Superspace, Vandermondes, and representations

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(Joint with Andy Wilson)

UCSD

Symmetric polynomials

S_n acts on $\mathbb{Q}[x_1, \ldots, x_n]$ by variable permutation

$f \in \mathbb{Q}[x_1, \ldots, x_n]$ is *symmetric* if $w \cdot f = f$ for all $w \in S_n$

$$e_d := \sum_{1 \leq i_1 < \cdots < i_d \leq n} x_{i_1} \cdots x_{i_d}$$

Thm: [Newton] $e_1, \ldots, e_n$ is an algebraically independent generating set of $\mathbb{Q}[x_1, \ldots, x_n]^{S_n}$.

Coinvariant ring

The *coinvariant ring* is

$$R_n := \mathbb{Q}[x_1, \ldots, x_n]/\langle \mathbb{Q}[x_1, \ldots, x_n]_+^{S_n} \rangle$$
$$= \mathbb{Q}[x_1, \ldots, x_n]/\langle e_1, \ldots, e_n \rangle.$$

Thm: [Chevalley] As ungraded S_n-modules,

$$R_n \cong \mathbb{Q}[S_n].$$

Thm: [Lusztig-Stanley] As graded S_n-modules,

$$\mathrm{grFrob}(R_n; q) = \sum_{T \in \mathrm{SYT}(n)} q^{\mathrm{maj}(T)} s_{\mathrm{shape}(T)}$$

Tableau statistics

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 8 \\ \hline 2 & 5 & 7 \\ \cline{1-3} 6 \\ \cline{1-1} \end{array}$$

- $\mathrm{maj}(T) = 1 + 4 + 5 = 10$
- $\mathrm{des}(T) = 3$

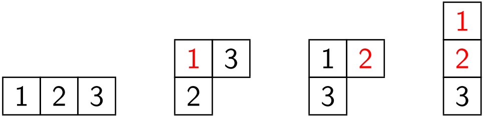
Lusztig-Stanley Theorem

$$R_n = \mathbb{Q}[x_1, \ldots, x_n]/\langle e_1, e_2, \ldots, e_n \rangle.$$

Thm: [Lusztig-Stanley] The *graded* S_n-isomorphism type of R_n is

$$\mathrm{grFrob}(R_n; q) = \sum_{T \in \mathrm{SYT}(n)} q^{\mathrm{maj}(T)} s_{\mathrm{shape}(T)}$$

Ex: $n = 3$



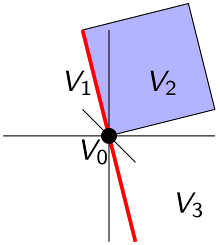
$$\mathrm{grFrob}(R_3; q) = q^0 s_{(3)} + q^1 s_{(2,1)} + q^2 s_{(2,1)} + q^3 s_{(1,1,1)}$$

Flags

Def: A *flag* in $\mathbb{C}^n$ is a sequence

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n$$

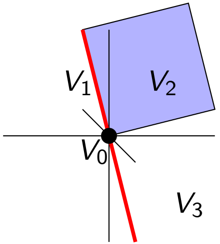
of subspaces with $\dim V_i = i$.



$\mathcal{F}\ell(n) = \{\text{all flags in } \mathbb{C}^n\} = GL_n(\mathbb{C})/B$ is the *flag variety*.

Cohomology

Def: $\mathcal{F}\ell(n) = GL_n(\mathbb{C})/B$



Thm: [Borel] The cohomology of $\mathcal{F}\ell(n)$ is

$$H^\bullet(\mathcal{F}\ell(n)) = R_n = \mathbb{Q}[x_1, \ldots, x_n]/\langle e_1, \ldots, e_n \rangle$$

Cohomology and R_n

Thm: [Borel] The cohomology of $\mathcal{F}\ell(n)$ is

$$H^\bullet(\mathcal{F}\ell(n)) = R_n = \mathbb{Q}[x_1, \ldots, x_n]/\langle e_1, \ldots, e_n \rangle.$$

$$\mathrm{Hilb}(R_n; q) = [n]!_q = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1})$$

$$\mathrm{Hilb}(R_4; q) = [4]!_q = 1 + 3q + 5q^2 + 6q^3 + 5q^4 + 3q^5 + q^6$$

$\mathcal{F}\ell(n)$ **is a smooth compact variety! So** ...

