Castelnuovo–Mumford regularity and excited Young diagrams

Colleen Robichaux
UCLA

joint work with Jenna Rajchgot and Anna Weigandt
Michigan State University
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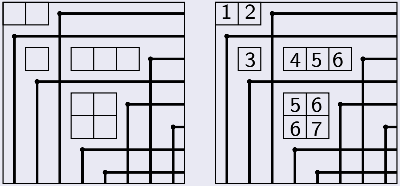
Diagrams and Reduced Words

The Rothe diagram for $w \in S_n$ is the collection of boxes

$$D(w) = \{(i,j) \in [n] \times [n] : j < w(i), i < w^{-1}(j)\}.$$

Filling boxes in row i of $D(w)$ with $i, i+1, \ldots$ gives a reduced word for $D(w)$ (reading labels bottom to top, left to right).

Example: $w = 31726845 = s_6 s_7 s_5 s_6 s_3 s_4 s_5 s_6 s_1 s_2$



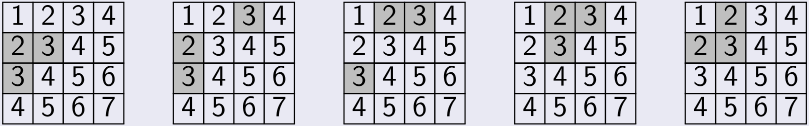
1	2	3	4	5	6	7	8
2	3	4	5	6	7	8	9
3	4	5	6	7	8	9	10
4	5	6	7	8	9	10	11
5	6	7	8	9	10	11	12
6	7	8	9	10	11	12	13
7	8	9	10	11	12	13	14
8	9	10	11	12	13	14	15

Reduced Words

Question

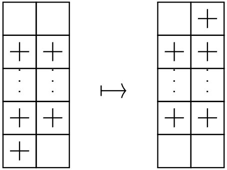
What are all of the ways to shade entries in this labelled grid to give a reduced word for w (reading labels bottom to top, left to right)?

Example: $w = 1432 = s_3 s_2 s_3$



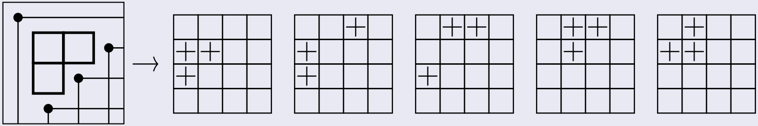
Reduced Pipe dreams (Bergeron–Billey '94)

The set of reduced pipe dreams $RPD(w)$ for $w \in S_n$ is the set of diagrams obtainable through successive moves



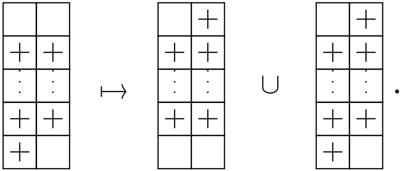
starting from left justified $D(w)$ in $[n] \times [n]$.

Example: $RPD(w)$ for $w = 1432$



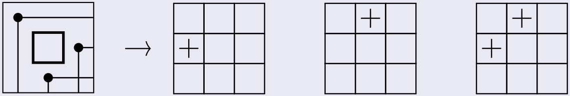
Pipe dreams (Fomin–Kirillov '94)

The set of pipe dreams $PD(w)$ for $w \in S_n$ is the set of diagrams obtainable through successive moves



starting from left justified $D(w) \subseteq [n] \times [n]$.

Example: $PD(w)$ for $w = 132$



Grothendieck polynomials

Here $PD(w)$ corresponds shadings of the labelled grid such that the *Demazure product* of the corresponding entries gives w.

These $PD(w)$ generate Grothendieck polynomials:

Theorem [Fomin–Kirillov '94]

$$\mathfrak{G}_w(x_1,\ldots,x_n) = \sum_{P\in PD(w)} (-1)^{(\#+'s)-\ell(w)} x^{wt(P)}$$

Here $x^{wt(P)} = x_1^{\#+'\text{s in row } 1} \ldots x_n^{\#+'\text{s in row } n}$.

