An action-packed introduction to homomesy

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Combinatorics & Graph Theory Seminar
Michigan State University
(Hosted on Zoom)

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Slides for this talk are available online (or will be soon) on my research webpage:

Google "Tom Roby"

Abstract: Dynamical algebraic combinatorics explores maps on sets of discrete combinatorial objects with particular attention to their orbit structure. Interesting counting questions immediately arise: How many orbits are there? What are their sizes? What is the period of the map if it's invertible? Are there any interesting statistics on the objects that are well-behaved under the map?

One particular phenomenon of interest is "homomesy", where a statistic on the set of objects has the same average for each orbit of an action. Along with its intrinsic interest as a kind of hidden "invariant", homomesy can be used to help understand certain properties of the action. Proofs of homomesy often lead one to develop tools that further our understanding of the underlying dynamics, e.g., by finding an equivariant bijection. These notions can be lifted to higher (piecewise-linear and birational) realms, of which the combinatorial situation is a discrete shadow, and the resulting identities are somewhat surprising. Maps that can be decomposed as products of "toggling" involutions are particularly amenable to this line of analysis.

This talk will be a introduction to these ideas, giving a number of examples.

Acknowledgments

This talks discusses joint work, mostly with Jim Propp, Darij Grinberg, and Mike Joseph.

I'm grateful to Mike Joseph and Darij Grinberg for sharing source code for slides from their earlier talks, which I shamelessly cannibalized.

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Please feel free to put questions and comments in the chat, and the moderator will convey them with appropriate timing and finese. Or someone else may answer them!

Some themes in dynamicala algebraic combinatorics

1. Periodicity/order;
2. Orbit structure;
3. Homomesy;
4. Equivariant bijections; and
5. Lifting from combinatorial to piecewise-linear and birational settings.

Cyclic rotation of binary strings

"Immer mit den einfachsten Beispielen anfangen." — David Hilbert

Cyclic rotation of binary strings

- Let $S_{n,k}$ be the set of length n binary strings with k 1s.
- Let $C_R : S_{n,k} \to S_{n,k}$ be rightward cyclic rotation.

Example

Cyclic rotation for $n = 6$, $k = 2$:

$$101000 \underset{C_R}{\longmapsto} 010100$$

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- *Periodicity* is clear here. The map has order $n = 6$.
- *Orbit structure* is very nice; every orbit size must divide n.
- *Homomesy?* Need a statistic, first.
- *Equivariant bijection?* No need.

Cyclic rotation of binary strings

An **inversion** of a binary string is a pair of positions (i, j) with $i < j$ such that there is a 1 in position i and a 0 in position j.

Example

Orbits of cyclic rotation for $n = 6$, $k = 2$:

String	Inv	String	Inv	String	Inv
101000	7	110000	8	100100	6
010100	5	011000	6	010010	4
001010	3	001100	4	001001	2
000101	1	000110	2		
100010	5	000011	0		
010001	3	100001	4		

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100010	5	000011	0		
010001	3	100001	4		
Average	**4**	**Average**	**4**	**Average**	**4**

Definition of Homomesy

Given

- a set S,
- an invertible map $\tau : S \to S$ such that every τ-orbit is finite,
- a function ("statistic") $f : S \to \mathbb{K}$ where $\mathbb{K}$ is a field of characteristic 0.

We say that the triple (S, τ, f) exhibits **homomesy** if there exists a constant $c \in \mathbb{K}$ such that for every τ-orbit $\mathcal{O} \subseteq S$,

$$\frac{1}{\#\mathcal{O}} \sum_{x \in \mathcal{O}} f(x) = c.$$

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In this case, we say that the function f is **homomesic** with average c (also called **c-mesic**) under the action of τ on S.

Theorem (Propp & R. [PrRo15, §2.3])

Let $\mathrm{inv}(s)$ *denote the number of inversions of* $s \in S_{n,k}$.

Then the function $\mathrm{inv} : S_{n,k} \to \mathbb{Q}$ *is homomesic with average* $\frac{k(n-k)}{2}$ *with respect to cyclic rotation on* $S_{n,k}$.

Theorem (Propp & R. [PrRo15, §2.3])

Let $\mathrm{inv}(s)$ *denote the number of inversions of* $s \in S_{n,k}$.

Then the function $\mathrm{inv} : S_{n,k} \to \mathbb{Q}$ *is homomesic with average* $\frac{k(n-k)}{2}$ *with respect to cyclic rotation on* $S_{n,k}$.

Proof.

Consider **superorbits** of length n. Show that replacing "01" with "10" in a string s leaves the total number of inversions in the superorbit generated by s unchanged (and thus the average since our superorbits all have the same length). ■

Cyclic rotation of binary strings

Example

$n = 6, \; k = 2$

String	Inv	String	Inv	String	Inv
101000	7	110000	8	100100	6
010100	5	011000	6	010010	4
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Cyclic rotation of binary strings

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010001	3	100001	4	001001	2
Average	**4**	**Average**	**4**	**Average**	**4**

Cyclic rotation of binary strings

Example

String	String	Inversions Change
101000	011000	-1
010100	001100	-1
001010	000110	-1
000101	000011	-1
100010	100001	-1
010001	110000	+5

There are other homomesic statistics as well:
Let $\mathbb{1}_j(s) := s_j$, the jth bit of the string s. Can you see why this is homomesic?

Bulgarian Solitaire

Homomesy: A more general definition

There are some cases where we find a similar phenomenon, but where the map no longer has finite orbits. Here is a more general definition of homomesy that is useful for some purposes.

Definition

Let τ be an self-map on a discrete set of objects S, and f be a statistic on S. We say f is **homomesic** if the value of

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} f(\tau^i(x)) = c$$

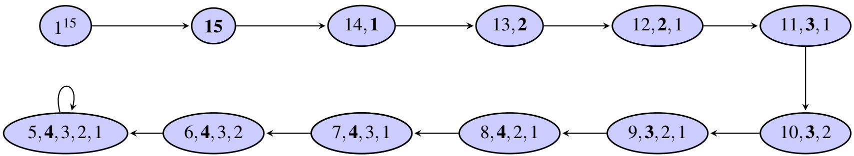
is **independent** of the starting point $x \in S$. (Also, f is c-mesic.)

This clearly reduces to the earlier definition in the case where we have an invertible action with finite orbits.

Example 2: Bulgarian solitaire

Given a way of dividing n identical chips into one or more heaps (represented as a partition λ of n), define $\sigma(\lambda)$ as the partition of n that results from removing a chip from each heap and putting all the removed chips into a new heap.

- First surfaced as a puzzle in Russia around 1980, and a solution by Andrei Toom in *Kvant*; later popularized in 1983 Martin Gardiner column; see survey of Brian Hopkins [Hop12].
- Initial puzzle: starting from any of 176 partitions of 15, one ends at $(5,4,3,2,1)$.



Bulgarian solitaire: "orbits" are now "trajectories"

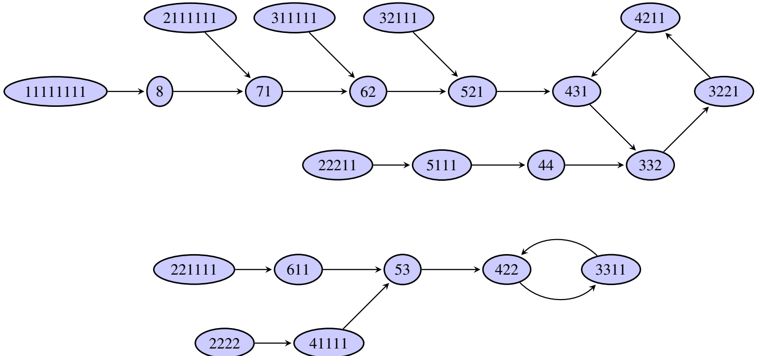
E.g., for $n = 8$, two trajectories are

$$53 \to 42\underline{2} \to \mathbf{\underline{3}311} \to \mathbf{\underline{4}22} \to \dots$$

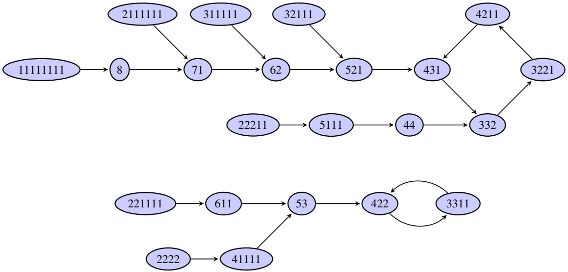
and

$$62 \to 5\underline{2}1 \to 4\underline{3}1 \to \mathbf{\underline{3}32} \to \mathbf{\underline{3}221} \to \mathbf{\underline{4}211} \to \mathbf{\underline{4}31} \to \dots$$

(the new heaps are underlined).



Bulgarian solitaire: homomesies



Let $\varphi(\lambda)$ be the number of parts of λ. In the forward orbit of $\lambda = (5,3)$, the average value of φ is $(4+3)/2 = 7/2$; while for $\lambda = (6,2)$, the average value of φ is $(3+4+4+3)/4 = 14/4 = 7/2$.

Proposition ("Bulgarian Solitaire has homomesic number of parts")

If $n = k(k-1)/2 + j$ with $0 \le j < k$, then for every partition λ of n, the ergodic average of φ on the forward orbit of λ is $k - 1 + j/k$.

($n = 8$ corresponds to $k = 4$, $j = 2$.) So the number-of-parts statistic on partitions of n is homomesic wrt/б; similarly for "size of (kth) largest part".

Ignoring transience

Since S is finite, every forward orbit is eventually periodic, and the ergodic average of φ for the forward orbit that starts at x is just the average of φ over the periodic orbit that x eventually goes into.

Ignoring transience

Since S is finite, every forward orbit is eventually periodic, and the ergodic average of φ for the forward orbit that starts at x is just the average of φ over the periodic orbit that x eventually goes into.

This definition also works in situations where S is infinite. But for rest of this talk, we'll restrict attention to maps τ that are invertible on S, where S is finite, so our initial definition (below) makes sense.

