Web invariants for noncrossing partitions

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Standard Young tableaux

A standard Young tableau is a filling of a Young diagram with n boxes with positive integers such that

- entries $1, \dots, n$ each appear exactly once and
- entries strictly increase across rows and down columns.

1	2	4	7
3	6		
5			

1	3	4
2	5	6
7		
8		

1	2	5	7
3	6	8	9
4	10	11	12

Promotion on SYT

Given a standard Young tableau T of shape λ with $|\lambda| = k$, form $p(T)$ using the following steps.

(1) Delete the entry 1 from the box in the upper lefthand corner of T and replace it with $\bullet$.

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 6 \\ \hline 3 & 5 & 7 \\ \hline 4 & 8 & 9 \\ \hline \end{array} \quad \mapsto \quad \begin{array}{|c|c|c|}\hline \bullet & 2 & 6 \\ \hline 3 & 5 & 7 \\ \hline 4 & 8 & 9 \\ \hline \end{array}$$

Promotion on SYT

Given a standard Young tableau T of shape λ with $|\lambda| = k$, form $p(T)$ using the following steps.

(2) Perform jeu de taquin.

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 6 \\ \hline 3 & 5 & 7 \\ \hline 4 & 8 & 9 \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|}\hline \bullet & 2 & 6 \\ \hline 3 & 5 & 7 \\ \hline 4 & 8 & 9 \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|}\hline 2 & \bullet & 6 \\ \hline 3 & 5 & 7 \\ \hline 4 & 8 & 9 \\ \hline \end{array}$$

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Promotion on SYT

Given a standard Young tableau T of shape λ with $|\lambda| = k$, form $p(T)$ using the following steps.

(2) Perform jeu de taquin.

$T =$

1	2	6
3	5	7
4	8	9

$\mapsto$

$\bullet$	2	6
3	5	7
4	8	9

$\mapsto$

2	$\bullet$	6
3	5	7
4	8	9

$\mapsto$

2	5	6
3	$\bullet$	7
4	8	9

$\mapsto$

2	5	6
3	7	$\bullet$
4	8	9

Promotion on SYT

Given a standard Young tableau T of shape λ with $|\lambda| = k$, form $p(T)$ using the following steps.

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3	5	7
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$\mapsto$

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3	5	7
4	8	9

$\mapsto$

2	$\bullet$	6
3	5	7
4	8	9

$\mapsto$

2	5	6
3	$\bullet$	7
4	8	9

$\mapsto$

2	5	6
3	7	$\bullet$
4	8	9

$\mapsto$

2	5	6
3	7	9
4	8	$\bullet$

Promotion on SYT

Given a standard Young tableau T of shape λ with $|\lambda| = k$, form $p(T)$ using the following steps.

(3) Delete the $\bullet$ and fill its box with $k+1$.

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 6 \\ \hline 3 & 5 & 7 \\ \hline 4 & 8 & 9 \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|}\hline \bullet & 2 & 6 \\ \hline 3 & 5 & 7 \\ \hline 4 & 8 & 9 \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|}\hline 2 & \bullet & 6 \\ \hline 3 & 5 & 7 \\ \hline 4 & 8 & 9 \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|}\hline 2 & 5 & 6 \\ \hline 3 & \bullet & 7 \\ \hline 4 & 8 & 9 \\ \hline \end{array} \mapsto$$

$$\begin{array}{|c|c|c|}\hline 2 & 5 & 6 \\ \hline 3 & 7 & \bullet \\ \hline 4 & 8 & 9 \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|}\hline 2 & 5 & 6 \\ \hline 3 & 7 & 9 \\ \hline 4 & 8 & \bullet \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|}\hline 2 & 5 & 6 \\ \hline 3 & 7 & 9 \\ \hline 4 & 8 & 10 \\ \hline \end{array}$$

Promotion on SYT

Given a standard Young tableau T of shape λ with $|\lambda| = k$, form $p(T)$ using the following steps.

(4) Subtract 1 from each entry.

