Conjectures on cohomology of Grassmannians

Summer 2020 Polymath Jr. group (arXiv:2011.03179)

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1. (q-)Binomials, grassmannians
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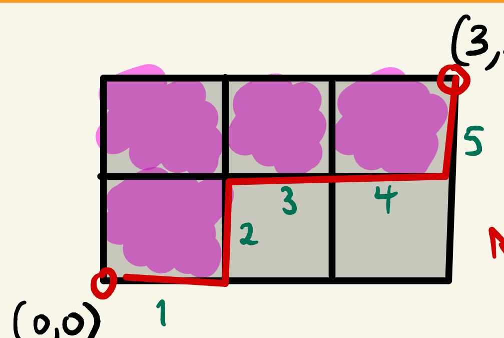
1. Binomials, q-binomials, grassmannians

Binomial coefficient $\binom{k+\ell}{\ell}$ = # ℓ- subsets of $\{1,2,\ldots,k+\ell\}$ $= \dfrac{(k+\ell)!}{\ell!\,k!}$

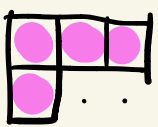
= # walks $(0,0) \to (k,\ell)$ taking unit steps north or east

= # Ferrers diagrams of partitions λ fitting in an $\ell \times k$ rectangle

e.g. $\ell = 2$
$k = 3$



$\longleftrightarrow$ $\lambda =$



vertical steps $\{2,5\} \subset \{1,2,3,4,5\}$

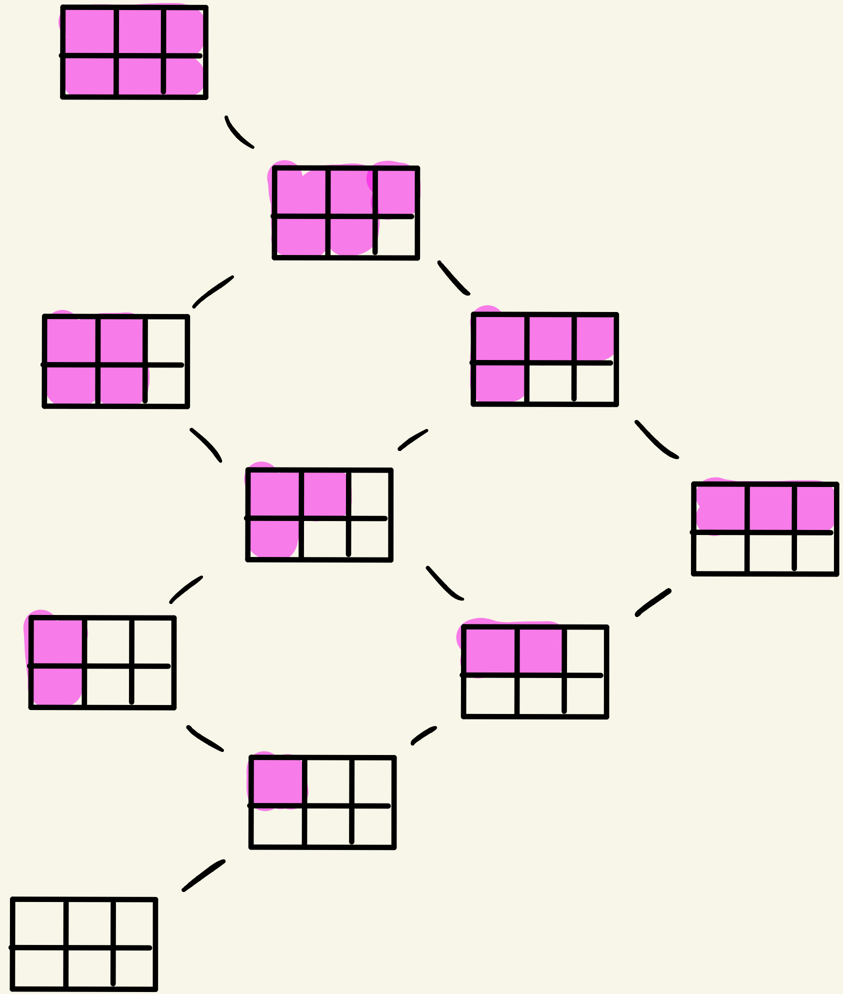
$l=2$

$h=3$

$$\binom{2+3}{2} = \binom{5}{2}$$

$$= \frac{5!}{2!\,3!}$$

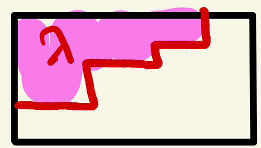
$$= \frac{5 \cdot 4}{2 \cdot 1} = 10$$



q-Binomial coefficient

$$\begin{bmatrix} k+\ell \\ \ell \end{bmatrix}_q = \sum_{\substack{\text{Ferrers diagrams} \\ \lambda \subseteq \ell \times k \text{ rectangle}}} q^{|\lambda|}$$

where $|\lambda| = \lambda_1 + \lambda_2 + \dots$ = # squares in its Ferrers diagram



$$= \frac{[k+\ell]!_q}{[\ell]!_q \, [k]!_q}$$

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where $[k]!_q := [k]_q [k-1]_q \cdots [2]_q [1]_q$, $[k]_q := 1 + q + q^2 + \dots + q^{k-1}$