Problem

Give an easily computable formula for $\deg(\mathfrak{G}_w(x_1,\ldots,x_n))$, where $w \in S_n$.

Degrees of Grothedieck polynomials

Initial work of Rajchgot-Ren-R.-St. Dizier-Weigandt '19 proved a formula for Grassmannian permutations.

Theorem [Pechenik-Speyer-Weigandt '21]

For $w \in S_n$,

$$\deg(\mathfrak{G}_w(x_1, \ldots, x_n)) = \mathrm{raj}(w) = \sum_{i \in [n]} r_i.$$

Here r_i counts the number of terms in $(w_i, w_{i+1}, \ldots, w_n)$ excluded from the longest increasing subsequence in w starting with w_i.

Example: $w = 2341756$

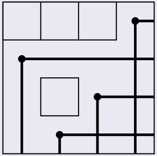
We compute $\deg(\mathfrak{G}_w(x_1, \ldots, x_n)) = 2 + 2 + 2 + 1 + 2 + 0 + 0 = 9$.

Matrix Schubert varieties

Matrix Schubert variety $\overline{X}_w$ has defining ideal

$$I_w = \langle r_w(i,j) + 1 \text{ minors of } \mathbf{z}_{i \times j}(w) \rangle.$$

Example: $w = 4132$



$$\xrightarrow{r_w} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix} \xrightarrow{\mathbf{z}(v)} \begin{pmatrix} z_{11} & z_{12} & z_{13} & z_{14} \\ z_{21} & z_{22} & z_{23} & z_{24} \\ z_{31} & z_{32} & z_{33} & z_{34} \\ z_{41} & z_{42} & z_{43} & z_{44} \end{pmatrix}$$

$$I_w = \langle z_{11},\ z_{12}, z_{13},\ z_{21}z_{32} - z_{22}z_{31},\ z_{11}z_{22} - z_{12}z_{21},\ z_{11}z_{32} - z_{12}z_{31} \rangle$$

We can study $\mathbb{C}[\overline{X}_w] = \mathbb{C}[\mathbf{z}]/I_w$.

Minimal free resolution

Consider the coordinate ring S/I. The **minimal free resolution**

$$0 \to \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{l,j}} \to \cdots \to \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{0,j}} \to S/I \to 0.$$

The **Castelnuovo–Mumford regularity** of S/I

$$\operatorname{reg}(S/I) := \max\{j - i \mid \beta_{i,j} \neq 0\}.$$

Combining results of Fulton '92, Knutson–Miller '05, and Buch '02:

Theorem

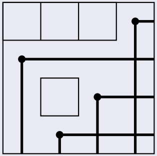
$$\operatorname{reg}(\mathbb{C}[\overline{X}_w]) = \deg(\mathfrak{G}_w(x_1, \ldots, x_n)) - \ell(w).$$

Kazhdan–Lusztig varieties of Woo–Yong '06

Kazhdan–Lusztig variety $\mathcal{N}_{v,w}$ has defining ideal

$$I_{v,w} = \langle r_w(i,j) + 1 \text{ minors of } \mathbf{z}_{i\times j}(v) \rangle.$$

Example: $w = 4132$, $v = 4231$



$$\xrightarrow{r_w} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix} \xrightarrow{\mathbf{z}(v)} \begin{pmatrix} z_{11} & z_{12} & z_{13} & 1 \\ z_{21} & 1 & 0 & 0 \\ z_{31} & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

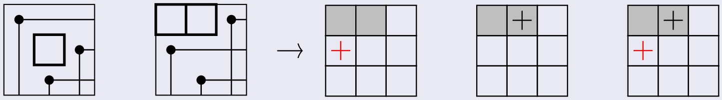
$$I_{v,w} = \langle z_{11},\ z_{12}, z_{13},\ z_{11} - z_{12}z_{21},\ -z_{12}z_{31},\ -z_{31} \rangle$$

Matrix Schubert varieties are examples of KL varieties.

Unspecialized Grothendiecks and pipe dreams

The set of unspecialized pipe dreams $PD(v,w)$ for v, w is the set of pipe dreams for w supported on left justified $D(v)$.