- The Hilbert series of R_n is palindromic. (Poincaré Duality)
- The Hilbert series of R_n is unimodal. (Hard Lefschetz)

Quotient Rings

Q: How do we study $R_n = \mathbb{Q}[x_1, \ldots, x_n]/\langle e_1, \ldots, e_n \rangle$?

Basic Problem: If $f \in \mathbb{Q}[x_1, \ldots, x_n]$ and $I \subseteq \mathbb{Q}[x_1, \ldots, x_n]$ is an ideal, how to decide if

$$f + I = 0 \Leftrightarrow f \in I?$$

Idea: Give an *alternative model* for R_n as a *subspace* of $\mathbb{Q}[x_1, \ldots, x_n]$.

Vandermondes and Modules

Def: The *Vandermonde* $\delta_n \in \mathbb{Q}[x_1, \ldots, x_n]$ is

$$\begin{aligned}\delta_n := &\prod_{1 \leq i < j \leq n} (x_i - x_j) \\ &= \varepsilon_n \cdot (x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^1 x_n^0)\end{aligned}$$

where $\varepsilon_n = \sum_{w \in \mathfrak{S}_n} \mathrm{sign}(w) \cdot w \in \mathbb{Q}[S_n]$.

Def: $V_n =$ smallest subspace of $\mathbb{Q}[x_1, \ldots, x_n]$ containing δ_n and closed under

$$\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_n$$

(graded S_n-module).

Bridge Thm: As graded S_n-modules, $V_n \cong R_n$.

Generalized Coinvariant Algebra

Def: [Haglund-R-Shimozono] For $k \leq n$, let $I_{n,k} \subseteq \mathbb{Q}[x_1, \ldots, x_n]$ be

$$I_{n,k} := \langle x_1^k, \ldots, x_n^k, e_n, e_{n-1}, \ldots, e_{n-k+1} \rangle$$

and let $R_{n,k} := \mathbb{Q}[x_1, \ldots, x_n]/I_{n,k}$.

- $k = n \Rightarrow$ recover $R_{n,n} = R_n$.
- $k = 1 \Rightarrow$ we have $R_{n,1} = \mathbb{Q}$.
- $R_{n,k} \cong \mathbb{Q}[\mathcal{OP}_{n,k}]$, where $\mathcal{OP}_{n,k}$ = ordered set ptns of $[n]$ into k blocks

Thm: [Haglund-R-Shimozono] For $k \leq n$ we have

$$\mathrm{grFrob}(R_{n,k}; q) = \sum_{T \in \mathrm{SYT}(n)} q^{\mathrm{maj}(T)} \begin{bmatrix} n - \mathrm{des}(T) - 1 \\ n - k \end{bmatrix}_q s_{\mathrm{shape}(T)}.$$

Diagonal Coinvariants

S_n acts on $\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ diagonally.

$$w.x_i := x_{w(i)}, \quad w.y_i := y_{w(i)}$$

Def: [Garsia-Haiman] The *diagonal coinvariants* are

$$DR_n := \mathbb{Q}[\mathbf{x}_n, \mathbf{y}_n] / \langle \mathbb{Q}[\mathbf{x}_n, \mathbf{y}_n]_+^{S_n} \rangle.$$

Thm: [Haiman] As an *ungraded* S_n-module, DR_n is isomorphic to the parking representation of S_n (up to sign twist) so that $\dim(DR_n) = (n+1)^{n-1}$.

Shuffle Theorem

Thm: [Haiman] As a *bigraded* S_n-module, we have

$$\mathrm{grFrob}(DR_n; q, t) = \nabla e_n$$

Problem: Find the Schur expansion $\nabla e_n = \sum_{\lambda \vdash n} c_\lambda(q, t) s_\lambda$.

Shuffle Thm: [Carlsson-Mellit] The *monomial* expansion of ∇e_n is

$$\nabla e_n = \sum_{P \in \mathrm{Park}_n} q^{\mathrm{area}(P)} t^{\mathrm{dinv}(P)} F_{\mathrm{iDes}(P)}$$

Delta Conjecture

Shuffle Thm: [Carlsson-Mellit] The *monomial* expansion of ∇e_n is

$$\nabla e_n = \sum_{P \in \mathrm{Park}_n} q^{\mathrm{area}(P)} t^{\mathrm{dinv}(P)} F_{\mathrm{iDes}(P)}$$

Delta Conj: [Haglund-Remmel-Wilson] Let $k \leq n$. We have

$$\Delta'_{e_{k-1}} e_n = \mathrm{Rise}_{n,k}(\mathbf{x}; q, t) = \mathrm{Val}_{n,k}(\mathbf{x}; q, t).$$

- Gives Shuffle Theorem when $k = n$
- First equality $\Delta'_{e_{k-1}} e_n = \mathrm{Rise}_{n,k}(\mathbf{x}; q, t)$ proven by D'Adderio-Mellit, Blasiak-Haiman-Morse-Pun-Seelinger.