$\ell = 2$

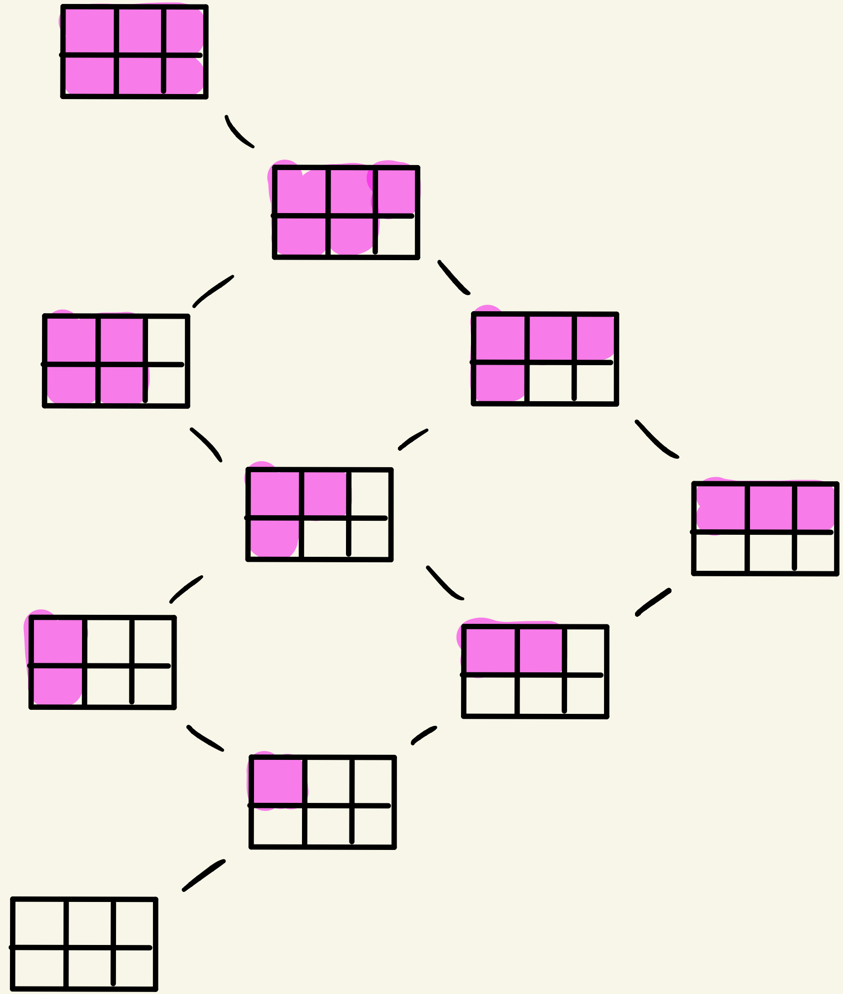
$k = 3$

$$\begin{bmatrix}2+3\\2\end{bmatrix}_q = \frac{[5]!_q}{[2]!_q\,[3]!_q}$$

$$= \frac{[5]_q\,[4]_q}{[2]_q\,[1]_q}$$

$$= (1+q+q^2+q^3+q^4)(1+q^2)$$

$$= 1+q+2q^2+2q^3+2q^4+q^5+q^6$$



q^6

$+\, q^5$

$+\, 2q^4$

$+\, 2q^3$

$+\, 2q^2$

$+\, q$

$+\, 1$

Pascal identity

$$\binom{k+\ell}{\ell} = \binom{k+\ell-1}{\ell-1} + \binom{k+\ell-1}{\ell}$$

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & \dots & & & & \vdots & & & & \dots &
\end{array}$$

$q=1$

q-Pascal identities

$$\begin{bmatrix}k+\ell\\ \ell\end{bmatrix}_q = q^{\ell}\begin{bmatrix}k+\ell-1\\ \ell\end{bmatrix}_q + \begin{bmatrix}k+\ell-1\\ \ell-1\end{bmatrix}_q$$

$$= \begin{bmatrix}k+\ell-1\\ \ell\end{bmatrix}_q + q^{k}\begin{bmatrix}k+\ell-1\\ \ell-1\end{bmatrix}_q$$

$$\begin{bmatrix} k+\ell \\ k \end{bmatrix}_q = \sum_{\lambda \subset \ell \times k \text{ rectangle}} q^{|\lambda|} = \# \, Gr\left(\ell, (\mathbb{F}_q)^{k+\ell}\right) \quad \text{if } q = p^d \text{ a prime power}$$