$T =$

1	2	6
3	5	7
4	8	9

$\mapsto$

•	2	6
3	5	7
4	8	9

$\mapsto$

2	•	6
3	5	7
4	8	9

$\mapsto$

2	5	6
3	•	7
4	8	9

$\mapsto$

2	5	6
3	7	•
4	8	9

$\mapsto$

2	5	6
3	7	9
4	8	•

$\mapsto$

2	5	6
3	7	9
4	8	10

$\mapsto$

1	4	5
2	6	8
3	7	9

Promotion on SYT

What are the orbit sizes of standard Young tableaux under promotion?

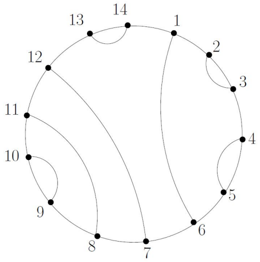
Let $p(T)$ be the result of performing promotion on standard Young tableau T.

Schützenberger: For rectangular T with n boxes, $p^n(T) = T$.

Our motivation for this project.

1	2	4	7	8	9	13
3	5	6	10	11	12	14

$\longleftrightarrow$

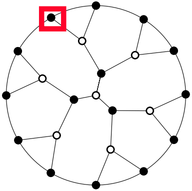


- Bijection between 2-row standard Young tableaux and noncrossing matchings (or sl_2 webs).
- Tableau promotion corresponds to rotation of diagram. (D. White)
- Tableau evacuation corresponds to reflection of diagram. (P.–Pechenik)

Our motivation for this project.

1	3	5	9
2	6	7	11
4	8	10	12

$\longleftrightarrow$



- Bijection between 3-row standard Young tableaux and sl_3 webs. (Khovanov–Kuperberg, Tymoczko)
- Tableau promotion corresponds to rotation of diagram. (Petersen–Pylyavskyy–Rhoades, Tymoczko)
- Tableau evacuation corresponds to reflection of diagram. (P.–Pechenik)

Our motivation for this project

An **increasing tableau** has strictly increasing rows and strictly increasing columns.

1	2	4	5
2	3	5	6
5	6	7	8

1	2	3
2	3	
3		

Increasing tableaux are a K-theoretic analogue of standard Young tableaux.

- K-promotion on increasing tableaux
- K-evacuation on increasing tableaux

K-Promotion on increasing tableaux

K-promotion looks like this:

1	2	3	5
2	4	5	6
3	6	8	9

.

K-Promotion on increasing tableaux

K-promotion looks like this:

•	2	3	5
2	4	5	6
3	6	8	9

.

K-Promotion on increasing tableaux

K-promotion looks like this:

2	•	3	5
•	4	5	6
3	6	8	9

.

K-Promotion on increasing tableaux

K-promotion looks like this:

2	3	$\bullet$	5
3	4	5	6
$\bullet$	6	8	9

.

K-Promotion on increasing tableaux

K-promotion looks like this:

2	3	5	$\bullet$
3	4	$\bullet$	6
$\bullet$	6	8	9

.

K-Promotion on increasing tableaux

K-promotion looks like this:

2	3	5	6
3	4	6	•
6	•	8	9

.

K-Promotion on increasing tableaux

K-promotion looks like this:

2	3	5	6
3	4	6	$\bullet$
6	8	$\bullet$	9

.

K-Promotion on increasing tableaux

K-promotion looks like this:

2	3	5	6
3	4	6	9
6	8	9	•

.

K-Promotion on increasing tableaux

K-promotion looks like this:

1	2	4	5
2	3	5	8
5	7	8	9

.

(Thomas–Yong, Pechenik)

K-Promotion on increasing tableaux

What are the orbit sizes of increasing tableaux under K-promotion?

In general, we don't know!

K-promotion

What is the order of K-promotion on rectangular increasing tableaux?

Conjecture (Cameron–Fon-der-Flaass 1995)

Let T be a packed, rectangular increasing tableau with largest entry q. If q is prime, then the length of the K-promotion orbit of T is a multiple of q (except in the weird case).

1	2	5
3	4	6
4	6	7

1	2	3	4
2	3	4	5
3	4	5	6

K-promotion

What is the order of K-promotion on rectangular increasing tableaux?