Definition ([PrRo15])

Given an (invertible) action τ on a finite set of objects S, call a statistic $f : S \to \mathbb{C}$ **homomesic** with respect to (S, τ) if the average of f over each τ-orbit $\mathcal{O}$ is the same constant c for all $\mathcal{O}$, i.e., $\frac{1}{\#\mathcal{O}} \sum_{s \in \mathcal{O}} f(s) = c$ does not depend on the choice of $\mathcal{O}$.

(Call f c-**mesic** for short.)

Coxeter Toggling
Independent Sets
of Path Graphs

Independent Sets of a Path Graph

Definition

An **independent set** of a graph is a subset of the vertices that does not contain any adjacent pair.

Let $\mathcal{I}_n$ denote the set of independent sets of the n-vertex path graph $\mathcal{P}_n$. We usually refer to an independent set by its **binary representation**.

Example

 is written 1010100.

Independent Sets of a Path Graph

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Example

 is written 1010100.

In this case, $\mathcal{I}_n$ refers to all binary strings with length n that do not contain the factor 11.

Toggles

Definition (Striker - generalized earlier concept of Cameron and Fon-der-Flaass)

For $1 \leq i \leq n$, the map $\tau_i : \mathcal{I}_n \to \mathcal{I}_n$, the **toggle at vertex i** is defined in the following way. Given $S \in \mathcal{I}_n$:

- if $i \in S$, τ_i removes i from S,
- if $i \notin S$, τ_i adds i to S, if $S \cup \{i\}$ is still independent,
- otherwise, $\tau_i(S) = S$.

Formally,

$$\tau_i(S) = \begin{cases} S \setminus \{i\} & \text{if } i \in S \\ S \cup \{i\} & \text{if } i \notin S \text{ and } S \cup \{i\} \in \mathcal{I}_n \\ S & \text{if } i \notin S \text{ and } S \cup \{i\} \notin \mathcal{I}_n \end{cases}.$$

Toggles

Proposition

Each toggle τ_i *is an involution, i.e.,* τ_i^2 *is the identity. Also,* τ_i *and* τ_j *commute if and only if* $|i - j| \neq 1$.

Definition

Let $\varphi := \tau_n \circ \cdots \circ \tau_2 \circ \tau_1$, which applies the toggles left to right.

Example

In $\mathcal{I}_5$, $\varphi(10010) = 01001$ by the following steps:

$$10010 \xmapsto{\tau_1} 00010 \xmapsto{\tau_2} 01010 \xmapsto{\tau_3} 01010 \xmapsto{\tau_4} 01000 \xmapsto{\tau_5} 01001.$$

Order & Orbits

- The order of this action grows quite fast as n increases and is difficult to describe in general. It is the LCM of the orbit sizes, which are not all divisors of some small number (relative to n):
  2, 3, 6, 15, 24, 231, 210, 1989, 240, 72105, 18018, 3354725, 3360
- For $n = 6$ orbit sizes are 3, 7, and 11, so order is LCM(3,7,11)$=$ 231.
- The number of orbits appeared to be OEIS A000358 , but we didn't understand why at first.
- This means that this action is unlikely to exhibit interesting Cyclic Sieving.
- But we can still find homomesy.

Homomesy

Here is an example φ-orbit in $\mathcal{I}_7$, containing 1010100. In this case, $\varphi^{10}(S) = S$.

	1	**2**	**3**	**4**	**5**	**6**	**7**
S	1	0	1	0	1	0	0
$\varphi(S)$	0	0	0	0	0	1	0
$\varphi^2(S)$	1	0	1	0	0	0	1
$\varphi^3(S)$	0	0	0	1	0	0	0
$\varphi^4(S)$	1	0	0	0	1	0	1
$\varphi^5(S)$	0	1	0	0	0	0	0
$\varphi^6(S)$	0	0	1	0	1	0	1
$\varphi^7(S)$	1	0	0	0	0	0	0
$\varphi^8(S)$	0	1	0	1	0	1	0
$\varphi^9(S)$	0	0	0	0	0	0	1

Homomesy

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	1	**2**	**3**	**4**	**5**	**6**	**7**
S	1	0	1	0	1	0	0
$\varphi(S)$	0	0	0	0	0	1	0
$\varphi^2(S)$	1	0	1	0	0	0	1
$\varphi^3(S)$	0	0	0	1	0	0	0
$\varphi^4(S)$	1	0	0	0	1	0	1
$\varphi^5(S)$	0	1	0	0	0	0	0
$\varphi^6(S)$	0	0	1	0	1	0	1
$\varphi^7(S)$	1	0	0	0	0	0	0
$\varphi^8(S)$	0	1	0	1	0	1	0
$\varphi^9(S)$	0	0	0	0	0	0	1
Total:	**4**	**2**	**3**	**2**	**3**	**2**	**4**

	1	**2**	**3**	**4**	**5**	**6**	**7**
S	1	0	1	0	1	0	0
$\varphi(S)$	0	0	0	0	0	1	0
$\varphi^2(S)$	1	0	1	0	0	0	1
$\varphi^3(S)$	0	0	0	1	0	0	0
$\varphi^4(S)$	1	0	0	0	1	0	1
$\varphi^5(S)$	0	1	0	0	0	0	0
$\varphi^6(S)$	0	0	1	0	1	0	1
$\varphi^7(S)$	1	0	0	0	0	0	0
$\varphi^8(S)$	0	1	0	1	0	1	0
$\varphi^9(S)$	0	0	0	0	0	0	1
Total:	**4**	**2**	**3**	**2**	**3**	**2**	**4**

Theorem (Joseph–R. [JR18])

Define $\mathbb{1}_i : \mathcal{I}_n \to \{0,1\}$ *to be the indicator function of vertex* i.

For $1 \leq i \leq n$, $\mathbb{1}_i - \mathbb{1}_{n+1-i}$ *is 0-mesic on* φ*-orbits of* $\mathcal{I}_n$.

Also $2\mathbb{1}_1 + \mathbb{1}_2$ *and* $\mathbb{1}_{n-1} + 2\mathbb{1}_n$ *are 1-mesic on* φ*-orbits of* $\mathcal{I}_n$.

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
$\varphi^2(S)$	1	0	1	0	0	0	1	0	1	0
$\varphi^3(S)$	0	0	0	1	0	0	0	0	0	1
$\varphi^4(S)$	1	0	0	0	1	0	1	0	0	0
$\varphi^5(S)$	0	1	0	0	0	0	0	1	0	1
$\varphi^6(S)$	0	0	1	0	1	0	0	0	0	0
$\varphi^7(S)$	1	0	0	0	0	1	0	1	0	1
$\varphi^8(S)$	0	1	0	1	0	0	0	0	0	0
$\varphi^9(S)$	0	0	0	0	1	0	1	0	1	0
$\varphi^{10}(S)$	1	0	1	0	0	0	0	0	0	1
$\varphi^{11}(S)$	0	0	0	1	0	1	0	1	0	0
$\varphi^{12}(S)$	1	0	0	0	0	0	0	0	1	0
$\varphi^{13}(S)$	0	1	0	1	0	1	0	0	0	1
$\varphi^{14}(S)$	0	0	0	0	0	0	1	0	0	0
Total:	**6**	**3**	**4**	**4**	**4**	**4**	**4**	**4**	**3**	**6**

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
$\varphi^2(S)$	1	0	1	0	0	0	1	0	1	0
$\varphi^3(S)$	0	0	0	1	0	0	0	0	0	1
$\varphi^4(S)$	1	0	0	0	1	0	1	0	0	0
$\varphi^5(S)$	0	1	0	0	0	0	0	1	0	1
$\varphi^6(S)$	0	0	1	0	1	0	0	0	0	0
$\varphi^7(S)$	1	0	0	0	0	1	0	1	0	1
$\varphi^8(S)$	0	1	0	1	0	0	0	0	0	0
$\varphi^9(S)$	0	0	0	0	1	0	1	0	1	0
$\varphi^{10}(S)$	1	0	1	0	0	0	0	0	0	1
$\varphi^{11}(S)$	0	0	0	1	0	1	0	1	0	0
$\varphi^{12}(S)$	1	0	0	0	0	0	0	0	1	0
$\varphi^{13}(S)$	0	1	0	1	0	1	0	0	0	1
$\varphi^{14}(S)$	0	0	0	0	0	0	1	0	0	0
Total:	6	3	4	4	4	4	4	4	3	6

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$\varphi^9(S)$	0	0	0	0	1	0	1	0	1	0
$\varphi^{10}(S)$	1	0	1	0	0	0	0	0	0	1
$\varphi^{11}(S)$	0	0	0	1	0	1	0	1	0	0
$\varphi^{12}(S)$	1	0	0	0	0	0	0	0	1	0
$\varphi^{13}(S)$	0	1	0	1	0	1	0	0	0	1
$\varphi^{14}(S)$	0	0	0	0	0	0	1	0	0	0
Total:	6	3	4	4	4	4	4	4	3	6

Idea of the proof that $\mathbb{1}_i - \mathbb{1}_{n+1-i}$ is 0-mesic: Given a 1 in an "orbit board", if the 1 is not in the right column, then there is a 1 either

- 2 spaces to the right,
- or 1 space diagonally down and right,

and never both.

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
$\varphi^2(S)$	1	0	1	0	0	0	1	0	1	0
$\varphi^3(S)$	0	0	0	1	0	0	0	0	0	1
$\varphi^4(S)$	1	0	0	0	1	0	1	0	0	0
$\varphi^5(S)$	0	1	0	0	0	0	0	1	0	1
$\varphi^6(S)$	0	0	1	0	1	0	0	0	0	0
$\varphi^7(S)$	1	0	0	0	0	1	0	1	0	1
$\varphi^8(S)$	0	1	0	1	0	0	0	0	0	0
$\varphi^9(S)$	0	0	0	0	1	0	1	0	1	0
$\varphi^{10}(S)$	1	0	1	0	0	0	0	0	0	1
$\varphi^{11}(S)$	0	0	0	1	0	1	0	1	0	0
$\varphi^{12}(S)$	1	0	0	0	0	0	0	0	1	0
$\varphi^{13}(S)$	0	1	0	1	0	1	0	0	0	1
$\varphi^{14}(S)$	0	0	0	0	0	0	1	0	0	0
Total:	6	3	4	4	4	4	4	4	3	6

Idea of the proof that $\mathbb{1}_i - \mathbb{1}_{n+1-i}$ is 0-mesic: This allows us to partition the 1's in the orbit board into **snakes** that begin in the left column and end in the right column.