Grassmannian of ℓ-dim'l subspaces in $(\mathbb{F}_q)^{k+\ell}$

$\ell = 2$

$k = 3$

row-reduced echelon forms

= Schubert cell decomposition of $Gr(\ell, \mathbb{F}^{k+\ell})$

$= Gr(2, \mathbb{F}^5)$

$$\begin{bmatrix} * & * & * & 0 & 1 \\ * & * & * & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} * & * & 0 & * & 1 \\ * & * & 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} * & * & 0 & 1 & 0 \\ * & * & 1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} * & 0 & * & * & 1 \\ * & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} * & 0 & * & 1 & 0 \\ * & 1 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & * & * & * & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} * & 0 & 1 & 0 & 0 \\ * & 1 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & * & * & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & * & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

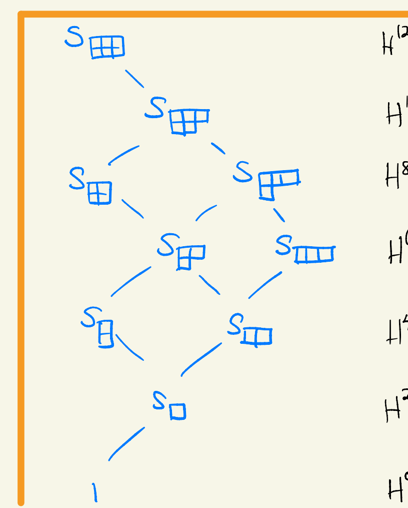
$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Taking $\mathbb{C}^{k+\ell}$ instead of $(\mathbb{F}_q)^{k+\ell}$, the complex Grassmannian

$X = Gr(\ell, \mathbb{C}^{k+\ell})$ has same Schubert cell decomposition

$\rightsquigarrow$ its cohomology $H^*(X) = H^*(X, \mathbb{Q})$

has Schubert $\mathbb{Q}$-basis $\{s_\lambda\}_{\lambda \subset \ell \times k \text{ rectangle}}$ with $s_\lambda \in H^{2|\lambda|}(X)$



$$\Rightarrow \mathrm{Hilb}(H^*(X), q) := \sum_{i \geq 0} \dim_{\mathbb{Q}} H^{2i}(X) \cdot q^i$$

$$= \sum_{\lambda \subset \ell \times k \text{ rectangle}} q^{|\lambda|} = \begin{bmatrix} k+\ell \\ \ell \end{bmatrix}_q$$

2. The cohomology ring

Let $R^{\ell,k}$:= cohomology ring $H^*(X,\mathbb{Q})$ for $X = Gr(\ell,\mathbb{C}^{k+\ell})$

$$\cong \mathbb{Q}[e_1, e_2, \ldots, e_\ell, h_1, h_2, \ldots, h_k] \Big/ \left(\begin{array}{l} e_1 - h_1, \\ e_2 - e_1 h_1 + h_2, \\ \vdots \\ \sum_{i+j=d} (-1)^j e_i h_j, \\ \vdots \\ e_\ell h_{k-1} - e_{\ell-1} h_k, \\ e_\ell h_k \end{array} \right)$$

Borel 1950's

where

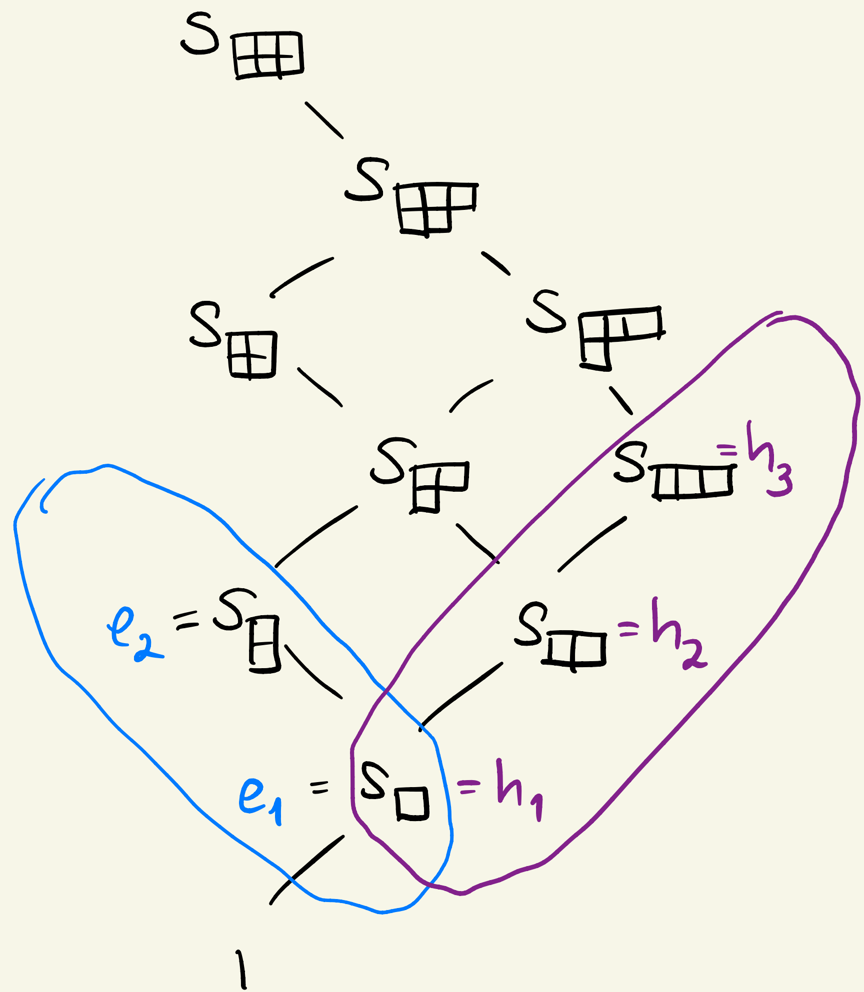
$e_i := s_{\text{column of } i \text{ boxes}}$, $h_j := s_{\text{row of } j \text{ boxes}}$