Theorem (P.–Pechenik 2020)

Let T be a rectangular increasing tableau with largest entry q and K-promotion orbit of size k. Then k and q share a prime divisor (except in the weird case).

1	2	5
3	4	6
4	6	7

1	2	3	4
2	3	4	5
3	4	5	6

K-promotion

What is the order of K-promotion on rectangular increasing tableaux?

Theorem (P.–Pechenik 2020)

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1	2	5
3	4	6
4	6	7

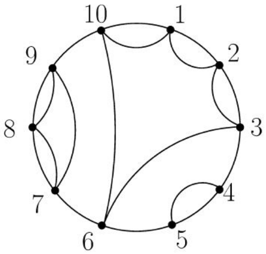
1	2	3	4
2	3	4	5
3	4	5	6

Things looks great for 2- and 3-row rectangles, and then not at all great beyond that.

Our motivation for this project.

1	2	3	4	6	7	8
2	3	5	6	8	9	10

$\longleftrightarrow$

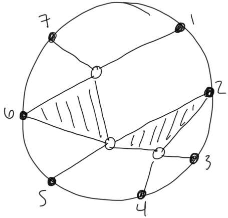


- Bijection between "packed" 2-row increasing tableaux and noncrossing partitions with no singletons.
- Tableau promotion corresponds to rotation of diagram. (Pechenik)
- Tableau evacuation corresponds to reflection of diagram. (Pechenik)

Our motivation for this project

1	2	3
3	5	6
4	6	7

$?? \longleftrightarrow ??$

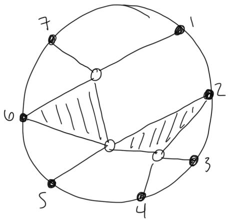


Question: Can we do this for 3-row increasing tableaux?

Our motivation for this project

1	2	3
3	5	6
4	6	7

$?? \longleftrightarrow ??$



Idea: The matchings and sl_3 webs have nice polynomials associated to them. Maybe we first need to understand what polynomials should be associated with noncrossing partitions.

Consider the following matrix of indeterminants and the set of polynomials in this variable set.

$$M_n = \begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ y_1 & y_2 & y_3 & \cdots & y_n \end{pmatrix}$$

e.g. $p_{13}(X, Y) = x_1 y_3 - x_3 y_1$

$SL_2(\mathbb{C})$ acts by left multiplication. For example,

$$\begin{pmatrix} 1 & 2 \\ .5 & 2 \end{pmatrix}\begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ y_1 & y_2 & y_3 & \cdots & y_n \end{pmatrix}=\begin{pmatrix} x_1+2y_1 & x_2+2y_2 & x_3+2y_3 & \cdots & x_n+2y_n \\ .5x_1+2y_1 & .5x_2+2y_2 & .5x_3+2y_3 & \cdots & .5x_n+2y_n \end{pmatrix}$$

$$p_{13} = x_1y_3 - x_3y_1 \longrightarrow (x_1+2y_1)(.5x_3+2y_3) - (x_3+2y_3)(.5x_1+2y_1)$$
$$= \cdots = x_1y_3 - x_3y_1$$

Which polynomials are invariant under this action?

$$M_n = \begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ y_1 & y_2 & y_3 & \cdots & y_n \end{pmatrix}$$

Which polynomials are invariant under this action?

Classical invariant theory: $\mathbb{C}[X, Y]^{SL_2(\mathbb{C})} = \langle\ 2 \times 2 \text{ minors }\rangle$

We can specify each 2×2 minor by a pair of columns, p_{ij}.

$$M_n = \begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ y_1 & y_2 & y_3 & \cdots & y_n \end{pmatrix}$$

Which polynomials are invariant under this action?

Classical invariant theory: $\mathbb{C}[X, Y]^{SL_2(\mathbb{C})} = \langle\ 2 \times 2 \text{ minors}\ \rangle$

We can specify each 2×2 minor by a pair of columns, p_{ij}.