Example: $PD(v,w)$ for $w = 132, v = 312$.



so $\#PD(v,w) = 1$.

Defined by Woo–Yong, the unspecialized Grothendiecks $\mathfrak{G}_{v,w}$ are

$$\mathfrak{G}_{v,w}(x_1,\ldots,x_n) = \sum_{P\in PD(v,w)} (-1)^{(\#+'s)-\ell(w)} x^{wt(P)}$$

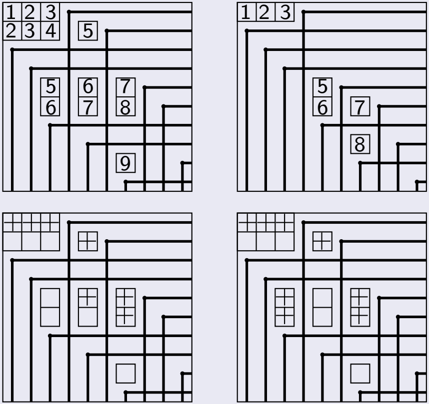
Theorem

$$\mathrm{reg}(\mathbb{C}[\mathcal{N}_{v,w}]) = \deg\ (\mathfrak{G}_{v,w}) - \ell(w).$$

Correspondence with subwords

The set $PD(v, w)$ bijects with subwords of v for w under the Demazure product.

Example: $v = 46128935(10)7$ and $w = 412368597(10)$.



321-avoiding permutations

321-avoiding permutations are permutations such that there is no $i < j < k$ such that $w_i > w_j > w_k$. For example, $w = 1746235$ is not 321-avoiding.

To simplify the problem of computing $\deg(\mathfrak{G}_{v,w})$, we restrict to 321-avoiding permutations v, w.

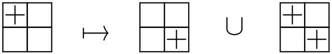
Helpful facts:

- 321-avoiding permutations are totally commutative.
- 321-avoiding permutations naturally correspond with skew-Young diagrams.

K-skew excited Young diagrams

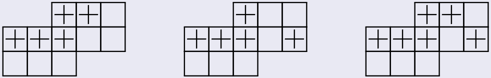
Let $v \geq w \in S_n$ be 321-avoiding. We can associate v with a skew Young diagram $\mathcal{R}_v$. Mark positions in $\mathcal{R}_v$ with $+$'s corresponding to the earliest subword of w in v.

A **K-skew excited Young diagram** of w in v is a diagram obtainable by applying K-excited moves



to the initial diagram for w. Call the set of these $\mathrm{SEYD}(v, w)$.

Example: Below are $\mathrm{SEYD}(v, w)$ for $v = 47128356$ and $w = 14273568$.



Unspecialized Grothendieck polynomials

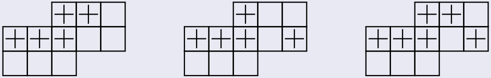
Restricting to 321-avoiding permutations:

$$\mathfrak{G}_{v,w}(x_1,\ldots,x_n) = \sum_{P \in \mathrm{SEYD}(v,w)} (-1)^{\#P-\ell(w)} x^{wt(P)}.$$

This gives

$$\deg(\mathfrak{G}_{v,w}) = \max\{\#P \mid P \in \mathrm{SEYD}(v,w)\}$$
$$\mathrm{reg}(\mathbb{C}[\mathcal{N}_{v,w}]) = \max\{\#P \mid P \in \mathrm{SEYD}(v,w)\} - \ell(w).$$

Example: $\mathrm{SEYD}(v,w)$ for $v = 47128356$ and $w = 14273568$.



Thus $\deg(\mathfrak{G}_{v,w})) = 6$, so $\mathrm{reg}(\mathbb{C}[\mathcal{N}_{v,w}]) = 6 - 5 = 1$.