This technique is similar to one used by Shahrzad Haddadan to prove homomesy in orbits of an invertible map called "winching" on k-element subsets of $\{1, 2, \ldots, n\}$.

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
$\varphi^2(S)$	1	0	1	0	0	0	1	0	1	0
$\varphi^3(S)$	0	0	0	1	0	0	0	0	0	1
$\varphi^4(S)$	1	0	0	0	1	0	1	0	0	0
$\varphi^5(S)$	0	1	0	0	0	0	0	1	0	1
$\varphi^6(S)$	0	0	1	0	1	0	0	0	0	0
$\varphi^7(S)$	1	0	0	0	0	1	0	1	0	1
$\varphi^8(S)$	0	1	0	1	0	0	0	0	0	0
$\varphi^9(S)$	0	0	0	0	1	0	1	0	1	0
$\varphi^{10}(S)$	1	0	1	0	0	0	0	0	0	1
$\varphi^{11}(S)$	0	0	0	1	0	1	0	1	0	0
$\varphi^{12}(S)$	1	0	0	0	0	0	0	0	1	0
$\varphi^{13}(S)$	0	1	0	1	0	1	0	0	0	1
$\varphi^{14}(S)$	0	0	0	0	0	0	1	0	0	0
Total:	6	3	4	4	4	4	4	4	3	6

Idea of the proof that $\mathbb{1}_i - \mathbb{1}_{n+1-i}$ is 0-mesic: Each snake corresponds to a composition of $n-1$ into parts 1 and 2. Also, any snake determines the orbit!

- 1 refers to 1 space diagonally down and right
- 2 refers to 2 spaces to the right

S	1	0	1	0	1	0	0	1	0	1
$\varphi(S)$	0	0	0	0	0	1	0	0	0	0
$\varphi^2(S)$	1	0	1	0	0	0	1	0	1	0
$\varphi^3(S)$	0	0	0	1	0	0	0	0	0	1
$\varphi^4(S)$	1	0	0	0	1	0	1	0	0	0
$\varphi^5(S)$	0	1	0	0	0	0	0	1	0	1
$\varphi^6(S)$	0	0	1	0	1	0	0	0	0	0
$\varphi^7(S)$	1	0	0	0	0	1	0	1	0	1
$\varphi^8(S)$	0	1	0	1	0	0	0	0	0	0
$\varphi^9(S)$	0	0	0	0	1	0	1	0	1	0
$\varphi^{10}(S)$	1	0	1	0	0	0	0	0	0	1
$\varphi^{11}(S)$	0	0	0	1	0	1	0	1	0	0
$\varphi^{12}(S)$	1	0	0	0	0	0	0	0	1	0
$\varphi^{13}(S)$	0	1	0	1	0	1	0	0	0	1
$\varphi^{14}(S)$	0	0	0	0	0	0	1	0	0	0
Total:	6	3	4	4	4	4	4	4	3	6

Red snake composition: 221121

Purple snake composition: 211212

Orange snake composition: 112122

Green snake composition: 121221

Blue snake composition: 212211

Brown snake composition: 122112

More Consequences of Snakes

Besides homomesy, this snake representation can be used to explain a lot about the orbits (particularly the orbit sizes, i.e. the number of independent sets in an orbit).

- When n is even, all orbits have odd size.
- "Most" orbits in $\mathcal{I}_n$ have size congruent to $3(n-1) \bmod 4$.
- The number of orbits of $\mathcal{I}_n$ (OEIS A000358)
- And much more...

Using elementary Coxeter theory, it's possible to extend our main theorem to other "Coxeter elements" of toggles. We get the same homomesy if we toggle exactly once at each vertex in **any** order.

Antichain Rowmotion on Posets

Rowmotion: an invertible operation on antichains

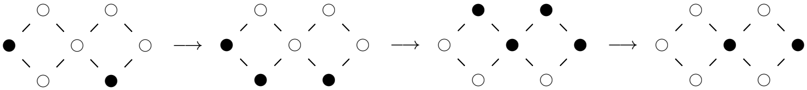
Let $\mathcal{A}(P)$ be the set of antichains of a finite poset P.

Given $A \in \mathcal{A}(P)$, let $\rho_{\mathcal{A}}(A)$ be the set of minimal elements of the complement of the *downward-saturation* of A (the smallest downset containing A).

$\rho_{\mathcal{A}}$ is invertible since it is a composition of three invertible operations:

$$\text{antichains} \longleftrightarrow \text{downsets} \longleftrightarrow \text{upsets} \longleftrightarrow \text{antichains}$$

$\rho_{\mathcal{A}}$:



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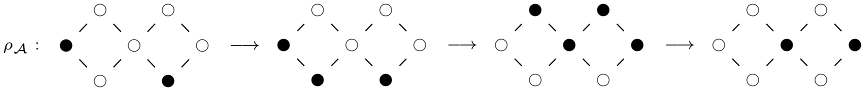


This map and its inverse have been considered with varying degrees of generality, by many people more or less independently (using a variety of nomenclatures and notations): Duchet, Brouwer and Schrijver, Cameron and Fon Der Flaass, Fukuda, Panyushev, Rush and Shi, and Striker and Williams, who named it **rowmotion**.

Panyushev's conjecture (AST's theorem)

Let Δ be a (reduced irreducible) root system in $\mathbf{R}^n$. (Pictures soon!)

Choose a system of positive roots and make it a poset of rank n by decreeing that y covers x iff $y - x$ is a simple root.

Theorem (Armstrong–Stump–Thomas [AST11], Conj. [Pan09])

Let $\mathcal{O}$ be an arbitrary $\rho_{\mathcal{A}}$-orbit. Then

$$\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} \#A = \frac{n}{2}.$$

In our language, the cardinality statistic is homomesic with respect to the action of rowmotion on antichains in root posets.

Picture of root posets

Here are the classes of posets included in Panyushev's conjecture.

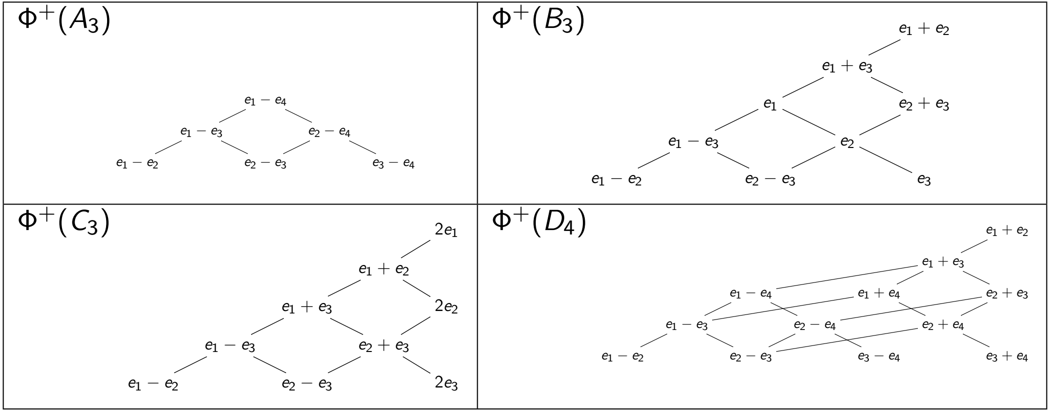
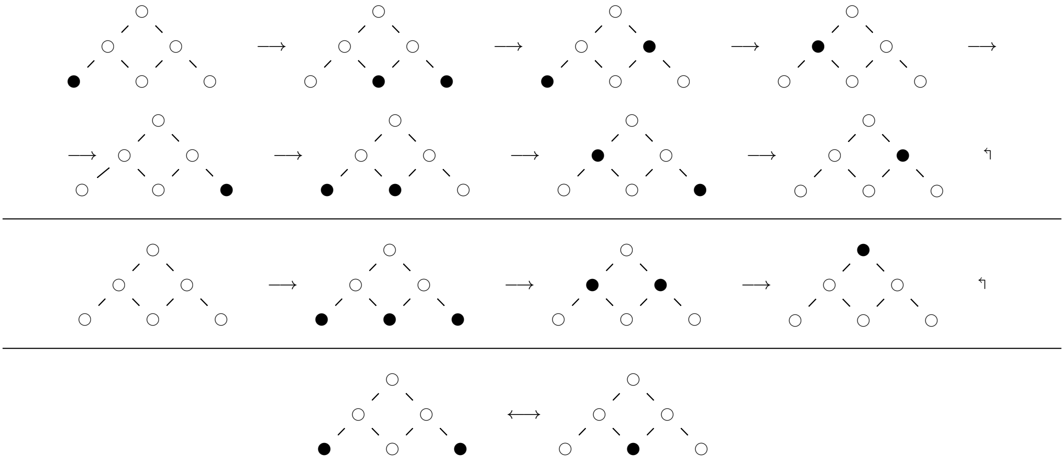


Figure: The positive root posets A_3, B_3, C_3, and D_4.

(Graphic courtesy of Striker-Williams.)