We are going to restrict to the setting where each column is used in exactly one minor.

$$p_{25}p_{13}p_{46}$$

Two ways to encode products of p_{ij}s

$$M_8 = \begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_8 \\ y_1 & y_2 & y_3 & \cdots & y_8 \end{pmatrix}$$

$p_{14}p_{27}p_{35}p_{68} =$

$$\begin{vmatrix} x_1 & x_4 \\ y_1 & y_4 \end{vmatrix} \begin{vmatrix} x_2 & x_7 \\ y_2 & y_7 \end{vmatrix} \begin{vmatrix} x_3 & x_5 \\ y_3 & y_5 \end{vmatrix} \begin{vmatrix} x_6 & x_8 \\ y_6 & y_8 \end{vmatrix}$$

Two ways to encode products of p_{ij}s

$$M_n = \begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ y_1 & y_2 & y_3 & \cdots & y_n \end{pmatrix}$$

$p_{14}p_{27}p_{35}p_{68} =$

$$\begin{vmatrix} x_1 & x_4 \\ y_1 & y_4 \end{vmatrix} \begin{vmatrix} x_2 & x_7 \\ y_2 & y_7 \end{vmatrix} \begin{vmatrix} x_3 & x_5 \\ y_3 & y_5 \end{vmatrix} \begin{vmatrix} x_6 & x_8 \\ y_6 & y_8 \end{vmatrix}$$

1	2	3	6
4	7	5	8

Two different bases

Restrict to using each column exactly once.

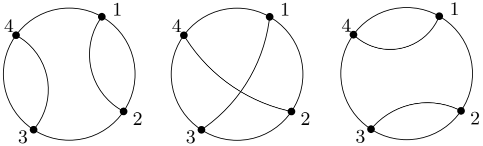
- standard Young tableau basis
- noncrossing matching basis

$p_{12}p_{34}$ $\quad$ $p_{13}p_{24}$ $\quad$ $p_{14}p_{23}$

1	3
2	4

1	2
3	4

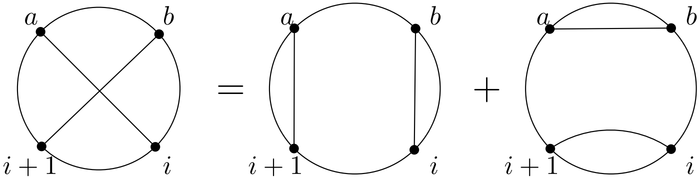
1	2
4	3



Two different bases

Restrict to using each column exactly once.

- standard Young tableau basis
- noncrossing matching basis



S_n action

S_n acts by right multiplication and permutes the columns of M_n. For example,

$$\begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ y_1 & y_2 & y_3 & \cdots & y_n \end{pmatrix} n1234\cdots(n{-}1) = \begin{pmatrix} x_n & x_1 & x_2 & \cdots & x_{n-1} \\ y_n & y_1 & y_2 & \cdots & y_{n-1} \end{pmatrix}$$

$$p_{13} = x_1y_3 - x_3y_1 \longrightarrow -p_{2n} = x_ny_2 - x_2y_n$$

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$$p_{13} = x_1y_3 - x_3y_1 \longrightarrow -p_{2n} = x_ny_2 - x_2y_n$$

Right action of S_n permuting columns in our restricted setting gives Specht module $S^{(m,m)}$, where $m = n/2$.
Noncrossing matchings and standard Young tableaux are bases.

3-row case

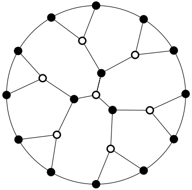
Now consider

$$M_n = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \\ z_1 & z_2 & \cdots & z_n \end{pmatrix}.$$

$$\vdots$$

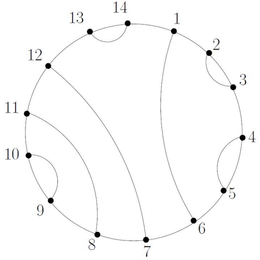
$$\vdots$$

1	3	5	6
2	4	7	9
8	10	11	12



- Bases of Specht module $S^{(m,m,m)}$, where $m = n/3$.

