What is the degree of a Grothendieck polynomial?

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The complete flag variety

The **complete flag variety** $\mathcal{F}\ell(\mathbb{C}^n)$ is the set of complete flags of nested vector subspaces

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n,$$

where $\dim V_i = i$.

Example

$$\text{Standard flag } \mathcal{SF}: \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \subset \begin{bmatrix} * \\ 0 \\ 0 \\ 0 \end{bmatrix} \subset \begin{bmatrix} * \\ * \\ 0 \\ 0 \end{bmatrix} \subset \begin{bmatrix} * \\ * \\ * \\ 0 \end{bmatrix} \subset \begin{bmatrix} * \\ * \\ * \\ * \end{bmatrix} = \mathbb{C}^4$$

Since $\mathcal{F}\ell(\mathbb{C}^n)$ has transitive action of GL_n, we can identify it with $\mathrm{GL}_n(\mathbb{C})/\mathrm{Stab}(\mathcal{SF}) = \mathrm{GL}_n(\mathbb{C})/U$, where $U =$ upper triangular matrices.

Matrix Schubert varieties

Bruhat decomposition: $\mathrm{GL}_n = \coprod_{w \in S_n} LwU$

Schubert cells: $X_w^\circ = LwU/U \subset \mathcal{F}\ell(\mathbb{C}^n)$

Schubert varieties: $X_w = \overline{X_w^\circ}$ give a complex cell decomposition of $F\ell(\mathbb{C}^n)$.

The **matrix Schubert variety** (Fulton 1992) $\tilde{X}_w = \overline{LwU} \subseteq \mathrm{Mat}(n)$ is defined by rank conditions on northwest submatrices.

Castelnuovo–Mumford regularity

- R a polynomial ring, $I \subseteq R$ a homogeneous ideal
- A **free resolution** of R/I is an exact diagram of graded R-modules

$$0 \to \bigoplus_{i\in\mathbb{Z}} R(-i)^{b_i^k} \to \cdots \to \bigoplus_{i\in\mathbb{Z}} R(-i)^{b_i^0} \to R/I \to 0$$

  that is exact.
- **Minimal free resolution** simultaneously minimizes all b_i^j
- k is the **projective dimension** of R/I. For R/I Cohen–Macaulay, this is the codimension of $\operatorname{Spec} R/I$ in $\operatorname{Spec} R$.
- The **Castelnuovo–Mumford regularity** of R/I is the greatest $i - j$ such that $b_i^j \neq 0$.

Regularity and K-polynomials

- Write $(R/I)_a$ for the degree a piece of R/I. The **Hilbert series** of R/I is the formal power series

$$H(R/I;t) = \sum_{a \in \mathbb{N}} \dim_{\mathbb{C}} (R/I)_a t^a = \frac{K(R/I;t)}{(1-t)^{n^2}}.$$

- For I prime and R/I Cohen–Macaulay,

$$\mathrm{reg}(R/I) = \deg(K(R/I;t)) - \operatorname{codim}(\operatorname{Spec} R/I).$$

Grothendieck polynomials and K-polynomials

- Start with the **longest** permutation in S_n

$$w_0 = n\, n-1 \ldots 1 \qquad \mathfrak{G}_{w_0}(\mathbf{x}) := x_1^{n-1} x_2^{n-2} \ldots x_{n-1}$$

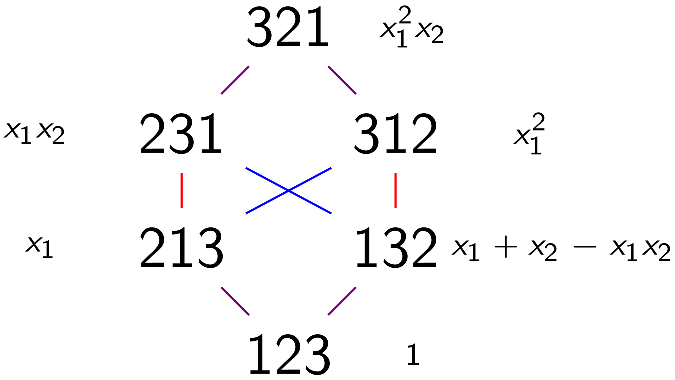
- **Grothendieck polynomials** are defined recursively by **divided difference operators:**

