Generalized Parking Function

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Joint work with François Bergeron and Yan Lanciault

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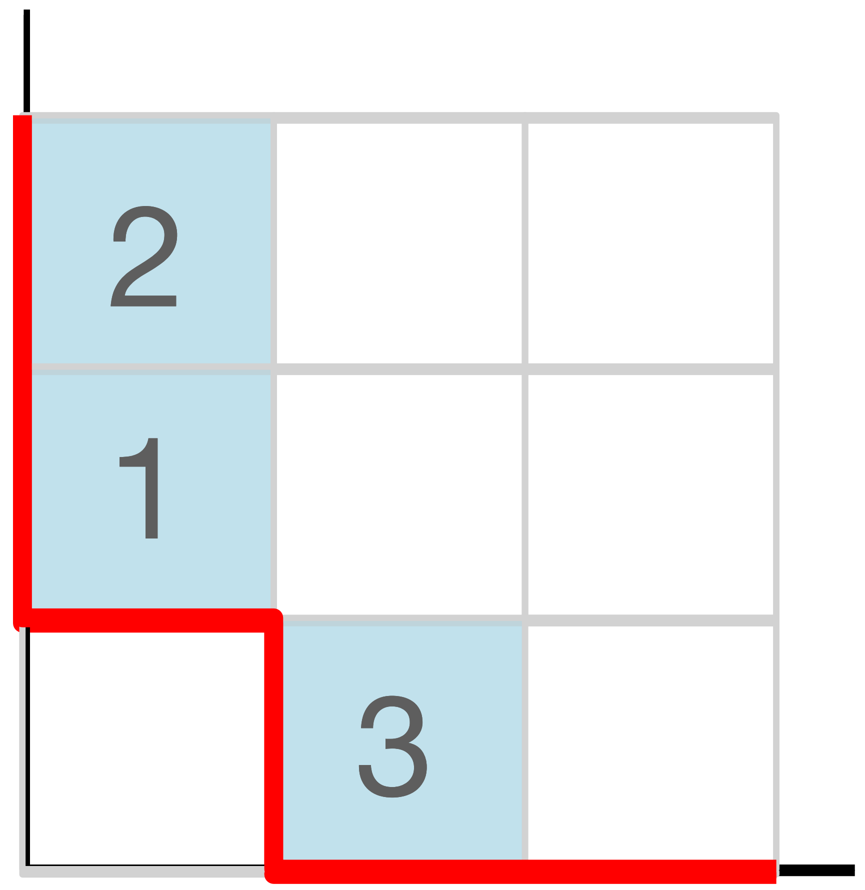
Classical Parking Function

A parking function is a sequence $a_1, \ldots, a_n$ such that if $b_1 \leq b_2 \leq \cdots \leq b_n$ is the increasing rearrangement of $a_1, \ldots, a_n$ then $b_i \leq i$