$p_{16}p_{23}p_{45}p_{7,12}p_{8,11}p_{9,10}p_{13,14}$ $\longleftrightarrow$

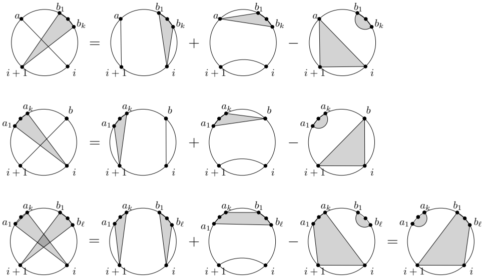


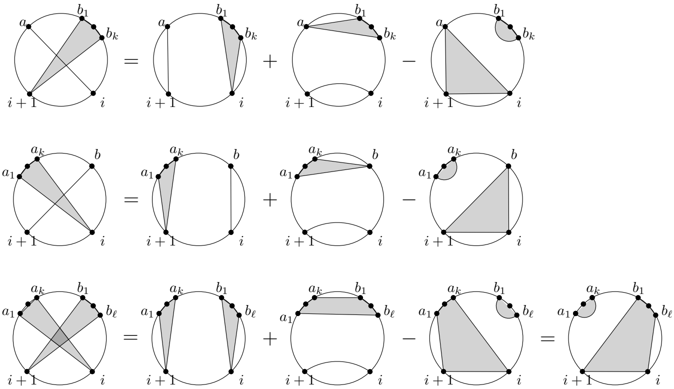
On to partitions with no singletons!

On to noncrossing partitions with no singletons!

Wish list:

- Want to be able to associate a polynomial to any partition with no singleton blocks.
- Want the long cycle to rotate noncrossing partitions.
- Agree with B. Rhoades's uncrossing rules up to sign.

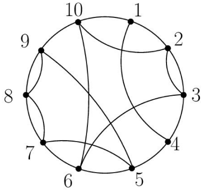




Notice

- Block size changes
- Number of blocks does not change

Invariants

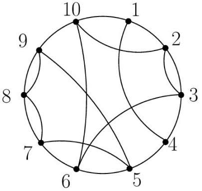


$$M = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1,10} \\ x_{21} & x_{22} & \cdots & x_{2,10} \\ x_{31} & x_{32} & \cdots & x_{3,10} \\ \vdots & & & \vdots \end{pmatrix}$$

Naive guess: $p_{14} p_{23610} p_{5789}$

1	2	5
4	3	7
	6	8
	10	9

Invariants



Jellyfish tableaux

- Blocks are columns
- Each column has entry in first two rows
- Exactly one entry in each row > 2

2	5	1
3	7	4
6		
10		
	8	
	9	

2	5	1
3	7	4
6		
	8	
10		
	9	

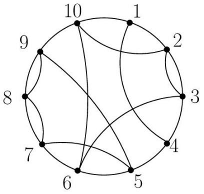
2	5	1
3	7	4
6		
	8	
	9	
10		

2	5	1
3	7	4
	8	
6		
10		
	9	

2	5	1
3	7	4
	8	
6		
	9	
10		

2	5	1
3	7	4
	8	
	9	
6		
10		

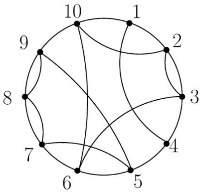
Invariants



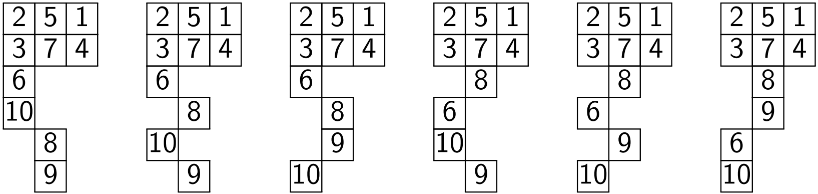
$$M = \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{16} & \cdots & x_{1,10} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{26} & \cdots & x_{2,10} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{36} & \cdots & x_{3,10} \\ x_{41} & x_{42} & x_{43} & \cdots & x_{46} & \cdots & x_{4,10} \\ x_{51} & x_{52} & x_{53} & \cdots & x_{56} & \cdots & x_{5,10} \\ x_{61} & x_{62} & x_{63} & \cdots & x_{66} & \cdots & x_{6,10} \end{pmatrix}$$