$$\overline{\partial_i} f := \frac{(1 - x_{i+1})f - s_i \cdot (1 - x_{i+1})f}{x_i - x_{i+1}}$$

$$\mathfrak{G}_{ws_i}(\mathbf{x}) := \overline{\partial_i} \mathfrak{G}_w(\mathbf{x}) \text{ if } w(i) > w(i+1)$$

- Setting $x_i \mapsto 1 - t$ gives the K-polynomial for the corresponding matrix Schubert variety
- The **Castelnuovo–Mumford polynomial** $\mathfrak{CM}_w(\mathbf{x})$ is the top-degree part of $\mathfrak{G}_w(\mathbf{x})$

Example Grothendieck polynomials



What is the degree of a Grothendieck polynomial?

- All of the previous was observed by Jenna Rajchgot, who then asked the title of this slide
- With Ren, Robichaux, St. Dizier, and Weigandt (2021), she gave a formula for the *Grassmannian* case

Theorem (P+Speyer+Weigandt)

For $w \in S_n$, we have $\deg \mathfrak{CM}_w(\mathbf{x}) = \mathrm{raj}(w)$, *the* **Rajchgot index** *of w.*

In particular, the Castelnuovo–Mumford regularity of the matrix Schubert variety $\tilde{X}_w$ is $\mathrm{raj}(w) - \mathrm{inv}(w)$.

Moreover, for any term order satisfying $x_1 < x_2 < \cdots < x_n$, the leading term of $\mathfrak{CM}_w(\mathbf{x})$ is a scalar multiple of the monomial $\mathbf{x}^{\mathrm{rajcode}(w)} = x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}$.

Rajchgot index and code

- Let $w = w(1)w(2)\cdots w(n)$
- For each k, find a longest increasing subsequence of $w(k)w(k+1)\cdots w(n)$ containing $w(k)$
- Let r_k be the number of terms from $w(k)w(k+1)\cdots w(n)$ omitted from this subsequence.
- $(r_1, \ldots, r_n) = \mathrm{rajcode}(w)$ is the **Rajchgot code** of w and its sum $\mathrm{raj}(w)$ the **Rajchgot index** of w.

Example

$w = 293417568$. A longest increasing subsequence starting from 2 is $2 \bullet 34 \bullet \bullet 568$, which omits three terms, so $r_1 = 3$. In full,

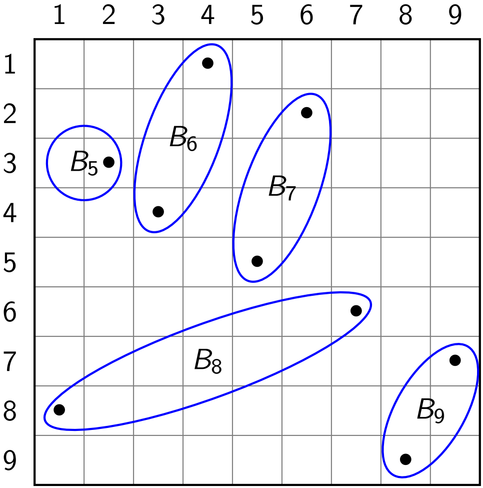
$$\mathrm{rajcode}(w) = (r_1, r_2, \ldots, r_9) = (3, 7, 2, 2, 1, 2, 0, 0, 0).$$