$d = 1, 2, \ldots, k+\ell$

with $e_0 = 1$, $e_i = 0$ if $i \notin \{0,1,2,\ldots,\ell\}$
$h_0 = 1$, $h_j = 0$ if $j \notin \{0,1,2,\ldots,k\}$

$$R^{2,3} = H^*(X), \quad X = Gr(2, \mathbb{C}^5)$$

$$\cong \frac{\mathbb{Q}[e_1, e_2, h_1, h_2, h_3]}{\left(\begin{array}{l} e_1 - h_1 \\ e_2 - e_1 h_1 + h_2, \\ \quad e_2 h_1 - e_1 h_2 + h_3, \\ \quad\quad e_2 h_2 - e_1 h_3, \\ \quad\quad\quad e_2 h_3 \end{array} \right)}$$



$R^{\ell,k}$ is generated by $e_1, e_2, \ldots, e_\ell$ or by $h_1, h_2, \ldots, h_k$

$$R^{\ell,k} \cong \mathbb{Q}[e_1, e_2, \ldots, e_\ell] \big/ (h_{k+1}, h_{k+2}, \ldots, h_{k+\ell})$$

interpret as Jacobi-Trudi determinants in $e_1, \ldots, e_k$

$$\cong \mathbb{Q}[h_1, h_2, \ldots, h_k] \big/ (e_{\ell+1}, e_{\ell+2}, \ldots, e_{\ell+k})$$

similar

$$R^{2,3} \cong \mathbb{Q}[e_1, e_2] \big/ (h_4, h_5)$$

$$h_4 = \det \begin{bmatrix} e_1 & e_2 & 0 & 0 \\ 1 & e_1 & e_2 & 0 \\ 0 & 1 & e_1 & e_2 \\ 0 & 0 & 1 & e_1 \end{bmatrix} \qquad h_5 = \det \begin{bmatrix} e_1 & e_2 & 0 & 0 & 0 \\ 1 & e_1 & e_2 & 0 & 0 \\ 0 & 1 & e_1 & e_2 & 0 \\ 0 & 0 & 1 & e_1 & e_2 \\ 0 & 0 & 0 & 1 & e_1 \end{bmatrix}$$

3. CONJECTURE and motivation

We were led to consider subalgebras of $R^{\ell,k}$ for $m = 0, 1, 2, \ldots$

$$R^{\ell,k,m} := \mathbb{Q}\text{-subalgebra of } R^{\ell,k} \text{ generated by } e_1, e_2, \ldots, e_m$$

$$\Big(= \text{subalgebra generated by } h_1, h_2, \ldots, h_m$$

$$= \text{subalgebra generated by all elements of degree} \le m \text{ in } R^{\ell,k} \Big)$$

$$\begin{array}{ccccccccc}
R^{\ell,k,0} & \subset & R^{\ell,k,1} & \subset & R^{\ell,k,2} & \subset & \ldots & \subset & R^{\ell,k,\min(\ell,k)} \\
\| & & \| & & \| & & & & \| \\
\mathbb{Q} & & \text{subalg. gen'd by } e_1 & & \text{subalg. gen'd by } e_1, e_2 & & & & R^{\ell,k}
\end{array}$$

CONJECTURE For $m = 1, 2, \ldots, \min(\ell, k)$

(R.-Tudose 2003)

$$\mathrm{Hilb}(R^{\ell,k,m}, q) := \sum_{d \geq 0} \dim_{\mathbb{Q}} (R^{\ell,k,m})_d \cdot q^d$$

$$= 1 + \sum_{i=1}^{m}{}' \; q^i \begin{bmatrix} k \\ i \end{bmatrix}_q \begin{bmatrix} \ell \\ i \end{bmatrix}'_{k,q}$$

usual q-binomial

a different q-analogue of $\binom{\ell}{i}$, depending on k:

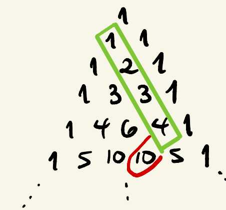
$$\begin{bmatrix} \ell \\ i \end{bmatrix}'_{k,q} := \sum_{j=0}^{\ell-i} q^{j(k-i+1)} \begin{bmatrix} i+j-1 \\ j \end{bmatrix}_q$$

REMARK

$$\begin{bmatrix} \ell \\ i \end{bmatrix}'_{k,q} := \sum_{j=0}^{\ell-i} q^{j(k-i+1)} \begin{bmatrix} i+j-1 \\ j \end{bmatrix}_q \quad \overset{q=1}{\rightsquigarrow} \quad \sum_{j=0}^{\ell-i} \binom{i+j-1}{j} = \binom{\ell}{i}$$

"Hockey-stick identity"

$$\binom{\ell}{i} = \binom{\ell-1}{i} + \binom{\ell-2}{i-1} + \binom{\ell-3}{i-2} \cdots + \binom{\ell-i}{0}$$



and hence at $q=1$, CONJ says

$$\dim_{\mathbb{Q}} R^{\ell,k,m} = \sum_{i=0}^{m} \binom{k}{i}\binom{\ell}{i}$$

consistent with $\dim_{\mathbb{Q}} R^{\ell,k} = \binom{k+\ell}{\ell} = \sum_{i\geq 0} \binom{k}{i}\binom{\ell}{i}$

Vandermonde convolution

Motivation for CONJ:

We actually only needed a weak consequence of CONJ, namely $e_m \cdot e_1^{\ell k - 2m}$ lies in $R^{\ell,k,m-1}$ for $3 \le m \le \ell$,

to simplify a proof of this result of M. Hoffman 1984:

THEOREM: If $\ell \neq k$, any graded ring endomorphism $R^{\ell,k} \longrightarrow R^{\ell,k}$ that acts as a nonzero scalar $c \neq 0$ on $(R^{\ell,k})_1$ must then scale every $(R^{\ell,k})_d$ by c^d

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This was part of a conjecture of O'Neill 1974 implying which Grassmannians $Gr(\ell, \mathbb{C}^{k+\ell})$ have the fixed-point property.

4. The frontal attack

Elimination theory -

Compute a Gröbner basis for this ideal

$$R^{\ell,k} = \mathbb{Q}[e_1, e_2, \ldots, e_\ell] / (h_{k+1}, h_{k+2}, \ldots, h_{k+\ell})$$

using a lexicographic order with $e_\ell > e_{\ell-1} > \ldots > e_3 > e_2 > e_1$.

Its standard monomials give $\mathbb{Q}$-bases for all $R^{\ell,k,m}$ at once.

EXAMPLE $\ell=2$, $k=3$ $\quad R^{2,3} \cong \mathbb{Q}[e_1,e_2]/(h_4,h_5)$

$\Big\Downarrow$ lex $e_2>e_1$ Gröbner basis calculation

$$\cong \mathbb{Q}[e_1,e_2]\Big/\Big(\underbrace{e_1^7}_{\text{some complicated } \mathbb{Q}[e_1,e_2]\text{- combination of } h_4,\,h_5},\ \underbrace{e_2e_1^3 - \text{lower terms}}_{\frac{1}{3}(h_5-3e_1\cdot h_4)},\ \underbrace{e_2^2-3e_1^2e_2+e_1^4}_{=h_4}\Big)$$

Standard monomials are those not divisible by red leading terms:

$$\{\underbrace{1,}_{\text{in } R^{2,3,0}}\ \Big|\ \underbrace{e_1,e_1^2,e_1^3,e_1^4,e_1^5,e_1^6,}_{\text{in } R^{2,3,1}}\ \Big|\ \underbrace{e_2,e_2e_1,e_2e_1^2}_{\text{in } R^{2,3,2}}\}$$

Guessing the form of the Gröbner basis for lex order with $e_\ell > e_{\ell-1} > \dots > e_3 > e_2 > e_1$ seems very hard

- see Polymath Jr REU report for conjectures when $\ell = 2, 3$

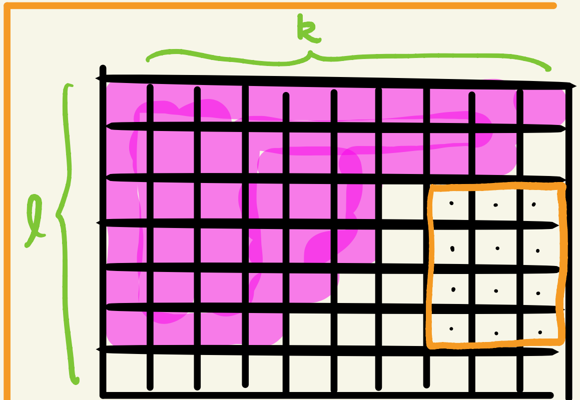
5. Reformulation via k-conjugation

Each summand in

$$\text{CONJ: } \operatorname{Hilb}(R^{\ell,k,m}, q) = 1 + \sum_{i=1}^{m} q^i \begin{bmatrix} k \\ i \end{bmatrix}_q \begin{bmatrix} \ell \\ i \end{bmatrix}'_{k,q}$$

has an interpretation:

$$q^i \begin{bmatrix} k \\ i \end{bmatrix}_q \begin{bmatrix} \ell \\ i \end{bmatrix}'_{k,q} \overset{\substack{\text{R. Tudose} \\ 2003}}{=} \sum_{\substack{\lambda \subset \ell\times k \text{ rectangle:} \\ \lambda \text{ is } i\text{-vacant}}} q^{|\lambda|}$$



