.

Schur functions

For $\lambda \vdash k$ let the Schur function s_λ be

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A symmetric function is **Schur-positive** if all coefficients in its expansion in the Schur basis are nonnegative.

Quasisymmetric functions

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Example.

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For a set of combinatorial objects A, equipped with a set map $\mathrm{Des} : A \mapsto 2^{[n-1]}$ let

$$\mathcal{Q}(A) := \sum_{a \in A} \mathcal{F}_{n,\mathrm{Des}(a)}.$$

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Question: For which such sets is $\mathcal{Q}(A)$ symmetric ? Schur-positive?

Schur-positivity

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Definition The descent set of a transitive / Gallai k-partition p of $\overrightarrow{K}_n$ / K_n is

$$\mathrm{Des}(p) := \{ i : (i, i+1) \text{ is a singleton in } p \}.$$

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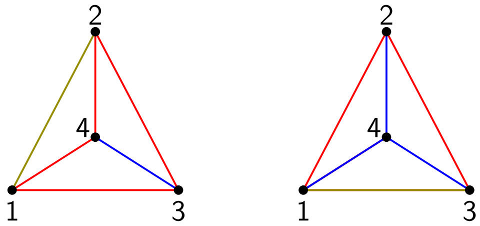

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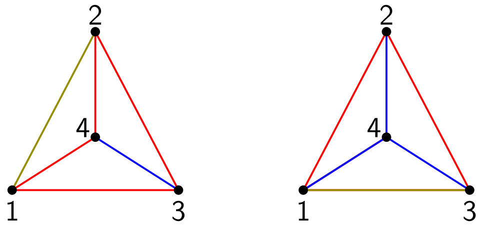

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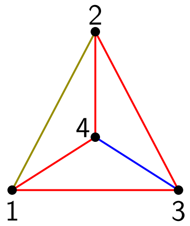

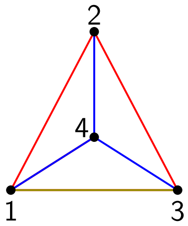

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Schur-positivity

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Denote the set of Gallai k-partitions of K_n by $G_{n,k}$.

Denote the set of transitive k-partitions of $\overrightarrow{K}_n$ by $T_{n,k}$.

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Theorem

For every $n, k \in \mathbb{N}$, the quasisymmetric functions

$$\mathcal{Q}(G_{n,k}) := \sum_{p \in G_{n,k}} \mathcal{F}_{\mathrm{Des}(p)}$$

and

$$\mathcal{Q}(T_{n,k}) := \sum_{p \in T_{n,k}} \mathcal{F}_{\mathrm{Des}(p)}$$

are Schur-positive.

Theorem

For every $n > 1$

$$\mathcal{Q}(T_{n,n-1}) = \mathrm{ch}\left(\chi^{(n-1,n-1)}\downarrow_{\mathfrak{S}_n}\right),$$

where $\chi^{(n-1,n-1)}$ *is the irreducible* $\mathfrak{S}_{2n-2}$*-character indexed by* $(n-1, n-1)$ *and* ch *is the Frobenius characteristic map from class functions to symmetric functions.*

Theorem

For every $n > 1$

$$\mathcal{Q}(G_{n,n-1}) = \mathrm{ch}\left(\left(\sum_{r=0}^{n-1} a_r \chi^{n-1+r,n-1-r}\right) \downarrow_{\mathfrak{S}_n} \right),$$

where a_r is the number of perfect matchings on $2r$ points on a line with no short chords.

Indecomposable 321-avoiding permutations

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Recall the descent set of a permutation π in the symmetric grop $\mathfrak{S}_n$

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Example Consider the permutations $[31254], [43152] \in \mathfrak{S}_5$

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decomposable , indecomposable

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Denote the set of indecomposable permutations in $\mathfrak{S}_n$ with no decreasing subsequence of length 3 by $\mathfrak{S}^*_n(321)$.

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Denote the set of indecomposable permutations in $\mathfrak{S}_n$ with no decreasing subsequence of length 3 by $\mathfrak{S}_n^*(321)$.

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For every $n > 1$

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Equivalently,

$$\sum_{p \in T_{n,n-1}} \mathbf{x}^{\mathrm{Des}(p)} = \sum_{\pi \in \mathfrak{S}_n^*(321)} \mathbf{x}^{\mathrm{Des}(\pi)},$$

where $\mathbf{x}^J := \prod_{j \in J} x_j$.