Delta Conjecture

Delta Conj: [Haglund-Remmel-Wilson] Let $k \leq n$. We have

$$\Delta'_{e_{k-1}} e_n = \mathrm{Rise}_{n,k}(\mathbf{x}; q, t) = \mathrm{Val}_{n,k}(\mathbf{x}; q, t).$$

Thm: [Haglund-Garsia-Remmel-R-Shimozono-Wilson-Yoo] We have

$$\Delta'_{e_{k-1}} e_n \mid_{t=0} = \mathrm{Rise}_{n,k}(\mathbf{x}; q, 0) = \mathrm{Rise}_{n,k}(\mathbf{x}; 0, q) = \mathrm{Val}_{n,k}(\mathbf{x}; q, 0) = \mathrm{Val}_{n,k}(\mathbf{x}; 0, q)$$

Thm: [Haglund-R-Shimozono] We have

$$\mathrm{grFrob}(R_{n,k}; q) = (\mathrm{rev}_q \circ \omega) \Delta'_{e_{k-1}} e_n \mid_{t=0}$$

Question: Subspace model for $R_{n,k}$?

Superspace

Defn: *Superspace* of rank n is the $\mathbb{Q}$-algebra Ω_n with generators

$$x_1, x_2, \ldots, x_n, \theta_1, \theta_2, \ldots, \theta_n$$

and relations

$$x_i x_j = x_j x_i, \quad x_i \theta_j = \theta_j x_i, \quad \theta_i \theta_j = -\theta_j \theta_i.$$

Fact: The algebra Ω_n is *doubly* graded:

- x-degree is *bosonic* degree
- θ-degree is *fermionic* degree.

Superspace Vandermondes

Defn: [R-Wilson] Let $k \leq n$. The *superspace Vandermonde* is

$$\delta_{n,k} := \varepsilon_n \cdot (x_1^{k-1} \cdots x_{n-k}^{k-1} x_{n-k+1}^{k-1} x_{n-k+2}^{k-2} \cdots x_{n-1}^{1} x_n^{0} \theta_1 \cdots \theta_{n-k})$$

Ex: We have

$$\begin{aligned} \delta_{3,2} &= \varepsilon_3 \cdot (x_1^1 x_2^1 x_3^0 \theta_1) \\ &= x_1 x_2 \theta_1 - x_1 x_2 \theta_2 - x_1 x_3 \theta_1 + x_1 x_3 \theta_3 + x_2 x_3 \theta_2 - x_2 x_3 \theta_3 \end{aligned}$$

N.B. $\delta_{n,k}$ is a *nonzero* element of Ω_n.

Superspace VanderModules

Defn: [R-Wilson] Let $k \leq n$. The *superspace Vandermonde* is

$$\delta_{n,k} := \varepsilon_n \cdot (x_1^{k-1} \cdots x_{n-k}^{k-1} x_{n-k+1}^{k-1} x_{n-k+2}^{k-2} \cdots x_{n-1}^{1} x_n^{0} \theta_1 \cdots \theta_{n-k})$$

Defn: [R-Wilson] Let $V_{n,k}$ be the smallest subspace of Ω_n containing $\delta_{n,k}$ which is closed under

$$\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_n$$

(graded S_n-module under x-degree).

Thm: [R-Wilson] For $k \leq n$ we have

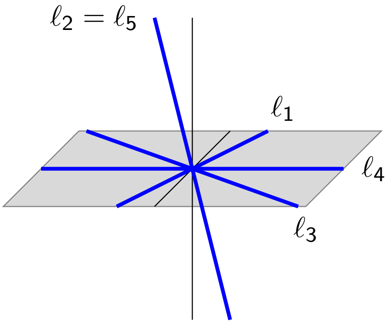
$$\mathrm{grFrob}(V_{n,k}; q) = (\mathrm{rev}_q \circ \omega)\mathrm{grFrob}(R_{n,k}; q) = \Delta'_{e_{k-1}} e_n \mid_{t=0} .$$

Q: What about *geometry*?