4-vacant because 4×3 box is largest $i\times(i-1)$ box missing here

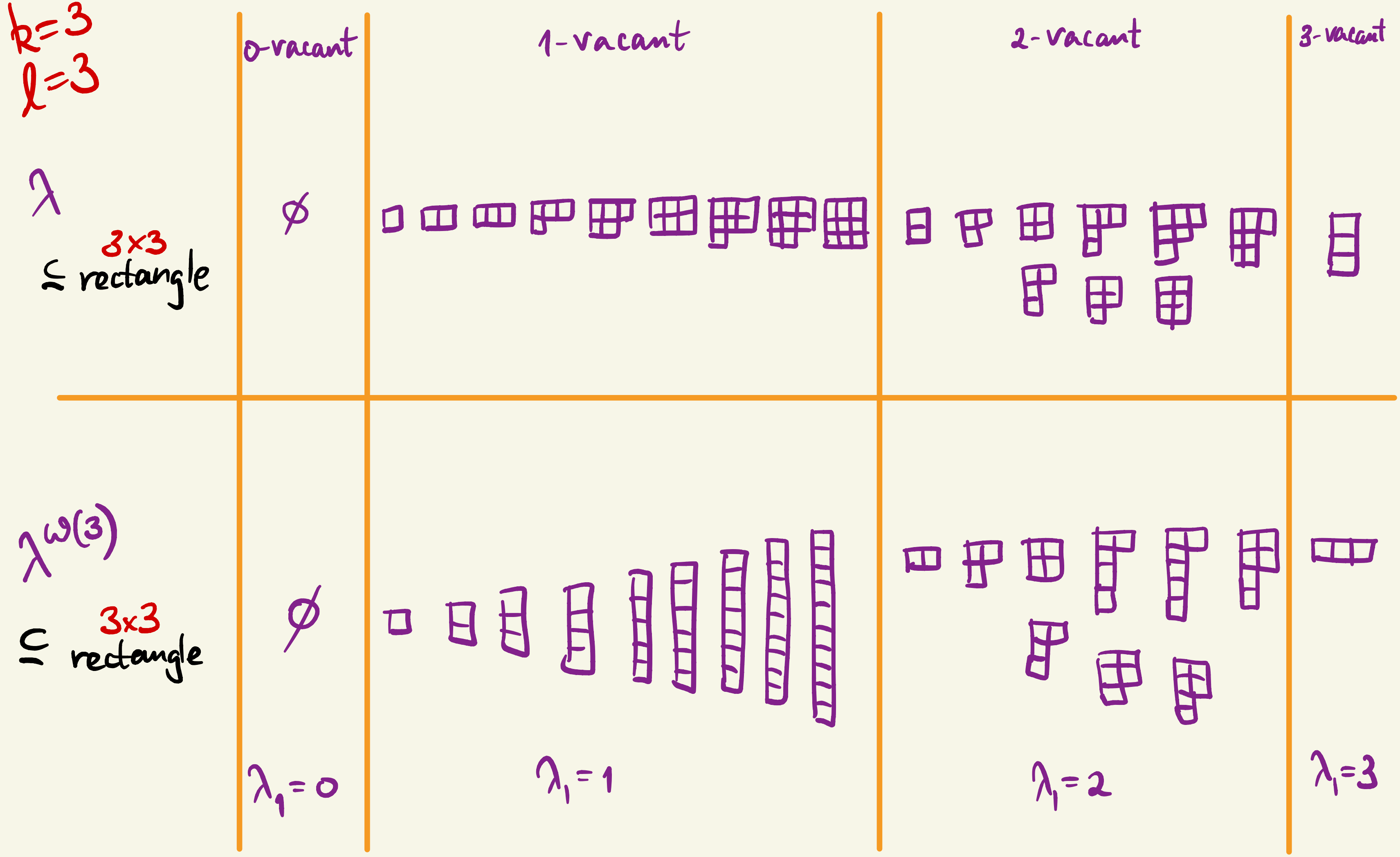
Reformulation:

$$q^{i}\begin{bmatrix} k \\ i \end{bmatrix}_q \begin{bmatrix} \ell \\ i \end{bmatrix}'_{k,q} \overset{\text{(R.-Tudose 2003)}}{=} \sum_{\substack{\lambda \subset \ell\times k \text{ rectangle}: \\ \lambda \text{ is } i\text{-vacant}}} q^{|\lambda|}$$

$$\overset{\text{(2020 Polymath Jr.)}}{=\!=} \sum_{\substack{\lambda^{\omega(k)} \subset \ell\times k \text{ rectangle}: \\ \lambda_1 = i}} q^{|\lambda|}$$

k-conjugation bijects these λ

k-conjugate of λ



k=3
l=3
0-vacant
1-vacant
2-vacant
3-vacant
λ
3x3
⊆ rectangle
λ^{ω(3)}
3x3
⊆ rectangle
∅
∅
λ₁=0
λ₁=1
λ₁=2
λ₁=3

What is k-conjugation?