Example of antichain rowmotion on A_3 root poset

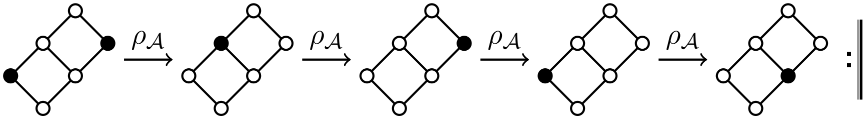
For the type A_3 root poset, there are 3 $\rho_{\mathcal{A}}$-orbits, of sizes 8, 4, 2:



Checking the average cardinality for each orbit we find that

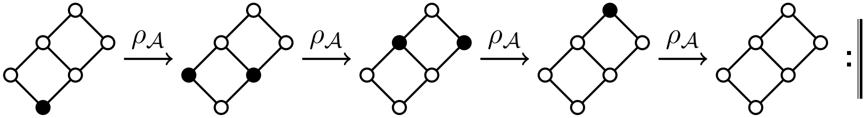
$$\frac{1+2+2+1+1+2+2+1}{8} = \frac{0+3+2+1}{4} = \frac{2+1}{2} = \frac{3}{2}.$$

Orbits of rowmotion on antichains of $[2] \times [3]$



2 1 1 1 1

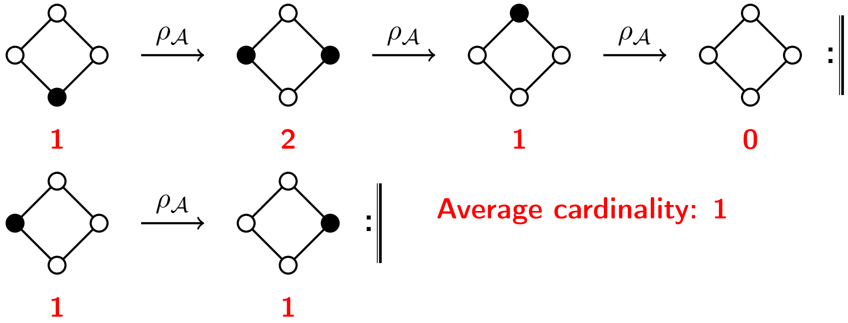
Average cardinality: 6/5



1 2 2 1 0

Average cardinality: 6/5

Orbits of rowmotion on antichains of $[2] \times [2]$



Average cardinality: 1

For antichain rowmotion on this poset, periodicity has been known for a long time:

Theorem (Brouwer–Schrijver 1974)

On $[a] \times [b]$, rowmotion is periodic with period $a + b$.

Theorem (Fon-Der-Flaass 1993)

On $[a] \times [b]$, every rowmotion orbit has length $(a + b)/d$, some d dividing both a and b.

Antichains in $[a] \times [b]$: cardinality is homomesic

For rectangular posets $[a] \times [b]$ (the type A *minuscule* poset, where $[k] = \{1, 2, \ldots, k\}$), the homomesy is easier to show than for root posets.

Theorem (Propp, R.)

Let $\mathcal{O}$ be an arbitrary $\rho_{\mathcal{A}}$-orbit in $\mathcal{A}([a] \times [b])$. Then $\dfrac{1}{\#\mathcal{O}} \displaystyle\sum_{A \in \mathcal{O}} \#A = \dfrac{ab}{a+b}$.

Antichains in $[a] \times [b]$: cardinality is homomesic

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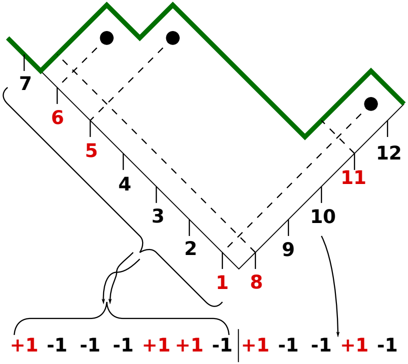

(Graphic courtesy of Ben Young.)

This proof uses an non-obvious equivariant bijection (the "Stanley–Thomas" word [Sta09, §2]) between antichains in $[a] \times [b]$ and binary strings, which carries the $\rho_{\mathcal{A}}$ map to cyclic rotation of bitstrings.

The figure shows the Stanley–Thomas word for a 3-element antichain in $\mathcal{A}([7] \times [5])$. Red $\leftrightarrow +1$, while Black $\leftrightarrow -1$.

Antichains in $[a] \times [b]$: cardinality is homomesic

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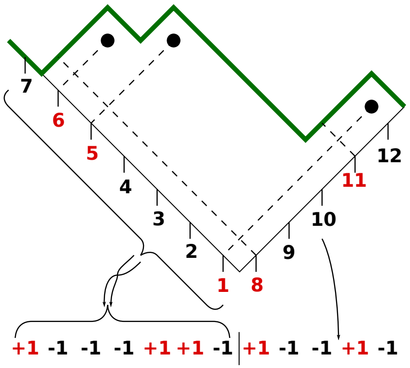
Let $\mathcal{O}$ be an arbitrary $\rho_{\mathcal{A}}$-orbit in $\mathcal{A}([a] \times [b])$. Then $\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} \#A = \frac{ab}{a+b}$.



(Graphic courtesy of Ben Young.)

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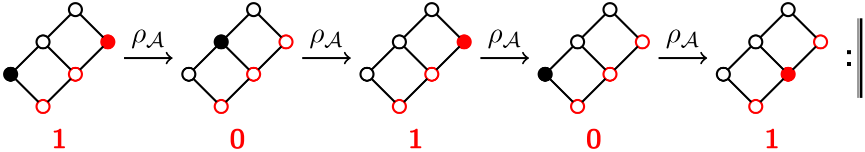
The figure shows the Stanley–Thomas word for a 3-element antichain in $\mathcal{A}([7] \times [5])$. Red $\leftrightarrow +1$, while Black $\leftrightarrow -1$.

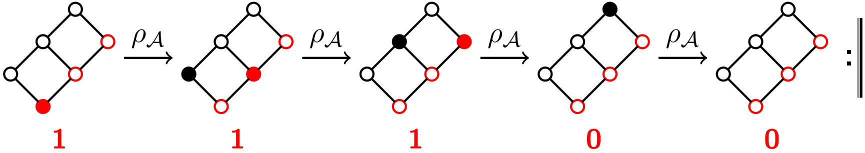
This bijection also allowed Propp–R. to derive refined homomesy results for fibers and antipodal points in $[a] \times [b]$.

Orbits of rowmotion on antichains of $[2] \times [3]$

Look at the cardinalities across a **positive fiber** such as the one highlighted in red.



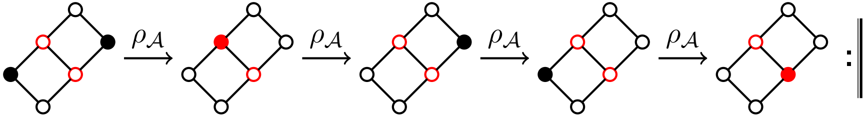
Average: 3/5



Average: 3/5

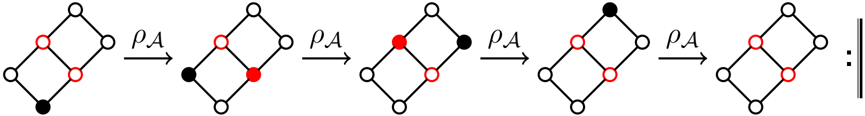
Orbits of rowmotion on antichains of $[2] \times [3]$

How about across a **negative fiber** such as the one highlighted in red.



0 1 0 0 1

Average: 2/5



0 1 1 0 0

Average: 2/5

Antichains in $[a] \times [b]$: fiber-cardinality is homomesic

For $(i, j) \in [a] \times [b]$, and A an antichain in $[a] \times [b]$, let $\mathbb{1}_{i,j}(A)$ be 1 or 0 according to whether or not A contains (i, j).

Also, let $f_i(A) = \sum_{j \in [b]} \mathbb{1}_{i,j}(A) \in \{0, 1\}$ (the cardinality of the intersection of A with the fiber $\{(i, 1), (i, 2), \ldots, (i, b)\}$ in $[a] \times [b]$), so that $\#A = \sum_i f_i(A)$.

Likewise let $g_j(A) = \sum_{i \in [a]} \mathbb{1}_{i,j}(A)$, so that $\#A = \sum_j g_j(A)$.

Theorem (Propp, R.)

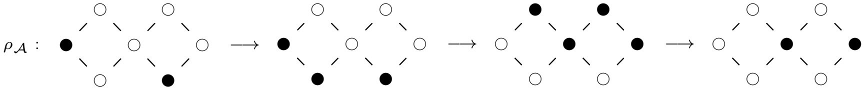
For all i, j,

$$\frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} f_i(A) = \frac{b}{a+b} \qquad \textit{and} \qquad \frac{1}{\#\mathcal{O}} \sum_{A \in \mathcal{O}} g_j(A) = \frac{a}{a+b}.$$

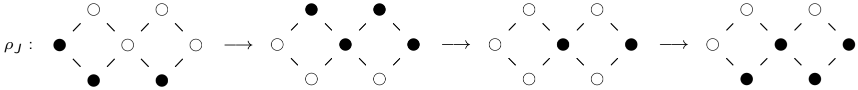
The indicator functions f_i and g_j are homomesic under $\rho_{\mathcal{A}}$, even though the indicator functions $\mathbb{1}_{i,j}$ aren't.

Rowmotion on order ideals

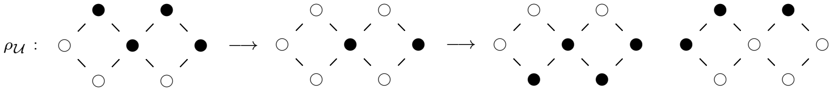
We've already seen examples of Rowmotion on antichains $\rho_{\mathcal{A}}$:



We can also define it as an operator ρ_J on $J(P)$, the set of order ideals of a poset P, by shifting the waltz beat by 1:



Or as an operator on the *up-sets (order filters)* $\mathcal{U}(P)$, of P:



Rowmotion via Toggling
(Rowmotion in Slow motion)

Toggling order ideals

Cameron and Fond-Der-Flaass showed how to write rowmotion on *order ideals* (equivalently *order filters*) as a product of simple involutions called *toggles*.