Spanning Lines

Def: [PR] For $k \leq n$, let $X_{n,k}$ be the space of *line configurations*

$$X_{n,k} := \{(\ell_1, \ldots, \ell_n) : \ell_i \text{ a line in } \mathbb{C}^k \text{ and } \ell_1 + \cdots + \ell_n = \mathbb{C}^k\}.$$



$\in X_{5,3}$

Thm: [Pawlowski-R] We have $H^\bullet(X_{n,k}) = R_{n,k}$.

Geometric Disappointment

Def: [PR] For $k \leq n$, let $X_{n,k}$ be the space of *line configurations*

$$X_{n,k} := \{(\ell_1, \ldots, \ell_n) : \ell_i \text{ a line in } \mathbb{C}^k \text{ and } \ell_1 + \cdots + \ell_n = \mathbb{C}^k\}.$$

Thm: [Pawlowski-R] We have $H^\bullet(X_{n,k}) = R_{n,k}$.

$X_{n,k}$ is smooth, but not compact!

$\mathrm{Hilb}(R_{n,k}; q)$ is not in general palindromic:

$$\mathrm{Hilb}(R_{3,2}; q) = 1 + 3q + 2q^2.$$

Q: Can this be salvaged?

Antisymmetric differentiation

For $1 \leq i \leq n$, let $\partial/\partial\theta_i$ be the $\mathbb{Q}[x_1, \ldots, x_n]$-endomorphism of Ω_n

$$\partial/\partial\theta_i : \theta_{j_1} \cdots \theta_{j_r} \mapsto \begin{cases} (-1)^{s-1}\theta_{j_1} \cdots \widehat{\theta_{j_s}} \cdots \theta_{j_r} & \text{if } i = j_s \\ 0 & \text{if } i \neq j_1, \ldots, j_s \end{cases}$$

$$(\partial/\partial\theta_2) \cdot (\theta_2\theta_4\theta_6) = \theta_4\theta_6$$
$$(\partial/\partial\theta_2) \cdot (\theta_1\theta_2\theta_6) = -\theta_1\theta_6$$
$$(\partial/\partial\theta_2) \cdot (x_1^2 x_4\theta_1\theta_2\theta_6) = -x_1^2 x_4\theta_1\theta_6$$

Rmk: The assignment $x_i \mapsto \partial/\partial x_i$ and $\theta_i \mapsto \partial/\partial\theta_i$ is an action of Ω_n on itself.

Doubly Graded Rings

Defn: [R-Wilson] For $k \leq n$, let $W_{n,k}$ be the smallest linear subspace of Ω_n which

- contains the Superspace Vandermonde $\delta_{n,k}$,
- is closed under $\partial/\partial x_1, \ldots, \partial/\partial x_n$, and
- is closed under $\partial/\partial \theta_1, \ldots, \partial/\partial \theta_n$.

A *doubly*-graded S_n-module *and* a doubly-graded ring:

$$W_{n,k} \cong \Omega_n / \mathrm{ann}(\delta_{n,k}).$$

Ex: The doubly graded S_4-module $W_{4,2}$ looks like

$$\theta\text{-deg} \begin{pmatrix} s_4 & s_4 + s_{31} & s_4 + s_{31} + s_{22} & s_{31} \\ s_{31} & 2s_{31} + s_{22} + s_{211} & s_{31} + s_{22} + 2s_{211} & s_{211} \\ s_{211} & s_{22} + s_{211} + s_{1111} & s_{211} + s_{1111} & s_{1111} \end{pmatrix}$$

$$x\text{-deg}$$

The modules $\mathbb{W}_{n,k}$

Defn: For $k \leq n$, let $\mathbb{W}_{n,k}$ be the smallest linear subspace of Ω_n containing $\delta_{n,k}$ which is closed under $\partial/\partial x_i$ and $\partial/\partial \theta_i$.

$$\mathbb{W}_{4,2} \leftrightarrow \begin{pmatrix} s_4 & s_4 + s_{31} & s_4 + s_{31} + s_{22} & s_{31} \\ s_{31} & 2s_{31} + s_{22} + s_{211} & s_{31} + s_{22} + 2s_{211} & s_{211} \\ s_{211} & s_{22} + s_{211} + s_{1111} & s_{211} + s_{1111} & s_{1111} \end{pmatrix}$$

- The maximum x-degree of $\mathbb{W}_{n,k}$ is $\binom{k}{2} + (n-k)(k-1)$.
- The maximum θ-degree of $\mathbb{W}_{n,k}$ is $n-k$.
- The θ-degree $n-k$ part of $\mathbb{W}_{n,k}$ is $V_{n,k}$.
- The θ-degree 0 part of $\mathbb{W}_{n,k}$ is $R_{n,k}$.