Algebras

Transitive and Gallai algebras

Transitive and Gallai algebras

The transitive algebra $\mathcal{T}_{n,k}$ is the commutative algebra over $\mathbb{C}$ generated by $\{x_{ij} : 1 \leq i < j \leq n\}$ subject to the relations

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Theorem: For all $n > 1$ and $k \geq 1$,

$$\dim(\mathcal{T}_{n,k}) = \#\{\text{transitive } k\text{-colorings of } \overrightarrow{K_n}\}$$

and

$$\dim(\mathcal{G}_{n,k}) = \#\{\text{Gallai } k\text{-colorings of } K_n\}.$$

Transitive and Gallai algebras

Transitive and Gallai algebras

The Hilbert series of a finitely generated filtered algebra $\mathcal{B}$ is

$$\mathrm{Hilb}(\mathcal{B}, q) := \sum_{j \geq 0} (\dim(\mathcal{B}_{\leq j}) - \dim(\mathcal{B}_{\leq j-1}))q^j ,$$

where $\mathcal{B}_{\leq j}$ is the degree j filtered component of $\mathcal{B}$.

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Theorem: For every $n \geq 2$

$$\mathrm{Hilb}(\mathcal{T}_{n,2}) = \sum_{k=0}^{n-1} s(n, n-k)\, q^k,$$

where $s(n,k)$ are the Stirling numbers of the first kind.

Transitive and Gallai algebras

Transitive and Gallai algebras

Conjecture: For all $n > 1$ and $k \geq 1$,

(a)

$$\mathrm{Hilb}(\mathcal{T}_{n,k}, q) = \sum_{j=1}^{n-1} P_{n,j}(q) \cdot [k]_j,$$

where $[k]_j := \prod_{i=0}^{j-1} \frac{q^{k-i}-1}{q-1}$ and $P_{n,1}(q), \ldots, P_{n,n-1}(q) \in \mathbb{Z}_{\geq 0}[q]$.

The leading coefficient satisfies $P_{n,n-1}(q) = C_{n-1} q^{\binom{n-1}{2}}$, where C_{n-1} is the Catalan number.

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(b) For all $n > 1$ and $k \geq 1$

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Remark: $Q_{n,j}(1)$ is equal to the number of Gallai j-partitions of the edge set of K_n.

Transitive and Gallai algebras

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Conjecture: In the above notation

$$Q_{n,n-1}(q) = q^{\binom{n}{2}-1} \sum_{j=0}^{n-1} E(n-1, j) q^{-j}.$$

Proofs

Schur-positivity - Proofs

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Proof For even n let $\mathbf{M}_n$ be the set of perfect matchings on n points.

Schur-positivity - Proofs

Theorem For every $n > 1$,
$\mathcal{Q}(G_{n,n-1}) = \mathrm{ch}((\sum_{r=0}^{n-1} a_r \chi^{n-1+r,n-1-r}) \downarrow_{\mathfrak{S}_n})$.

Proof For even n let $\mathbf{M}_n$ be the set of perfect matchings on n points.
For a perfect matching $m \in \mathbf{M}_n$ associate the short chord set

$$\mathrm{Short}(m) := \{i \in [n-1] : \ (i, i+1) \in m\}.$$

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Theorem (Marmor)

For every even n

$$\sum_{m \in \mathbf{M}_n} \mathcal{F}_{n,\mathrm{Short}(m)} = \sum_{r=0}^{n-1} a_r s_{(n-1+r,n-1-r)},$$

where a_r is the number of perfect matchings on $2r$ points on a line with no short chords.

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We present a bijection $f : G_{n,n-1} \to \mathbf{M}_{2n-2}$, under which

$$\mathrm{Des}(p) = \mathrm{Short}(f(p)) \cap [n-1] \qquad (\forall p \in G_{n,n-1}).$$

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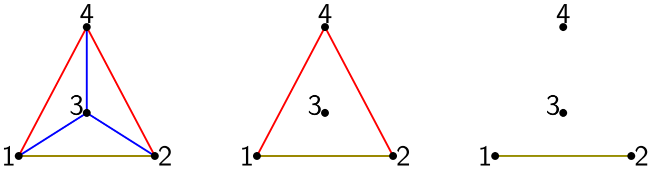
Recall

Gallai Theorem. For every $p \in G_{n,n-1}$ there exists a unique block whose edges span a complete bipartite graph of order n.

