Singularities of Regular Hessenberg Varieties

Martha Precup
WUSTL

joint work with E. Insko
and A. Woo

Flags

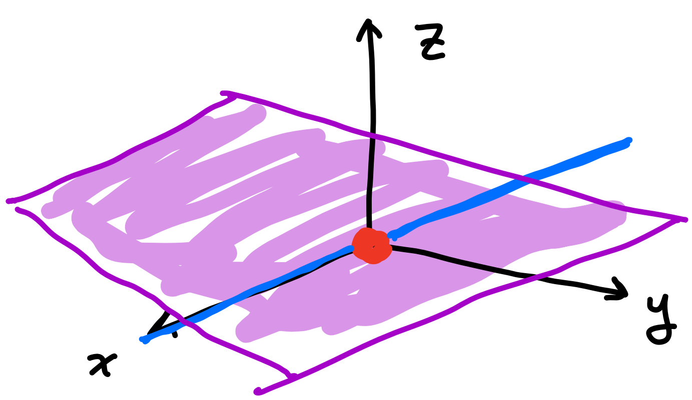
A **full flag** in $\mathbb{C}^n$ is a nested sequence of subspaces:

$$V_\bullet = (0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n = \mathbb{C}^n).$$

The **flag variety** is the collection $\mathcal{B}$ of all full flags in $\mathbb{C}^n$.

Ex: $E_\bullet = (0 < \mathbb{C}\{e_1\} < \mathbb{C}\{e_1, e_2\} < \cdots)$

is the **standard flag**.



Rmk: $\mathcal{B}$ is the flag variety of
$G = GL_n(\mathbb{C})$...

1. $G \circlearrowright \mathcal{B}$ by $g V_\bullet := (0 \subset g(V_1) \subset g(V_2) \subset \cdots)$
2. The action is transitive.
3. $$\mathrm{Stab}_G(E_\bullet) = \begin{bmatrix} * & * & & * \\ 0 & * & & \\ 0 & 0 & \ddots & \\ \vdots & \vdots & & \\ 0 & 0 & & * \end{bmatrix} =: B$$

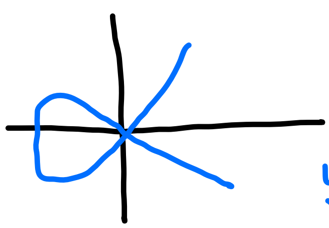
subgp of upper Δ matrices.

$\mathcal{B} \simeq G/B \rightsquigarrow$ G = reductive alg. gp
B = Borel subgp.

Fact: $\mathcal{B}$ is a smooth variety.

no corners / cusps / self-intersections

locally, the zero set of some polynomials.



$y^2 - x^2(x+1) = 0$

Subvarieties of $\mathcal{B}$

(I) Schubert varieties

Given a permutation $w \in S_n$, the corresponding <u>Schubert variety</u> is

$$X_w := \overline{B \underbrace{wE_\bullet}} \subseteq \mathcal{B}$$

$$wE_\bullet = (0 \subset \mathbb{C}\{e_{w(1)}\} \subset \mathbb{C}\{e_{w(1)}, e_{w(2)}\} \subset \cdots)$$

"permutation flag"

<u>FACTS</u>: ① Irreducible, $\dim(X_w) = \ell(w)$

$$\ell(w) = \#\{i<j \mid w(i) > w(j)\}$$

② (Lakshmibai-Sandhya) X_w is smooth iff <u>w avoids the patterns 4231 and 3412.</u>

↳ no subsequence of w in same relative order.

<u>Ex</u>: $w = 631524 \in S_6 \Rightarrow X_w$ is singular!

4 2 3 1

RMK: Pattern avoidance also characterizes:

- Gorenstein (Woo-Yong)
- bundle structures (Richmond+)
- LCI (Úlfarsson-Woo) ... etc.

(II) Hessenberg varieties

Let $X \in M_n(\mathbb{C})$ and $h : [n] \longrightarrow [n]$
s.t. $h(i) \geq i$ and $h(i-1) \leq h(i)$.

↳ Hessenberg function.

(1 2 3) Catalan
(2 2 3) many!
(1 3 3)...