Symmetries of $\mathbb{W}_{n,k}$

Defn: For $k \leq n$, let $\mathbb{W}_{n,k}$ be the smallest linear subspace of Ω_n containing $\delta_{n,k}$ which is closed under $\partial/\partial x_i$ and $\partial/\partial\theta_i$.

$$\mathbb{W}_{4,2} \leftrightarrow \begin{pmatrix} s_4 & s_4 + s_{31} & s_4 + s_{31} + s_{22} & s_{31} \\ s_{31} & 2s_{31} + s_{22} + s_{211} & s_{31} + s_{22} + 2s_{211} & s_{211} \\ s_{211} & s_{22} + s_{211} + s_{1111} & s_{211} + s_{1111} & s_{1111} \end{pmatrix}$$

Thm: [R-Wilson] The action of ω on $\mathbb{W}_{n,k}$ is the same as 180° rotation.

Super Poincaré Duality: [R-Wilson] Multiplication in complementary bidegrees in $W_{n,k}$ is a perfect pairing.

Conjectural Lefschetz

Look at the bigraded *Hilbert series* of $\mathbb{W}_{n,k} \ldots$

$$\mathrm{Hilb}(\mathbb{W}_{5,3}; q, z) \leftrightarrow \begin{pmatrix} 1 & 5 & 15 & 29 & 39 & 35 & 20 & 6 \\ 4 & 19 & 50 & 77 & 77 & 50 & 19 & 4 \\ 6 & 20 & 35 & 39 & 29 & 15 & 5 & 1 \end{pmatrix}$$

Conj: The rows and columns of $\mathrm{Hilb}(\mathbb{W}_{n,k}; q, z)$ are log concave (and thus unimodal).

(Super Hard Lefschetz Theorem?)

Ordered Superpartitions

Defn: A *superpartition* is a set partition $\{B_1, \ldots, B_k\}$ of $[n]$ where non-minimal elements of $B_1, \ldots, B_k$ may be **barred**.

$$\{\{1, \bar{5}, 6\}, \{2, \bar{4}\}, \{3\}\}$$

Defn: Let $\mathcal{OSP}_{n,k}$ be the family of *ordered* superpartitions of $[n]$ into k blocks.

$$(2, \bar{4} \mid 3 \mid 1, \bar{5}, 6)$$

$$|\mathcal{OSP}_{n,k}| = 2^{n-k} \cdot k! \cdot \mathrm{Stir}(n, k)$$

Ordered Superpartitions and $\mathbb{W}$-modules

Thm: [R-Wilson] The vector space dimension of $\mathbb{W}_{n,k}$ equals $|\mathcal{OSP}_{n,k}|$. In fact, $\mathbb{W}_{n,k}$ has a *monomial basis* indexed by $\mathcal{OSP}_{n,k}$.

$$(6, \bar{7} \mid 1, 4 \mid 5 \mid 2, \bar{3}, 8) \leftrightarrow (2, 0, \bar{1}, 2, 0, 0, \bar{0}, 3)$$
$$\leftrightarrow x_1^2 \cdot x_2^0 \cdot (x_3^1 \theta_3) \cdot x_4^2 \cdot x_5^0 \cdot x_6^0 \cdot (x_7^0 \theta_7) \cdot x_8^3$$

Rmk: Have a recursive formula for $\mathrm{Hilb}(\mathbb{W}_{n,k}; q, z)$. Still cannot prove unimodality! Sagan-Swanson have a *similar* conjectural basis of SR_n.