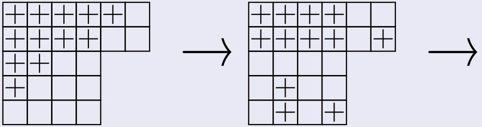
Computing CM-regularity of certain KL varieties

Theorem [Rajchgot–R.–Weigandt '23]

For $v_\rho, w_\nu \in S_n$ Grassmannian with same descent,

$$\mathrm{reg}(\mathbb{C}[\mathcal{N}_{v_\rho,w_\nu}]) = \sum_{i=1}^{n} \#\mathrm{antidiag}(\mathrm{T}(\rho,\nu)|_{\geq \mathrm{i}}).$$

Example: $v_{(5,4,2,1,0)}, w_{(6,6,4,4,4)} \mapsto T((5,4,2,1,0),(6,6,4,4,4))$



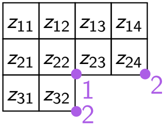
0	0	0	0	1
0	0	0	0	
1	2			
1				

gives $\mathrm{reg}(\mathbb{C}[\mathcal{N}_{v_\rho,w_\nu}]) = 1 + 2 + 1 = 4.$

Application: one-sided mixed ladder determinantal ideals

A ladder L is a Young diagram filled with indeterminates z_{ij}. The ideal $I_L \subseteq \mathbb{C}[L]$ is generated by NW minors of L determined by marked points on its boundary. This defines the one-sided mixed ladder determinantal variety $\mathbb{C}[L]/I_L$.

For example, we can take L:



These are Grassmannian KL-varieties

$$\mathbb{C}[L]/I_L \cong \mathbb{C}[\mathcal{N}_{v_\rho, w_\nu}].$$

Corollary [Rajchgot–R.–Weigandt '23]

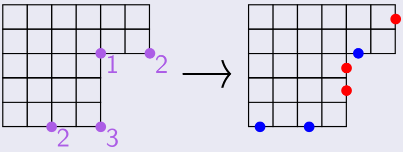
$$\mathrm{reg}(\mathbb{C}[L]/I_L) = \sum_{i=1}^{n} \#\mathrm{antidiag}(\mathrm{T}(\rho, \nu)|_{\geq \mathrm{i}}).$$

One-sided ladders and lattice paths

To each one-sided ladder, we can associate families of non-intersecting NE-oriented lattice paths.

The marked points on horizontal edges determine starting points of the paths and the marked points on vertical edges determine ending points of the paths.

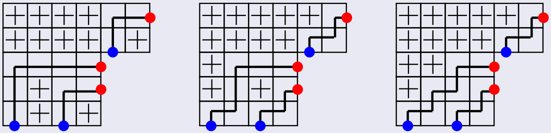
Example: Constructing lattice paths from L



Bijection between lattice paths and SEYD

Lattice paths in the region L naturally biject with SEYD's by drawing $+$'s in each cell not occupied by a path.

Example



In this setting, maximizing K-excited moves translates to maximizing elbows  in the region L.

CM-regularity and lattice paths

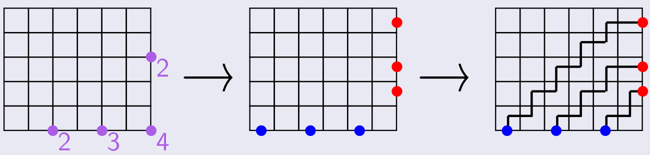
Reframing Conca '95, we compute the regularities of these determinantal ideals I_L using lattice paths:

Theorem [Conca '95, Krattenhaler–Ghorpade '15]

For I_L cogenerated by NW-minors of an $n \times m$ matrix

$$\mathrm{reg}(\mathbb{C}[L]/I_L) = \max_{P \in NILP(L)} \#\{\text{elbows } \lrcorner \text{ in } P\}.$$

Example: $P \in NILP(L)$ with maximal number of elbows



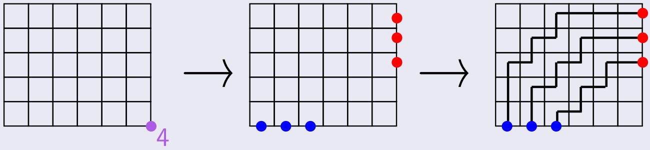
gives $\mathrm{reg}(\mathbb{C}[L]/I_L) = 7$.