The bijection

Example



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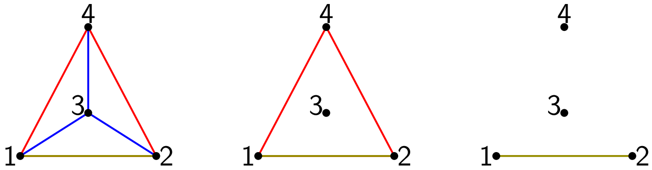
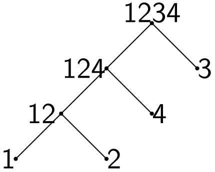


Draw the corresponding binary partition tree:



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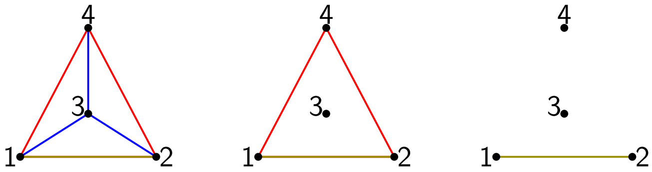

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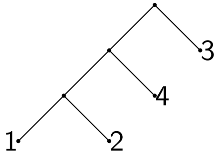

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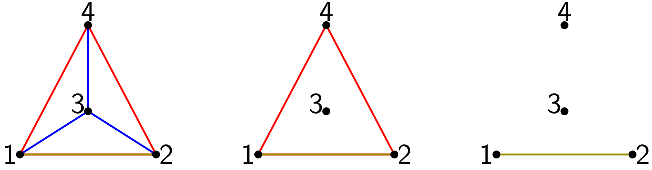

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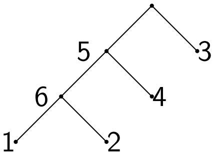

Figure: The resulting $(1, 2)(4, 6)(3, 5) \in \mathbf{M}_6$

□

Schur-positivity - Proofs

Theorem For every $n > k \geq 1$,
$\mathcal{Q}(T_{n,k})$ and $\mathcal{Q}(G_{n,k})$ are Schur-positive.

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Definition A subset $J \subseteq [n-1]$ is sparse if it does not contain any consecutive pair of elements.

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Definition A subset $J \subseteq [n-1]$ is sparse if it does not contain any consecutive pair of elements.

Lemma (Marmor)

Let A be a set equipped with a descent set map and assume that for every $a \in A$, $\mathrm{Des}(a)$ is sparse.
If for every sparse $J \subseteq [n-1]$, the cardinality of the set $\{c \in A : \ J \subseteq \mathrm{Des}(c)\}$ depends on the size of J only, then $\mathcal{Q}(A)$ is Schur-positive.

□

Schur-positivity - Proofs

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Lemma

For every $n > k \geq 1$ and $\emptyset \neq J \subseteq [n-1]$ with no consecutive elements

$$|\{p \in G(n,k) : \ J \subseteq \mathrm{Des}(p)\}| = g(n-|J|, k-|J|)$$

and

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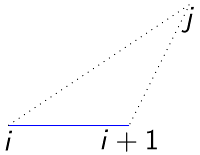
Proof of Lemma



Assume that $(i, i+1)$ is the only blue edge. Then for every $j \neq i, i+1$ the colors of (i, j) and $(i+1, j)$ are the same. Contract the edge $(i, i+1)$.

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In particular,

$$|\{c \in G(n,n-1) : \ J \subseteq \mathrm{Des}(c)\}| = (2n-2|J|-3)!!$$

and

$$|\{c \in T(n,n-1) : \ J \subseteq \mathrm{Des}(c)\}| = C_{n-1-|J|}.$$

Schur-positivity - Proofs

Dyck paths and $\mathrm{SYT}(n, n)$

Recall the natural bijection from $\mathrm{SYT}(n, n)$ to Dyck paths of length $2n$ sending the descent set to the peak set.

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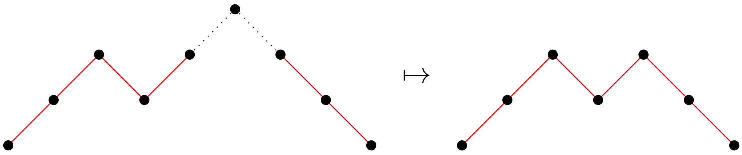

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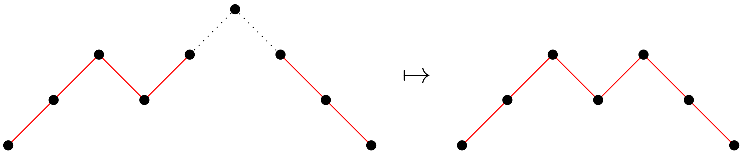

It follows that
for every $n \geq 1$ and $J \subseteq [n-1]$ with no consecutive entries

$$|\{T \in \mathrm{SYT}(n-1, n-1) : \; J \subseteq \mathrm{Des}(T)\}| = C_{n-1-|J|}.$$

□

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Thank you!