Graded Frobenius image

Thm: [R-Wilson] The *anticommutative* singly graded Frobenius image of $\mathbb{W}_{n,k}$ is

$$\mathrm{grFrob}(\mathbb{W}_{n,k}; z) = \sum_{(\lambda^{(1)}, \ldots, \lambda^{(k)})} z^{n - \lambda^{(1)}_1 - \cdots - \lambda^{(k)}_1} \cdot s_{\lambda^{(1)}} \cdots s_{\lambda^{(k)}}$$

where $(\lambda^{(1)}, \ldots, \lambda^{(k)})$ range over k-tuples of nonempty hooks with

$$|\lambda^{(1)}| + \cdots + |\lambda^{(k)}| = n$$

Rmk: Have a superpartition formula for the monomial expansion of $\mathrm{grFrob}(\mathbb{W}_{n,k}; q, z)$. Bigraded Schur expansion unknown.

Superspace Coinvariants

The Fields Institute Combinatorics Group has studied the *Superspace Coinvariant Ring*

$$\begin{aligned} SR_n &:= \Omega_n / \langle (\Omega_n)_+^{S_n} \rangle \\ &= \Omega_n / \langle e_1, \ldots, e_n, de_1, \ldots, de_n \rangle \end{aligned}$$

where $d := \theta_1(\partial/\partial x_1) + \cdots + \theta_n(\partial/\partial x_n)$ is the *total derivative*.

Fields Conj: [N. Bergeron, Machacek, Zabrocki, ...] We have

$$\mathrm{grFrob}(SR_n; q, z) = \sum_{k=1}^{n} z^{n-k} \cdot \Delta'_{e_{k-1}} e_n \,|_{t=0} \,.$$

Fields Strategy

Fields Conj: We have

$$\mathrm{grFrob}(SR_n; q, z) = \sum_{k=1}^{n} z^{n-k} \cdot \Delta'_{e_{k-1}} e_n \mid_{t=0} .$$

Obs: We have a linear map

$$\varphi : V_{n,1} \oplus \cdots \oplus V_{n,n} \hookrightarrow \Omega_n \twoheadrightarrow SR_n$$

Conj: [R-Wilson] The map φ is a bijection.

(Would imply Fields Conjecture.)

Zabrocki Conjecture

Fields Conj: We have

$$\operatorname{grFrob}(SR_n; q, z) = \sum_{k=1}^{n} z^{n-k} \cdot \Delta'_{e_{k-1}} e_n \mid_{t=0} .$$

Q: What about $\Delta'_{e_{k-1}} e_n$ itself?

Conj: [Zabrocki] Let $Y_n = \{y_1, \ldots, y_n\}$ be n new commuting variables and consider

$$DSR_n := \mathbb{Q}[X_n, Y_n, \Theta_n] / \langle \mathbb{Q}[X_n, Y_n, \Theta_n]_+^{S_n} \rangle.$$

We have

$$\operatorname{grFrob}(DSR_n; q, t, z) = \sum_{k=1}^{n} z^{n-k} \cdot \Delta'_{e_{k-1}} e_n.$$

Polarization Conjecture

Def: [R-Wilson] Let $\mathbb{V}_{n,k}$ be the smallest subspace of $\mathbb{Q}[X_n, Y_n, \Theta_n]$ which

- contains the Superspace Vandermonde $\delta_{n,k}$ (in x, θ),
- is closed under $\partial/\partial x_i$ and $\partial/\partial y_i$, and
- is closed under the *polarization operators*

$$y_1(\partial/\partial x_1)^j + \cdots + y_n(\partial/\partial x_n)^j \quad (j \geq 1)$$

Fact: $\mathbb{V}_{n,k}$ is a doubly graded S_n-module in x-degree and y-degree.

Conj: [R-Wilson] For any $k \leq n$ we have

$$\mathrm{grFrob}(\mathbb{V}_{n,k}; q, t) = \Delta'_{e_{k-1}} e_n.$$

Thanks for listening!

- J. Haglund, B. Rhoades, and M. Shimozono. Ordered set partitions, generalized coinvariant algebras, and the Delta Conjecture. *Adv. Math.*, **329** (2018) 851–915.
- B. Pawlowski and B. Rhoades. A flag variety for the Delta Conjecture. *Trans. Amer. Math. Soc.*, **372** (2019), 8195–8248.
- B. Rhoades and A. T. Wilson. Vandermondes in superspace. *Trans. Amer. Math. Soc.*, **373** (2020), 4483–4516.
- B. Rhoades and A. T. Wilson. Set superpartitions and superspace duality modules. Accepted, *Forum Math. Sigma*, 2022.