CM-regularity and lattice paths

Using this lattice path formula, we can re-derive the formula for the classical case, i.e., for I_L generated by the $(k+1)$-minors of an $n \times m$ rectangle L:

$$\mathrm{reg}(\mathbb{C}[L]/I_L) = nm - (n-k)(m-k) - k \cdot \max(n,m).$$

Example: $P \in NILP(L)$ with maximal number of elbows



gives $\mathrm{reg}(\mathbb{C}[L]/I_L) = 6 = 5 \cdot 6 - 2 \cdot 3 - 3 \cdot 6$.

CM-regularity of one-sided ladders and lattice paths

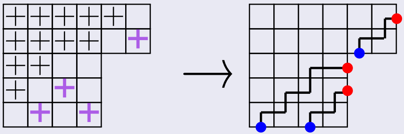
Following work of Krattenhaler–Prohaska '99 and Ghorpade '02:

Theorem [Krattenhaler–Ghorpade '15, Rajchgot–R.–Weigandt '23]

For a one-sided ladder L

$$\mathrm{reg}(\mathbb{C}[L]/I_L) = \max_{P \in NILP(L)} \#\{\text{elbows } \lrcorner \text{ in } P\}.$$

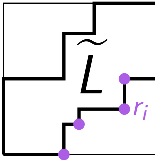
Example: $P \in NILP(L)$ with maximal number of elbows



gives $\mathrm{reg}(\mathbb{C}[L]/I_L) = 4 = \mathrm{reg}(\mathbb{C}[\mathcal{N}_{v_\rho, w_\gamma}])$.

Two-sided mixed ladder determinantal ideals

A two-sided ladder $\widetilde{L}$ is a skew-Young diagram filled with z_{ij}'s. $I_{\widetilde{L}}$ is the ideal generated by the NW r_i minors of $\widetilde{L}$. This defines the two-sided mixed ladder determinantal variety $\mathbb{C}[\widetilde{L}]/I_{\widetilde{L}}$.



Theorem [Escobar-Fink-Rajchgot-Woo ('24+)]

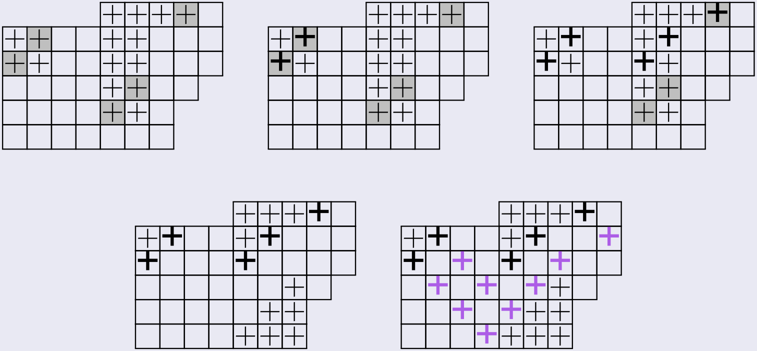
For particular $v, w \in S_n$ 321-avoiding

$$\mathbb{C}[\widetilde{L}]/I_{\widetilde{L}} \cong \mathbb{C}[\mathcal{N}_{v,w}].$$

We give an algorithm to construct a maximal $P \in \mathrm{SEYD}(v, w)$ for $v, w \in S_n$ 321-avoiding.

Algorithm example computing $\mathrm{reg}(\mathbb{C}[\mathcal{N}_{v,w}])$

Example: Constructing a maximal skew-excited Young diagram given by certain 321-avoiding $v, w \in S_{15}$.



gives $\mathrm{reg}(\mathbb{C}[\mathcal{N}_{v,w}]) = 9$.