Designed by Lapointe, Lascoux & Morse (2003)

as an involution $\lambda \leftrightarrow \lambda^{\omega(k)}$ on $\{\text{k-bounded partitions } \lambda\}$ i.e. $\lambda_1 \leq k$

to have this property:

the fundamental involution on symmetric functions Λ

$$\omega\left(s_\lambda^{(k)}\right) = s_{\lambda^{\omega(k)}}^{(k)}$$

where $\{s_\lambda^{(k)}\}$ is their k-Schur function basis for

$$\mathbb{Q}[h_1, h_2, \dots, h_k] \subset \mathbb{Q}[h_1, h_2, \dots]$$

$$\mathbb{Q}[h_1, h_2, \dots, h_k] = \mathbb{Q}[e_1, e_2, \dots, e_k]$$

$$\mathbb{Q}[h_1, h_2, \dots] = \Lambda_{\mathbb{Q}}$$

symmetric functions

It's an interesting composite of three maps:

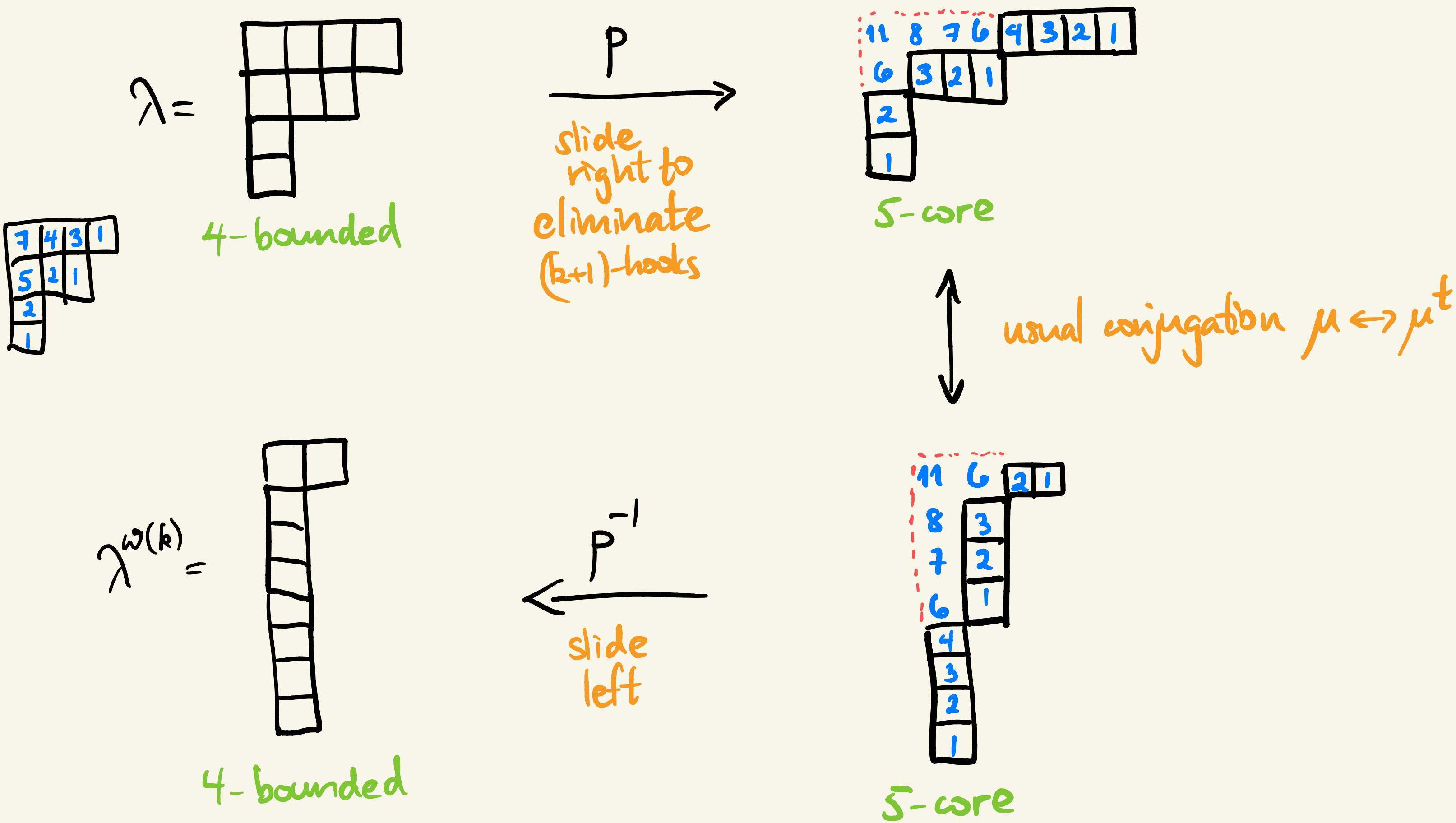
$$\begin{Bmatrix}\text{k-bounded}\\\text{partitions}\end{Bmatrix} \xrightarrow[\sim]{p} \begin{Bmatrix}\text{(k+1)-core}\\\text{partitions}\end{Bmatrix} \xrightarrow{\text{usual conjugation}} \begin{Bmatrix}\text{(k+1)-core}\\\text{partitions}\end{Bmatrix} \xrightarrow{p^{-1}} \begin{Bmatrix}\text{k-bounded}\\\text{partitions}\end{Bmatrix}$$