Definition (Cameron and Fon-Der-Flaass 1995)

Let $J(P)$ be the set of order ideals of a finite poset P.
Let $e \in P$. Then the **toggle** corresponding to e is the map $T_e : J(P) \to J(P)$ defined by

$$T_e(U) = \begin{cases} U \cup \{e\} & \text{if } e \notin U \text{ and } U \cup \{e\} \in J(P), \\ U \setminus \{e\} & \text{if } e \in U \text{ and } U \setminus \{e\} \in J(P), \\ U & \text{otherwise.} \end{cases}$$

Theorem (Cameron and Fon-Der-Flaass 1995)

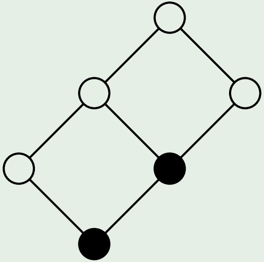
Applying the toggles T_e from top to bottom along a linear extension of P gives rowmotion on order ideals of P.

Toggling Up-sets

Theorem (Cameron and Fon-Der-Flaass 1995)

Applying the toggles T_e from top to bottom on P gives rowmotion on order ideals of P.

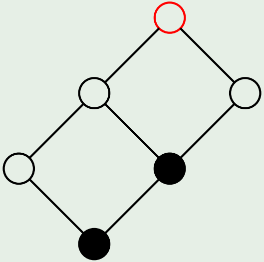
Example



Theorem (Cameron and Fon-Der-Flaass 1995)

Applying the toggles T_e from top to bottom on P gives rowmotion on order ideals of P.

Example



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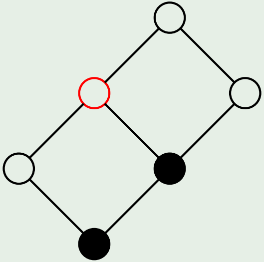

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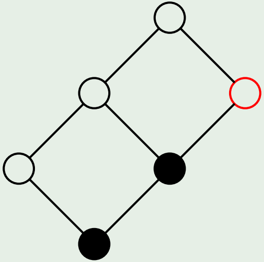

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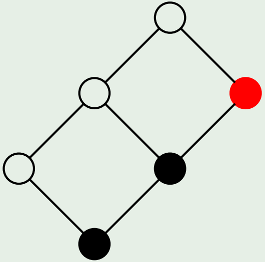

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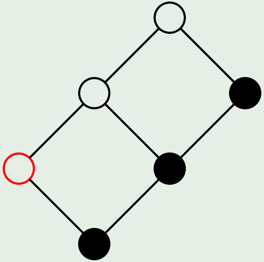

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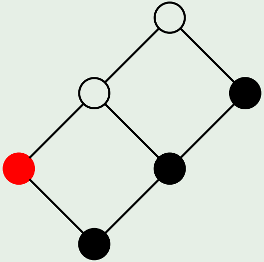

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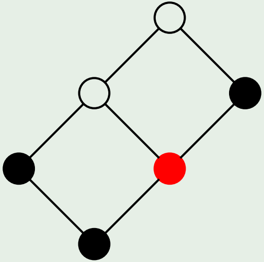

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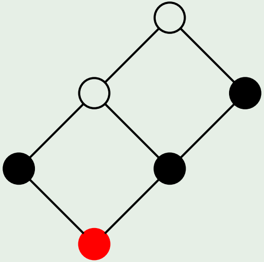

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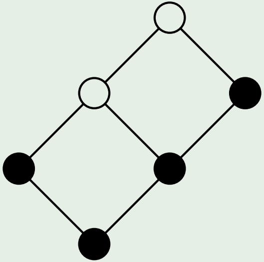


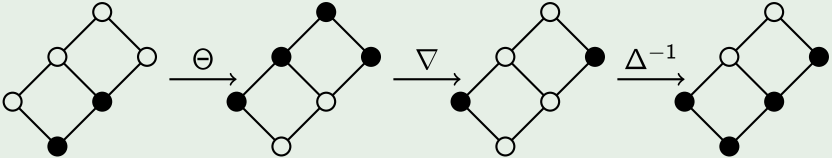
Rowmotion

We define the group action of **rowmotion** on the set of order ideals $J(P)$ via the map $\mathrm{Row} : J(P) \to J(P)$ given by the following three step process.

Start with an order ideal, and

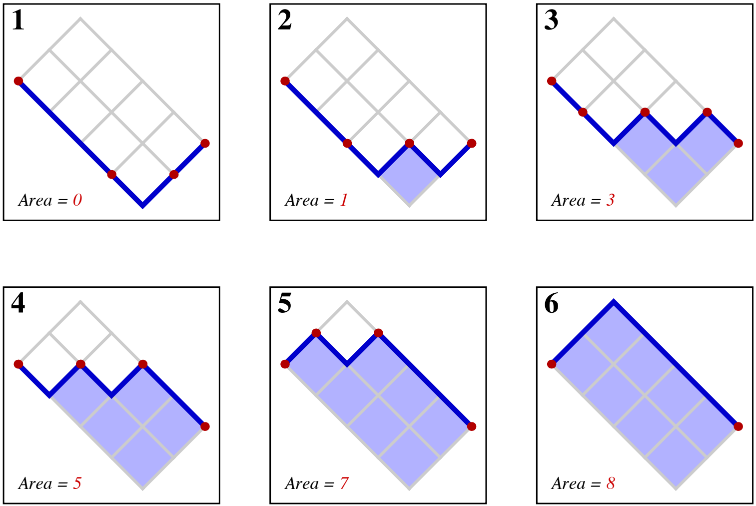
1. Θ: Take the complement (giving an order filter)
2. ∇: Take the minimal elements (giving an antichain)
3. Δ^{-1}: Saturate downward (giving a order ideal)

Example



Rowmotion on $[4] \times [2]$: Orbit 1

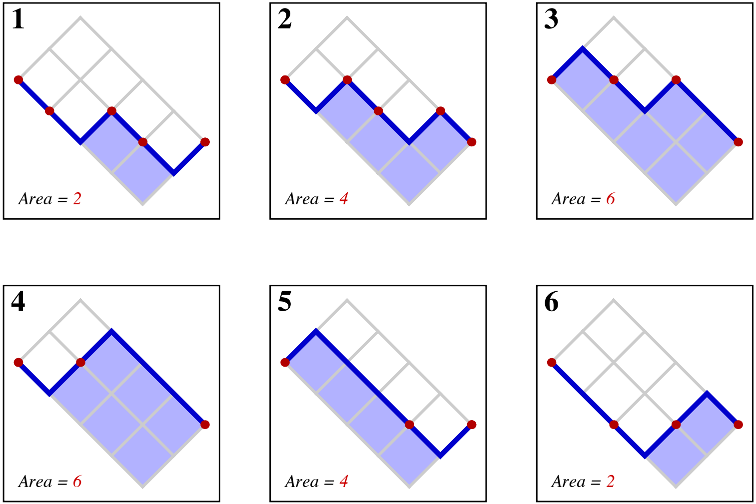
Rowmotion on $[4] \times [2]$: Orbit 1



$$(0+1+3+5+7+8)\ /\ 6 = 4$$

Rowmotion on $[4] \times [2]$: Orbit 2

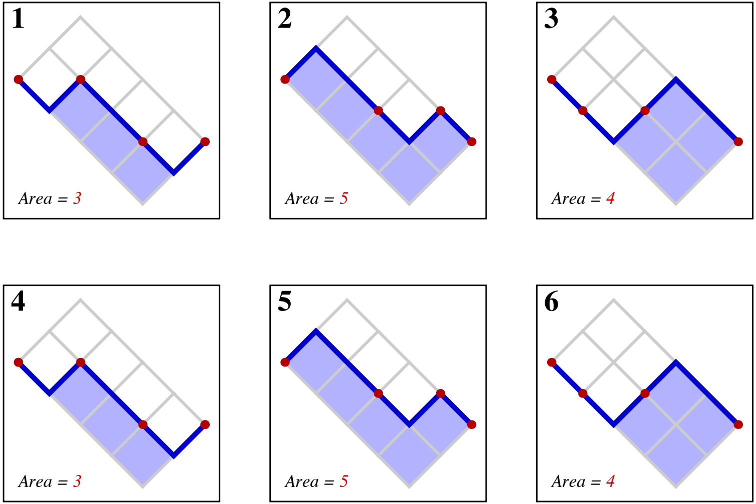
Rowmotion on $[4] \times [2]$: Orbit 2



$$(2+4+6+6+4+2)\ /\ 6 = 4$$

Rowmotion on $[4] \times [2]$: Orbit 3

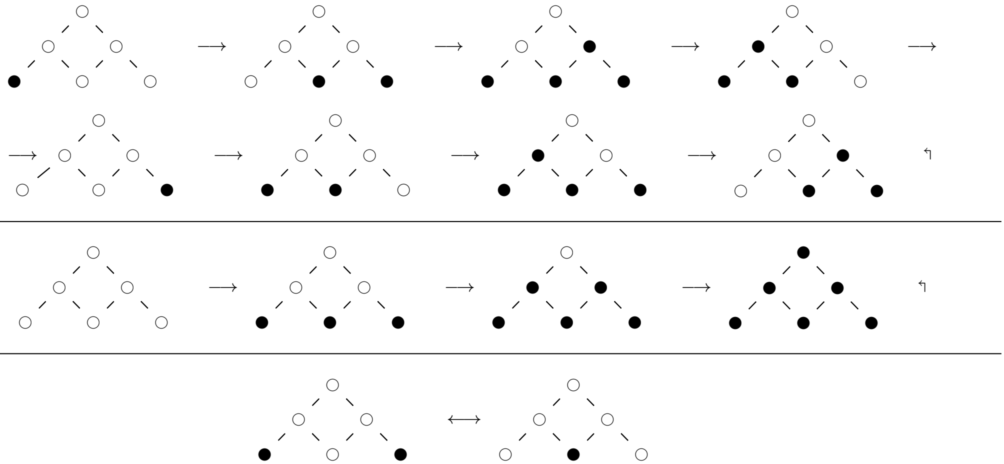
Rowmotion on $[4] \times [2]$: Orbit 3



$$(3+5+4+3+5+4)\ /\ 6 = 4$$

Example of order ideal rowmotion on A_3 root poset

For the type A_3 root poset, there are 3 ρ_J-orbits, of sizes 8, 4, 2:

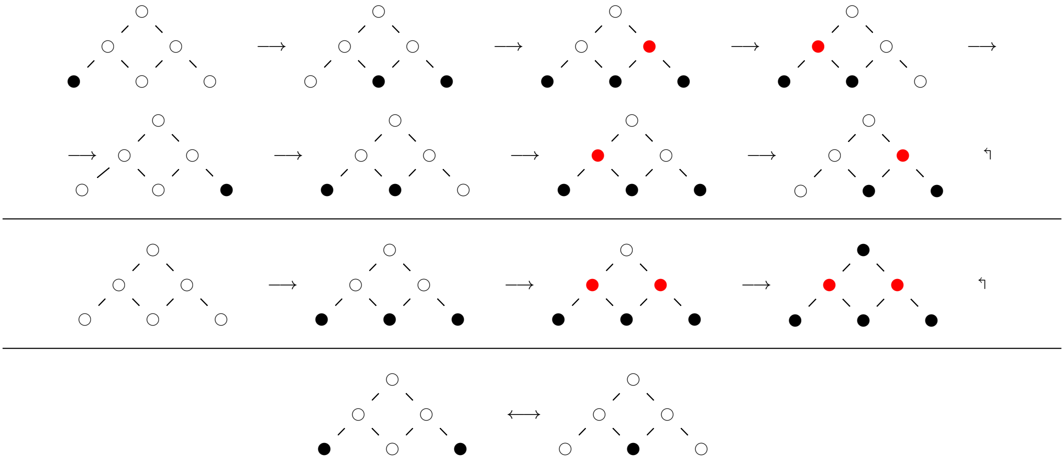


Checking the average cardinality for each orbit we find that

$$\frac{1+2+4+3+1+2+4+3}{8} = \frac{5}{2}; \quad \frac{0+3+5+6}{4} = \frac{7}{2}; \quad \frac{2+1}{2} = \frac{3}{2}. \text{ Darn!}$$

Example of order ideal rowmotion on A_3 root poset

For the type A_3 root poset, there are 3 $\rho_{\mathcal{A}}$-orbits, of sizes 8, 4, 2:



Checking the average rank-alternating cardinality for each orbit we find:

$$\frac{1+2+2+1+1+2+2+1}{8} = \frac{1+2+2+1}{4} = \frac{2+1}{2} = \frac{3}{2} \text{ Yay!}$$

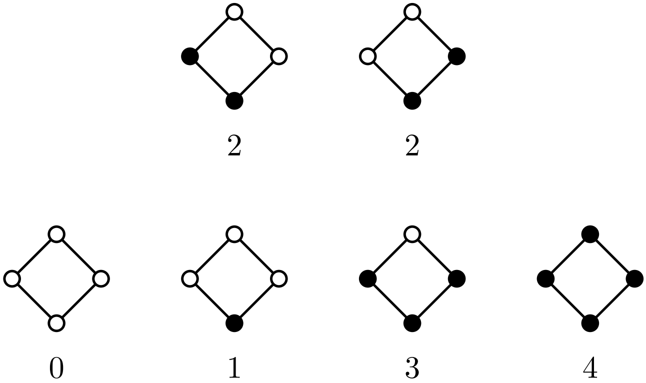
Root posets of type A: rank-signed cardinality is homomesic

Theorem (Haddadan)

Let P be the root poset of type A_n. If we assign an element $x \in P$ weight $\mathrm{wt}(x) = (-1)^{\mathrm{rank}(x)}$, and assign an order ideal $I \in J(P)$ weight $f(I) = \sum_{x \in I} \mathrm{wt}(x)$, then f is homomesic under rowmotion and promotion, with average $n/2$.

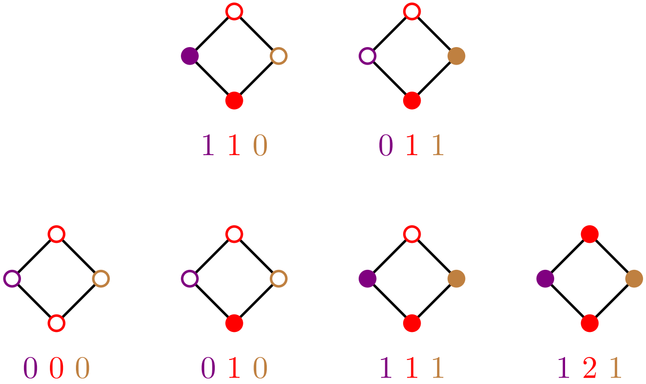
Ideals in $[a] \times [b]$: the case $a = b = 2$

Again we have an orbit of size 2 and an orbit of size 4:



Within each orbit, the average order ideal has cardinality $ab/2 = 2$.

Ideals in $[a] \times [b]$: file-cardinality is homomesic



Within each orbit, the average order ideal has

$1/2$ of a violet element, 1 red element, and $1/2$ of a brown element.

Ideals in $[a] \times [b]$: file-cardinality is homomesic

For $1 - b \leq k \leq a - 1$, define the kth **file** of $[a] \times [b]$ as

$$\{(i,j) : 1 \leq i \leq a,\ 1 \leq j \leq b,\ i - j = k\}.$$

For $1 - b \leq k \leq a - 1$, let $h_k(I)$ be the number of elements of I in the kth file of $[a] \times [b]$, so that $\#I = \sum_k h_k(I)$.

Theorem (Propp, R.)

For every ρ_J-orbit $\mathcal{O}$ in $J([a] \times [b])$:

- $$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} h_k(I) = \begin{cases} \frac{(a-k)b}{a+b} & \text{if } k \geq 0 \\ \frac{a(b+k)}{a+b} & \text{if } k \leq 0. \end{cases}$$
- $$\frac{1}{\#\mathcal{O}} \sum_{I \in \mathcal{O}} \#I = \frac{ab}{2}.$$

Piecewise-linear and birational liftings

Generalizing to the piecewise-linear setting

The decomposition of classical rowmotion into toggles allows us to define a **piecewise-linear (PL)** version of rowmotion acting on functions on a poset.

For a finite poset P, let $\widehat{P}$ denote P with an extra minimal element $\widehat{0}$ and an extra maximal element $\widehat{1}$ adjoined.

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For each $x \in P$, define the flip-map $\sigma_x : \mathcal{O}(P) \to \mathcal{O}(P)$ sending f to the unique f' satisfying

$$f'(y) = \begin{cases} f(y) & \text{if } y \neq x, \\ \min_{z \gtrdot x} f(z) + \max_{w \lessdot x} f(w) - f(x) & \text{if } y = x, \end{cases}$$

where $z \gtrdot x$ means z covers x and $w \lessdot x$ means x covers w.

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where $z \gtrdot x$ means z covers x and $w \lessdot x$ means x covers w.

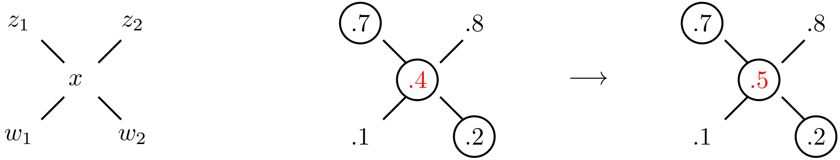
Note that the interval $[\min_{z \gtrdot x} f(z), \max_{w \lessdot x} f(w)]$ is precisely the set of values that $f'(x)$ could have so as to satisfy the order-preserving condition.

If $f'(y) = f(y)$ for all $y \neq x$, the map that sends

$$f(x) \qquad \text{to} \qquad \min_{z \gtrdot x} f(z) + \max_{w \lessdot x} f(w) - f(x)$$

is just the affine involution that swaps the endpoints of the interval.

Example of flipping at a node

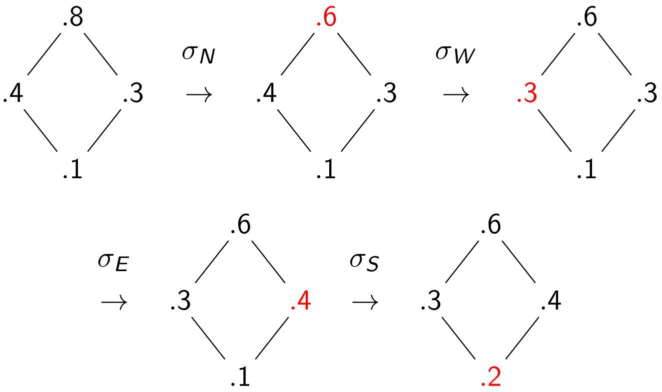


$$\min_{z \gtrdot x} f(z) + \max_{w \lessdot x} f(w) = .7 + .2 = .9$$

$$f(x) + f'(x) = .4 + .5 = .9$$

Composing flips

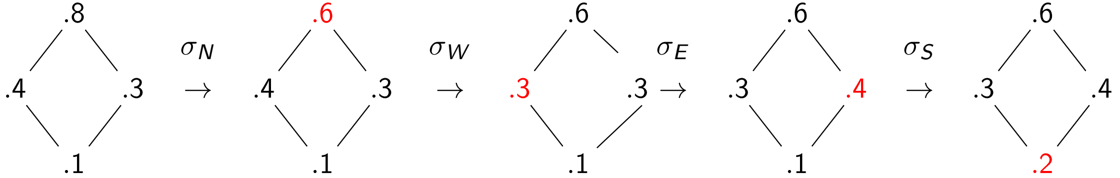
Just as we can apply toggle-maps from top to bottom, we can apply flip-maps from top to bottom, to get *piecewise-linear rowmotion*:



(We successively flip at $N = (1,1)$, $W = (1,0)$, $E = (0,1)$, and $S = (0,0)$ in order.)