CM-regularity of two-sided ladders and lattice paths

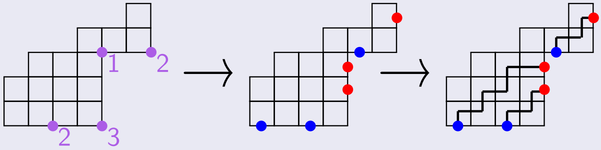
Generalizing work of Krattenhaler–Ghorpade '15 combined with Woo–Yong '12, we can compute $\mathrm{reg}(\mathbb{C}[\widetilde{L}]/I_{\widetilde{L}})$ using lattice paths:

Theorem [Krattenhaler–Ghorpade '15, R. '23]

For a two-sided ladder $\widetilde{L}$

$$\mathrm{reg}(\mathbb{C}[\widetilde{L}]/I_{\widetilde{L}}) = \max_{P \in NILP(\widetilde{L})} \#\{\text{unforced elbows in } P\}.$$

Example: $P \in NILP(\widetilde{L})$ with maximal number of elbows



gives $\mathrm{reg}(\mathbb{C}[\widetilde{L}]/I_{\widetilde{L}}) = 3$.

Conclusions

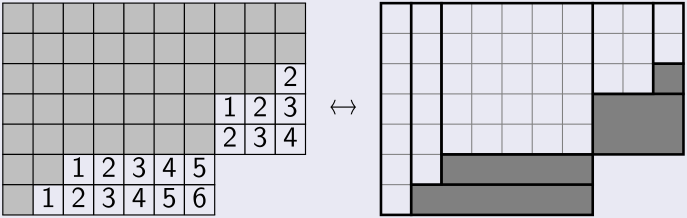
- We can express $\mathrm{reg}(\mathbb{C}[\mathcal{N}_{v,w}])$ in terms of excited Young diagrams and $\ell(w)$.
- For v, w Grassmannian, we obtain a tableaux-based formula to compute $\mathrm{reg}(\mathbb{C}[\mathcal{N}_{v,w}])$.
- For v, w 321-avoiding, we obtain an algorithm to compute $\mathrm{reg}(\mathbb{C}[\mathcal{N}_{v,w}])$.
- We connect our formulas to the combinatorics of lattice paths to compute regularities for ladder determinantal ideals.

Finding μ maximal

Let $R(\lambda) = R_1 \cup \ldots \cup R_r$ partition the columns of λ according to length and set $\lambda^{(k)} = \cup_{j=1}^{k} R_j$. Then if $(i, \lambda_i) \in R_{h_i}$,

$$\mu_i = \lambda_i + \mathrm{sv}(\lambda^{(h_i - 1)}).$$

Example: $\lambda = (10, 10, 9, 7, 7, 2, 1)$



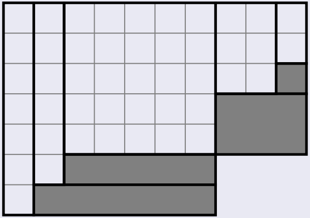
Regularity Formula for w_λ Grassmannian

Theorem [Rajchgot-Ren-R-St.Dizier-Weigandt (2019)]

Suppose $w_\lambda \in S_n$ has descent k. Then

$$\operatorname{reg}(S/I_{w_\lambda}) = \sum_{1\leq i\leq k} \operatorname{sv}(\lambda^{(h_i-1)})$$

Example: $\lambda = (10, 10, 9, 7, 7, 2, 1)$



Thus $\operatorname{reg}(S/I_{w_\lambda})$
$= 1 + 3 + 3 + 5 + 6 = 18$

Previous results

Then following special cases were known:

- classical determinantal ideals (Gräbe '88)
  - For $(k+1)$-minors of an $n \times m$ matrix, the regularity is $nm - (n-k)(m-k) - k \cdot \max(n, m)$
- Ideals cogenerated by NW-minors of an $n \times m$ matrix (Conca '95)
  - given by RSK
  - extended/reframed by Krattenhaler–Ghorpade '15 in terms of lattice paths.
    - In fact, $\mathcal{K}(X(L); \mathbf{t})$ is determinantal in terms of these lattice paths, as established by Abhyankar–Kulkarni '89 and Herzog–Trung '92.

Grassmannian permutations

A permutation $w \in S_n$ is Grassmannian if it has a unique descent k, i.e. if $i \neq k$, then $w_i < w_{i+1}$. To each Grassmannian permutation $w \in S_n$, we can uniquely associate a partition λ with k parts.

Example: $w = 24813567$ and $\lambda = (5, 2, 1)$