i.e. no hook-length equal to k+1

$$\lambda \longmapsto p(\lambda) \longmapsto (p(\lambda))^t \longmapsto p^{-1}\left((p(\lambda))^t\right)$$

$$p^{-1}\left((p(\lambda))^t\right) \stackrel{!!}{=} \lambda^{\omega(k)}$$

EXAMPLE $k=4$



λ =
4-bounded
p
slide right to eliminate (k+1)-hooks
5-core
usual conjugation μ ⟷ μ^t
p^{-1}
slide left
λ^{ω(k)} =
4-bounded
5-core

The reformulation suggests better conjectures that would imply the one from 2003:

CONJS (Polymath Jr. REU 2020) The images of either

$$\left\{ s_\lambda^{(\lambda_1)} \right\}_{\lambda_1 \leq k,\ \lambda^{\omega(k)} \subseteq \ell \times k \text{ rectangle}}$$

or

$$\left\{ \underset{h_{\lambda_1} h_{\lambda_2} \cdots}{\overset{}{h_\lambda}} \right\}_{\lambda_1 \leq k,\ \lambda^{\omega(k)} \subseteq \ell \times k \text{ rectangle}}$$

give a $\mathbb{Q}$-basis for $R^{\ell,k}$ that restrict to $\mathbb{Q}$-bases for each subalgebra $R^{\ell,k,m}$

6. Lagrangian analogue

Replace $X = Gr(\ell, \mathbb{C}^{k+\ell})$

with $X = LG(n, \mathbb{C}^{2n})$

$= \{$ maximal isotropic $\mathbb{C}$-subspaces of $\mathbb{C}^{2n}$ with a symplectic form $\langle \cdot , \cdot \rangle$ $\}$

$X = LG(n, \mathbb{C}^{2n})$ has cohomology ring $R^n_{LG} = H^*(X, \mathbb{Q})$

with $R^n_{LG} \cong \mathbb{Q}[e_1, e_2, \dots, e_n] \Big/ \left(e_i^2 + 2\sum_{k=1}^{n-i} (-1)^k e_{i+k} e_{i-k} \right)_{i=1,2,\dots,n}$

with $e_0 = 1$
$e_i = 0$ if $i \notin \{1,2,\dots,n\}$

and $\mathrm{Hilb}(R^n_{LG}, q) = [2]_q [2]_{q^2} [2]_{q^3} \cdots [2]_{q^n}$

$= (1+q)(1+q^2)(1+q^3)\cdots(1+q^n)$ $\overset{q=1}{\rightsquigarrow} 2^n$

Let $R_{LG}^{n,m} :=$ $\mathbb{Q}$-subalgebra of R_{LG}^{n} generated by $e_1, e_2, \ldots, e_m$

CONJ: (Polymath Jr. REU 2020)

$$\mathrm{Hilb}(R_{LG}^{n,m}, q) = 1 + \sum_{\substack{1 \le i \le m \\ i \text{ odd}}}' \left[\begin{matrix} n+1 \\ i+1 \end{matrix}\right]''_q$$

where $\left[\begin{matrix} n+1 \\ i+1 \end{matrix}\right]''_q := q^i \sum_{j=0}^{n-i} q^{\binom{j+1}{2}} \left[\begin{matrix} i+j \\ i \end{matrix}\right]_q$

$q=1$

$$\binom{n+1}{i+1} = \sum_{j=0}^{n-i} \binom{i+j}{i}$$

hockey-stick identity again

[consistent with $2^n = \sum_{i \text{ odd}} \binom{n+1}{i+1}$]

QUESTIONS

1. k-conjugation, k-Schur functions $s_\lambda^{(k)}$ are related to cohomology of affine Grassmannians. Is there a relation to these subalgebras in cohomology of usual Grassmannians?

2. Lagrangian Grassmannian is connected with shifted shapes and Schur's P, Q-functions. Is there a k-conjugation relevant here?

Thanks for your attention!