2	5	1
3	7	4
6		
10		
	8	
	9	

Invariants



$$M = \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{16} & \cdots & x_{1,10} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{26} & \cdots & x_{2,10} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{36} & \cdots & x_{3,10} \\ x_{41} & x_{42} & x_{43} & \cdots & x_{46} & \cdots & x_{4,10} \\ x_{51} & x_{52} & x_{53} & \cdots & x_{56} & \cdots & x_{5,10} \\ x_{61} & x_{62} & x_{63} & \cdots & x_{66} & \cdots & x_{6,10} \end{pmatrix}$$

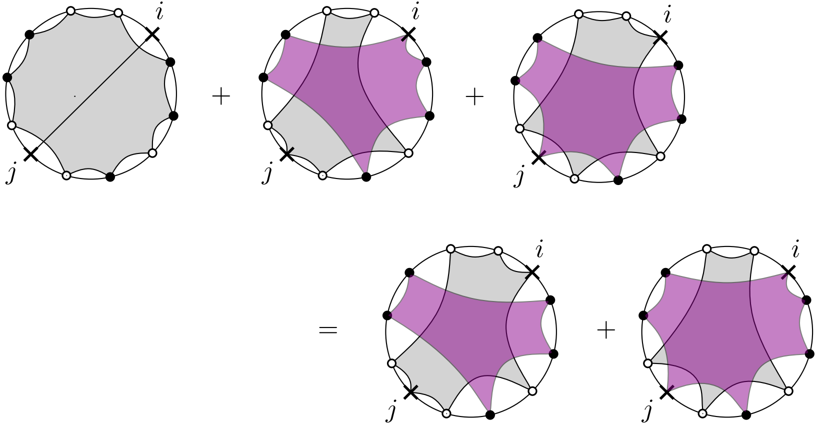


$S^{(d,d,1^\ell)}$

Theorem (Pechenik 2014, Rhoades 2017, Kim–Rhoades 2022, P–Pechenik–Striker 2022)

Noncrossing partitions of $2d+\ell$ into d parts with no singletons give a basis for the Specht module $S^{(d,d,1^\ell)}$. The long cycle $n12\cdots(n-1)$ acts by rotation and w_0 by reflection.

Crossing rules



Invariants of what?

$$M = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ x_{d1} & x_{d2} & \cdots & x_{dn} \end{pmatrix}$$

Which polynomials in $\mathbb{C}[M]$ are invariant under the following subgroup of $SL_d(\mathbb{C})$?

$$P = \left\{ \begin{pmatrix} a_{11} & a_{12} & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3d} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ a_{d1} & a_{d2} & a_{d3} & \cdots & a_{dd} \end{pmatrix} \;\middle|\; \det = 1 \right\}$$

$\mathbb{C}[M]^P = \langle d \times d$ and top-justified 2×2 minors$\rangle$

Generalization

In Fraser–P.–Pechenik–Striker (2023+), we associate r-jellyfish tableaux to each ordered set partition with blocks sizes at least r.

$r = 3$: $(1\ 4\ 12\ 15 \mid 3\ 8\ 10\ 16 \mid 5\ 6\ 9\ 13 \mid 2\ 7\ 11\ 14)$

1	3	6	2
4	8	5	7
12	10	9	11
	16		
		13	
15			
			14

Generalization

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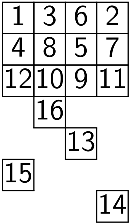
$r = 3$: (1 4 12 15 | 3 8 10 16 | 5 6 9 13 | 2 7 11 14)



For $r > 2$, the set of noncrossing ordered set partitions is linearly independent inside a certain Specht module but does not span. The long cycle $n12 \cdots (n-1)$ acts by rotation and w_0 by reflection.

Future/Current Work

- What can we do with 3-row increasing tableaux now?
- How do we formally connect this to recent work of J. Kim and B. Rhoades?

Thank you!

- C. Fraser, R. Patrias, O. Pechenik, and J. Striker. *Web invariants for flamingo Specht modules,* arXiv preprint arXiv:2308.07256.
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