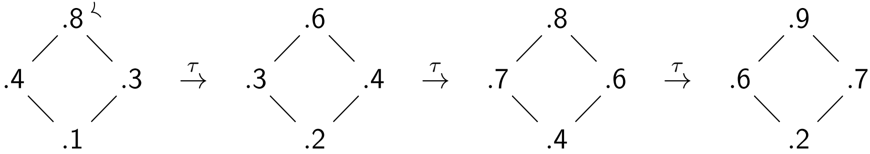
Composing flips and example of PL rowmotion orbit

We can apply flip-maps from top to bottom (successively flipping at $N = (1,1)$, $W = (1,0)$, $E = (0,1)$, and $S = (0,0)$ in order.), to get *piecewise-linear rowmotion*:



Here's an orbit of this map ($\tau = \sigma_S \circ \sigma_E \circ \sigma_W \circ \sigma_N$), which again has period 4.

τ



De-tropicalizing to birational maps

In the so-called *tropical semiring*, one replaces the standard binary ring operations $(+, \cdot)$ with the tropical operations $(\max, +)$. In the piecewise-linear (PL) category of the order polytope studied above, our flipping-map at x replaced the value of a function $f : P \to [0,1]$ at a point $x \in P$ with f', where

$$f'(x) := \min_{z \gtrdot x} f(z) + \max_{w \lessdot x} f(w) - f(x)$$

We can "detropicalize" this flip map and apply it to an assignment $f : P \to \mathbb{R}(x)$ of *rational functions* to the nodes of the poset, using that

$\min(z_i) = -\max(-z_i)$, to get the **birational toggle map**

$$(T_x f)(x) = f'(x) = \frac{\sum_{w \lessdot x} f(w)}{f(x) \sum_{z \gtrdot x} \frac{1}{f(z)}}$$

Birational rowmotion: definition

- For a field $\mathbb{K}$, a **$\mathbb{K}$-labelling of P** will mean a function $f : \widehat{P} \to \mathbb{K}$. We always set $f(\widehat{0}) = f(\widehat{1}) = 1$.
- For any $v \in P$, define the **birational v-toggle** as the rational map

$$T_v : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}} \text{ defined by } (T_v f)(w) = \frac{\sum\limits_{\widehat{P} \ni u \lessdot v} f(u)}{f(v) \sum\limits_{\widehat{P} \ni u \gtrdot v} \frac{1}{f(u)}} \text{ for } w = v.$$

(We leave $(T_v f)(w) = f(w)$ when $w \neq v$.)

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(We leave $(T_v f)(w) = f(w)$ when $w \neq v$.)

- This is a **local change** only to the label at v, and $T_v^2 = id$ (on the range of T_v).
- We define **birational rowmotion** as the rational map

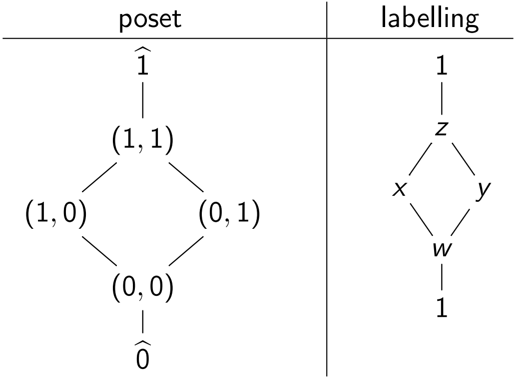
$$\rho_B := T_{v_1} \circ T_{v_2} \circ \ldots \circ T_{v_n} : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}},$$

where $(v_1, v_2, \ldots, v_n)$ is a linear extension of P.

Birational rowmotion: example

Example:

Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:



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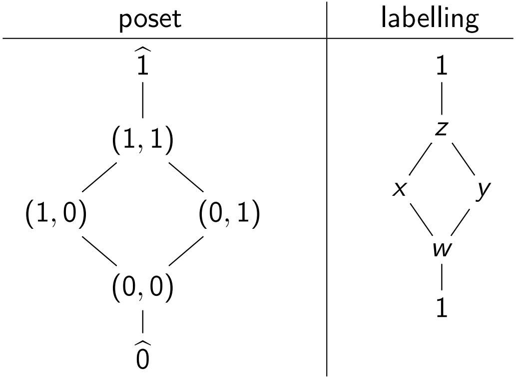

We have $\rho_B = T_{(0,0)} \circ T_{(0,1)} \circ T_{(1,0)} \circ T_{(1,1)}$

using the linear extension $((1, 1), (1, 0), (0, 1), (0, 0))$.

That is, toggle in the order "top, left, right, bottom".

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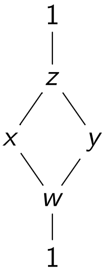
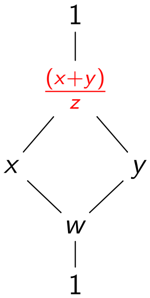
original labelling f	labelling $T_{(1,1)} f$
	

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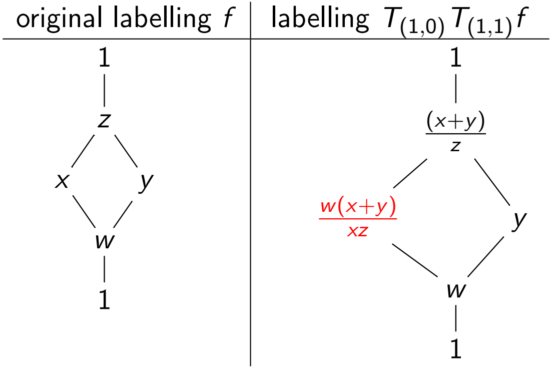

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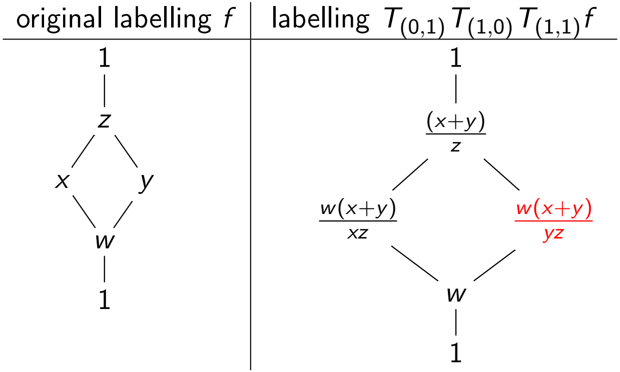

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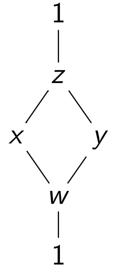
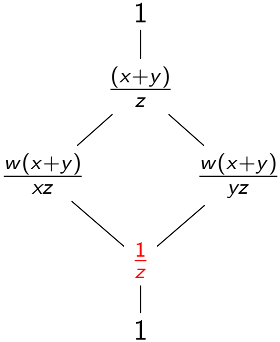
Example:

Let us "rowmote" a (generic) $\mathbb{K}$-labelling of the 2×2-rectangle:

original labelling f	labelling $T_{(0,0)} T_{(0,1)} T_{(1,0)} T_{(1,1)} f = \rho_B f$
	

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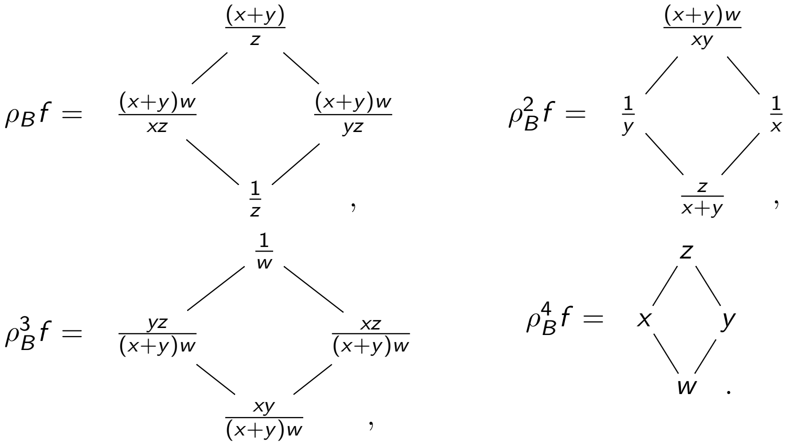

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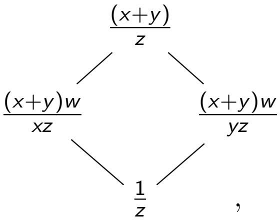
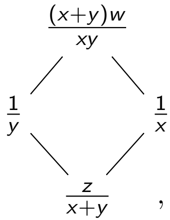
$\rho_B f =$



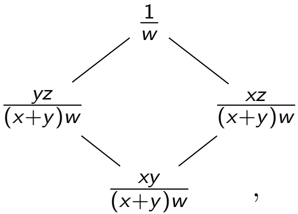
,

$\rho_B^2 f =$



,

$\rho_B^3 f =$



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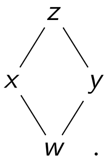

.

Notice that $\rho_B^4 f = f$, which generalizes to $\rho_B^{r+s+2} f = f$ for $P = [0, r] \times [0, s]$ [Grinberg-R 2015]. Notice also "antipodal reciprocity".

Why study this generalization?

Motivations and Connections

- Classical rowmotion is closely related to the Auslander-Reiten translation in quivers arising in certain special posets (e.g., rectangles) [Yil17].
- This generalization implies the results at the PL and combinatorial level (but not vice-versa).
- Birational rowmotion can be related to Y-systems of type $A_m \times A_n$ described in Zamolodchikov periodicity [Rob16, §4.4].
- The orbits of these maps all have natural *homomesic* statistics [PrRo15, EiPr13+, EiPr14].
- Periodicity of these systems is generally nontrivial to prove.

Birational homomesy on files of $J([0,r] \times [0,s])$

The poset $[0,1] \times [0,1]$ has three files, $\{(1,0)\}$, $\{(0,0),(1,1)\}$, and $\{(0,1)\}$.