The leading monomial of $\mathfrak{CM}_w(\mathbf{x})$ is $x_1^3 x_2^7 x_3^2 x_4^2 x_5 x_6^2$ and the degree of $\mathfrak{CM}_w(\mathbf{x})$ is $\mathrm{raj}(w) = 17$. Since $\mathrm{inv}(w) = 12$,

$$\mathrm{reg}(\tilde{X}_w) = \mathrm{raj}(w) - \mathrm{inv}(w) = 17 - 12 = 5.$$

The shape of a permutation

$w = 462357918, \pi(w) = \{\{2\},\{3,4\},\{5,6\},\{1,7\},\{8,9\}\}$



Other main results

Unlike $\mathfrak{G}_w(\mathbf{x})$, many $\mathfrak{CM}_w(\mathbf{x})$ coincide up to scalar multiple. Distinct $\mathfrak{CM}_w(\mathbf{x})$ are counted by Bell numbers.

Theorem (P+Speyer+Weigandt)

Double Castelnuovo–Mumford polynomials factor into **Rajchgot polynomials** *as*

$$\mathfrak{CM}_w(\mathbf{x};\mathbf{y}) = \mathfrak{R}_{\pi(w)}(\mathbf{x})\mathfrak{R}_{\pi(w^{-1})}(\mathbf{y}).$$

Moreover, for any term order satisfying

$$x_1 < x_2 < \cdots < x_n \quad \textit{and} \quad y_1 < y_2 < \cdots < y_n,$$

$\mathfrak{CM}_w(\mathbf{x};\mathbf{y})$ *has leading term exactly* $\mathbf{x}^{\mathrm{rajcode}(w)}\mathbf{y}^{\mathrm{rajcode}(w^{-1})}$

Two more characterizations of Rajchgot index

Theorem (P+Speyer+Weigandt)

For all $w \in S_n$, $\mathrm{inv}(w) \leq \mathrm{raj}(w)$ with equality exactly when w is dominant (132-avoiding), and $\mathrm{maj}(w) \leq \mathrm{raj}(w)$ with equality exactly when w is fireworks ($3-12$-avoiding).

Rajchgot index can be computed from major index on Bruhat intervals.

Theorem (P+Speyer+Weigandt)

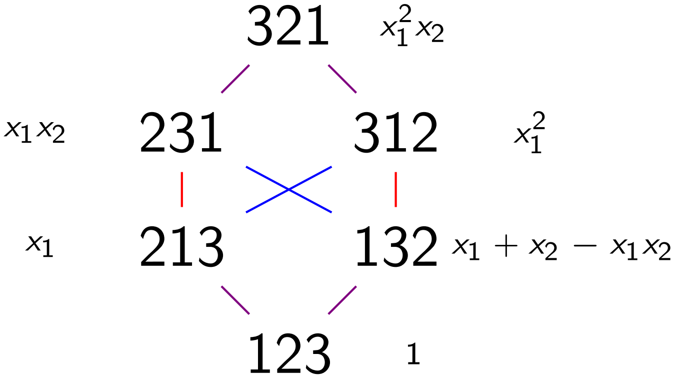
For all $w \in S_n$,

$$\mathrm{raj}(w) = \max\{\mathrm{maj}(v) : v \leq_R w\} = \max\{\mathrm{maj}(u^{-1}) : u \leq_L w\},$$

where $\leq_L$ and $\leq_R$ denote the left and right weak orders, respectively.

Idea of proof that $\deg \mathfrak{CM}_w(\mathbf{x}) = \mathrm{raj}(w)$

- Not hard to see that $\deg \mathfrak{CM}_w(\mathbf{x}) = \mathrm{raj}(w)$ for *dominant permutations* (132-avoiding)
- Also not hard to see $\deg \mathfrak{CM}_w(\mathbf{x}) = \mathrm{raj}(w)$ for *layered permutations* (231- and 312-avoiding)
- Both $\deg \mathfrak{CM}_w(\mathbf{x})$ and $\mathrm{raj}(w)$ are weakly increasing in 2-sided weak order
- Show that every w sits between a layered permutation and a dominant permutation with the same Rajchgot index



Thank you!!