The Hessenberg variety corresponding to X and h is:

$$\mathcal{H}ess(X,h) := \{ V_\bullet \in \mathcal{B} \mid X(V_i) \subseteq V_{h(i)} \ \forall i \}.$$

FACTS/EX :

① $h(i)=n \ \forall i \Rightarrow \mathcal{H}ess(X,h) = \mathcal{B}$

② $h(i)=i \ \forall i \Rightarrow \mathcal{H}ess(X,h)$ is a Springer fiber.

③ In general, $\mathcal{H}ess(X,h)$ is reducible and $\dim \mathcal{H}ess(X,h) = ?$.

④ Given a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_\ell)$ of n, let $X_\lambda \in M_n(\mathbb{C})$ be a regular matrix of Jordan type λ.

Ex. $\lambda = (2,2) \vdash 4$

$$X_{(2,2)} = \left[\begin{array}{cc|cc} c_1 & 1 & & \\ 0 & c_1 & \multicolumn{2}{c}{0} \\ \hline & & c_2 & 1 \\ \multicolumn{2}{c|}{0} & 0 & c_2 \end{array}\right] \qquad c_1 \neq c_2$$

Regular Hessenberg varieties

$\mathcal{H}ess(X_\lambda, h)$ is a <u>regular Hessenberg variety</u>...

- $\mathcal{H}ess(X_\lambda, h)$ is irreducible iff $h(i) > i$ for all $i < n$
- $\dim \mathcal{H}ess(X_\lambda, h) = \sum_i (h(i) - i)$.

<u>Q</u>: For which λ, h is $\mathcal{H}ess(X_\lambda, h)$ smooth?

$\underline{h_0(i) := i+1 \ \forall i < n}$ $\longleftrightarrow$ $\underline{h(i) = n \ \forall i}$

$\mathcal{H}ess(X_\lambda, h_0)$ smooth? $\longleftrightarrow$ $\mathcal{B}$ smooth

$\mathcal{H}ess(X_\lambda, h_0) \cap \overline{BwE_\bullet}$ smooth? $\longleftrightarrow$ $\overline{BwE_\bullet} = X_w$ pattern avoidance

Singular permutation flags $wE_\bullet$ in $\mathcal{H}ess(X_\lambda, h_0)$

The partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell)$ defines a set partion of $[n]$:

$$J_\lambda := \{1, 2, \ldots, \lambda_1\} \sqcup \{\lambda_1 + 1, \lambda_1 + 2, \ldots, \lambda_1 + \lambda_2\} \sqcup \cdots \sqcup \{\lambda_1 + \cdots + \lambda_{\ell-1} + 1, \ldots, n\}$$

"J_λ - blocks"

For each $w \in S_n$, let w^λ be the permutation obtained by reordering entries in each J_λ-block to be increasing.

Ex: $\lambda = (4,2,2) \vdash 8$

$$J_\lambda = \{1,2,3,4\} \sqcup \{5,6\} \sqcup \{7,8\}$$

$$w = 67814532 \rightsquigarrow w^\lambda = 57812634$$

<u>THM</u>: (Insko-P.-Woo) $wE_\bullet \in \mathcal{H}ess(X_\lambda, h_0)$ is smooth iff

① w^λ is a permutation of the J_λ-blocks AND

② w avoids the patterns 123 and 2143 in each J_λ-block.

<u>EX</u>: $n=8, \lambda=(4,2,2)$

$w = \underline{65}\ \underline{78}\ \underline{1432} \rightsquigarrow w^\lambda = 56\ 78\ 1234$

avoid all bad patterns

$\Rightarrow wE_\bullet$ is a smooth pt!

$w = 87\ 65\ \underline{12}4\underline{3}$

$\Rightarrow wE_\bullet$ is singular pt!

COR: For all $n \geq 3$, $\mathcal{H}ess(X_\lambda, h_0)$ is smooth iff $\lambda = (1,1,\ldots,1)$.

FURTHER WORK / Q's:

- There is a combinatorial criterion for determining which "Hessenberg-Schubert varieties" $\overline{\mathcal{H}ess(X_\lambda, h_0) \cap BwE_\bullet}$ are singular — and it is not pattern avoidance!

- Describe the singular locus of $\mathcal{H}ess(X_\lambda, h_0)$.
- What happens to $\mathcal{H}ess(X_\lambda, h)$ for λ fixed as h varies?