Multiplying over all iterates of birational rowmotion in a given file:

$$\prod_{k=1}^{4} \rho_B^k(f)(1,0) = \frac{(x+y)w}{xz} \quad \frac{1}{y} \quad \frac{yz}{(x+y)w} \quad (x) = 1,$$

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$$\prod_{k=1}^{4} \rho_B^k(f)(0,0)\rho_B^k(f)(1,1) = \frac{1}{z} \;\; \frac{x+y}{z} \;\; \frac{z}{x+y} \;\; \frac{(x+y)w}{xy} \;\; \frac{xy}{(x+y)w} \;\; \frac{1}{w} \;\; (w) \;\; (z) = 1,$$

$$\prod_{k=1}^{4} \rho_B^k(f)(0,1) = \frac{(x+y)w}{yz} \;\; \frac{1}{x} \;\; \frac{xz}{(x+y)w} \;\; (y) = 1.$$

Each of these products equalling one is the manifestation, for the poset of a product of two chains, of **homomesy along files** at the birational level.

Birational homomesy on files of $J([0,r] \times [0,s])$

Theorem ([GrRo15b, Thm. 30, 32])

(1) The birational rowmotion map ρ_B on the product of two chains $P = [0,r] \times [0,s]$ is periodic, with period $r+s+2$.

(2) The birational rowmotion map ρ_B on the product of two chains $P = [0,r] \times [0,s]$ satisfies the following reciprocity: $\rho_B^{i+j+1} f(i,j) = 1/\rho_B^0 f(r-i, s-j) = \frac{1}{x_{r-i,s-j}}$.

Theorem (Musiker-R [MR19])

Given a file F in $[0,r] \times [0,s]$, $\displaystyle\prod_{k=0}^{r+s+1} \prod_{(i,j)\in F} \rho_B^k f(i,j) = 1.$

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Given a file F in $[0,r]\times[0,s]$, $$\prod_{k=0}^{r+s+1}\prod_{(i,j)\in F}\rho_B^k f(i,j)=1.$$

The proof of this involves constructing a complicated formula for the ρ_B^k in terms of families of non-intersecting lattice paths, from which one can also deduce periodicity and the other geometric homomesies of this map, first proved by Grinberg-R [GrRo15b, Thm. 32].

Noncommutative Liftings

Much of this story lifts to skew fields, where the variables are not assumed to commute.

- In this setting toggles are no longer involutions, but the NC analogue of ρ_B can be defined, and their inverses can be included in the study.
- Periodicity miraculously still appears to hold, though we have no proofs and computer experiments are much more challenging.
- In parallel with the lifting of ρ_J to ρ_B, there is a lifting of $\rho_{\mathcal{A}}$ via Stanley's Chain polytope to birational (*BAR-motion*) and NC (*NAR-motion*) [JR19+].
- The Stanley–Thomas word which we used to show periodicity and homomesy for $\rho_{\mathcal{A}}$ lifts all the way to the NC setting, where it still shows homomesy. However, it does not show periodicity outside the combinatorial realm, since it no longer losslessly encodes the labelings [JR20+].

Rowmotion on Fence Posets

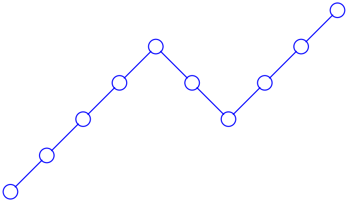
Fence posets

Let $\alpha = (\alpha_1, \ldots, \alpha_t)$ be a *composition* of $N - 1$ into t parts (all $\alpha_i \in \mathbb{P}$). The **fence poset F**(α) has $N = 1 + \sum \alpha_i$ elements (segments of lengths α_i) and covering relations

$$a_1 \lessdot a_2 \lessdot \cdots \lessdot a_{\alpha_1+1} \gtrdot a_{\alpha_1+2} \gtrdot \cdots \gtrdot a_{\alpha_1+\alpha_2+1} \lessdot \cdots$$

- The lattice of order ideals $J(F(\alpha))$ comes up in cluster algebras and can be used to define q-analogues of rational numbers.
- Rowmotion on order ideals and antichains of $F(\alpha)$ is currently being studied by a subgroup of a recent BIRS workshop on d.a.c.
- Bruce presented the problem based on the following result he proved.

$F(4,2,3) =$



Theorem (B. Sagan, Unpub)

Rowmotion on the antichains of $F(a-1, b-1)$ has the following properties.

(1) *All orbits have size $\ell = \mathrm{LCM}(a, b)$ except for one which has size $\ell + 1$.*

(2) *The number of orbits is $\mathrm{GCD}(a, b)$.*

(3) *The number of antichain elements in the orbits of size ℓ is $m = \dfrac{2ab - a - b}{\mathrm{GCD}(a, b)}$.*
The number of antichain elements in the other orbit is $m + 1$.

(4) *The total size of the order ideals in the orbits of size ℓ is $\dfrac{\ell(a+b-2)}{2}$.*
The total size of the order ideals in the orbit of length $\ell + 1$ is $\dfrac{(\ell+2)(a+b-2)}{2} + 1$.

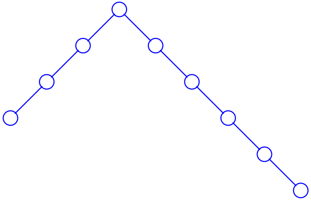
Antichain homomesies for rowmotion on two-segment fences

Theorem

Label the elements of $F(a-1, b-1)$ above by $1, 2, 3, \ldots, a$ (going up) then $a+1, \ldots, a+b-1$, and let χ_j be the indicator function of node j. Then

1. *The statistic $\chi_i - \chi_j$ is 0-mesic for i and j unshared elements in the same segment.*
2. *The statistics $a * \chi_1 + \chi_a$ and $b * \chi_{a+b-1} + \chi_a$ are 1-mesic.*

$F(3,5) =$



Unilink fences: $F(1, 1, \dots, 1)$

One of Bruce's initial questions was what about $F(1, 1, \dots, 1)$. It turned out that this case had already been explored. Let $\mathcal{Z}_n = F(1^{n-1})$ (with n elements).

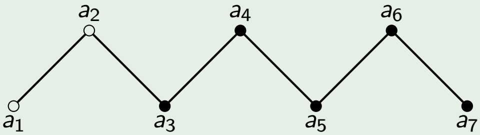
Proposition (Joseph–R.)

We have a bijection $\eta : \mathcal{I}_n \to J(\mathcal{Z}_n)$ *by*
$\eta(S) := \{a_i \mid i \in [n], i \text{ odd}, i \notin S\} \cup \{a_i \mid i \in [n], i \text{ even}, i \in S\}$.

Example

Let $n = 7$ and $S = 1001010 = \{1, 4, 6\}$. Then $a_1 \notin \eta(S)$ and $a_3, a_5, a_7 \in \eta(S)$ because $1 \in S$ and $3, 5, 7 \notin S$. Also, $a_2 \notin \eta(S)$ and $a_4, a_6 \in \eta(S)$, since $2 \notin S$ and $4, 6 \in S$.

$$1001010 \quad \stackrel{\eta}{\longmapsto}$$



Equivariant bijection between promotion on $\mathcal{Z}_n$ and toggling independent sets of P_n.

This map η is an equivariant bijection at the level of *toggles*, giving an equivariant bijection with *(Striker–Williams) promotion*, which toggles at each element of Z_n from left to right. Promotion and rowmotion are two examples of *Coxeter-toggling*; any two such maps are conjugate in the toggle group [Striker–Williams].

Proposition (Joseph–R.)

*For every $i \in [n]$, $\eta \circ \tau_i = t_i \circ \eta$. Thus, $\eta \circ \varphi = \mathrm{Pro} \circ \eta$, making η an **equivariant** bijection, as shown in the following commutative diagrams.*

$$
\begin{array}{ccc}
\mathcal{I}_n & \xrightarrow{\ \eta\ } & J(\mathcal{Z}_n) \\
\tau_i \downarrow & & \downarrow t_i \\
\mathcal{I}_n & \xrightarrow[\ \eta\ ]{} & J(\mathcal{Z}_n)
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathcal{I}_n & \xrightarrow{\ \eta\ } & J(\mathcal{Z}_n) \\
\varphi \downarrow & & \downarrow \mathrm{Pro} \\
\mathcal{I}_n & \xrightarrow[\ \eta\ ]{} & J(\mathcal{Z}_n)
\end{array}
$$

Homomesies for Coxeter toggling $J(Z_n)$

Theorem (Joseph–R.)

Let w be a Coxeter element in $\mathrm{Tog}(\mathcal{Z}_n)$*. Let* $\chi_{a_j} : J(Z_n) \to \{0,1\}$ *be the indicator function of* a_j*. Then on w-orbits in $J(Z_n)$, the following statistics are homomesic.*

- *If n is odd, then* $\chi_{a_j} - \chi_{a_{n+1-j}}$ *is 0-mesic for every* $j \in [n]$*. Also* $2\chi_{a_1} - \chi_{a_2}$ *and* $2\chi_{a_n} - \chi_{a_{n-1}}$ *are both 1-mesic.*
- *If n is even, then* $\chi_{a_j} + \chi_{a_{n+1-j}}$ *is 1-mesic for every* $j \in [n]$*. Also* $2\chi_{a_1} - \chi_{a_2}$ *is 1-mesic and* $2\chi_{a_n} - \chi_{a_{n-1}}$ *is 0-mesic.*

Proof.

From the definition of η, it is clear that for any $S \in \mathcal{I}_n$,

$$\chi_{a_j}(\eta(S)) = \begin{cases} \chi_j(S) & \text{if } j \text{ is even} \\ 1 - \chi_j(S) & \text{if } j \text{ is odd} \end{cases}.$$

The rest of the proof follows the from equivariant bijection and our earlier work on toggling $\mathcal{I}(P_n)$ ■

Summary and Take Aways

- Studying dynamics on objects in algebraic combinatorics is interesting, particularly with regard to our THEMES:
1) *Periodicity/order*; 2) *Orbit structure*; 3) *Homomesy* 4) *Equivariant bijections*

- Examples of cyclic sieving are also ripe for homomesy hunting.

- Situations in which maps can be built out of toggles seem particularly fruitful.

- Combinatorial objects are often discrete "shadows" of continuous PL objects, which in turn reflect algebraic dynamics. But combinatorial tools are still frequently useful, even at this level.

Slides for this talk are available online at

Google "Tom Roby".

Thanks very much for coming to this talk!

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