1,1,2



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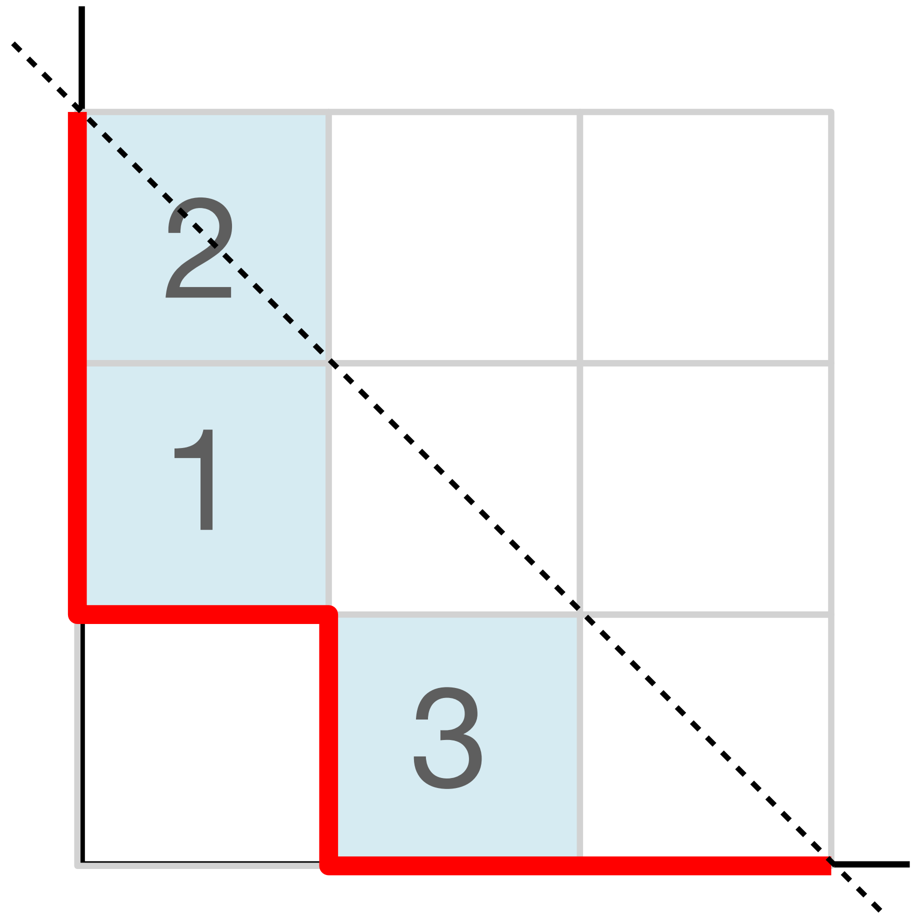
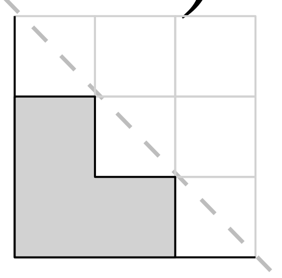
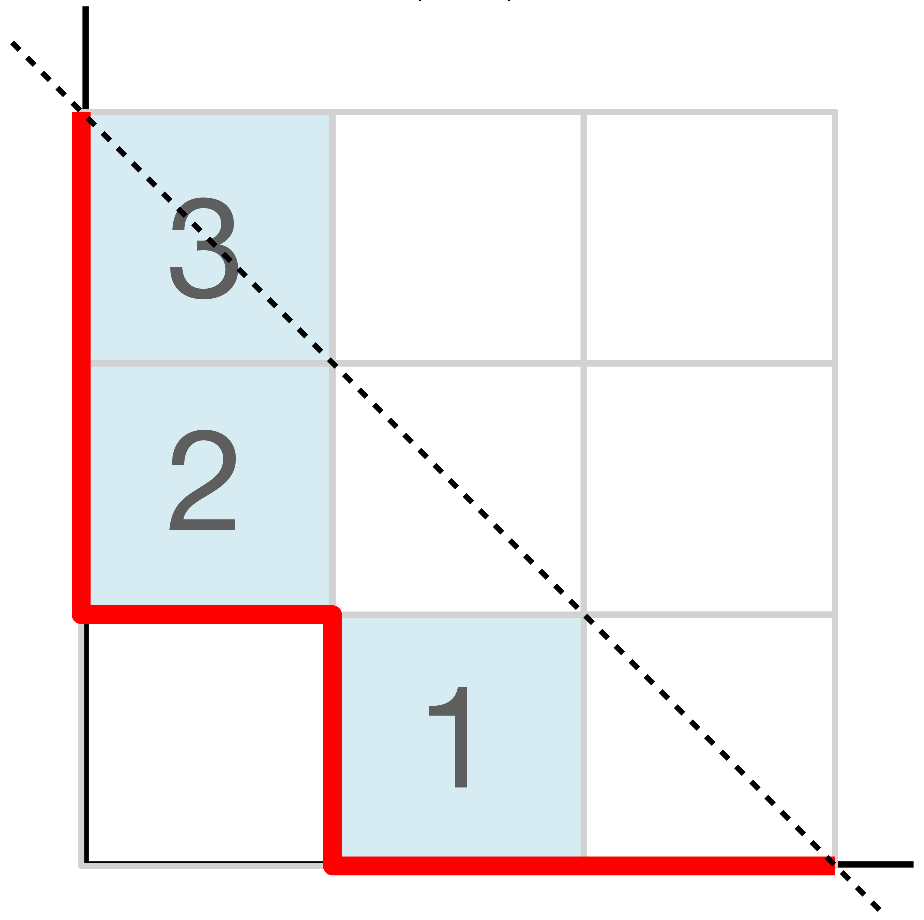


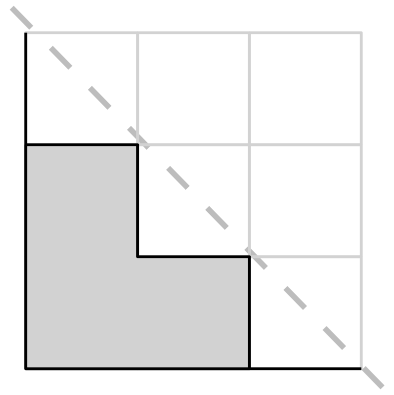
$$\mathrm{PF}_n = \bigcup_{\alpha} \mathrm{SYT}(\alpha + 1^n/\alpha)$$

where $\alpha \subseteq$ 

S_n action!

2,1,1

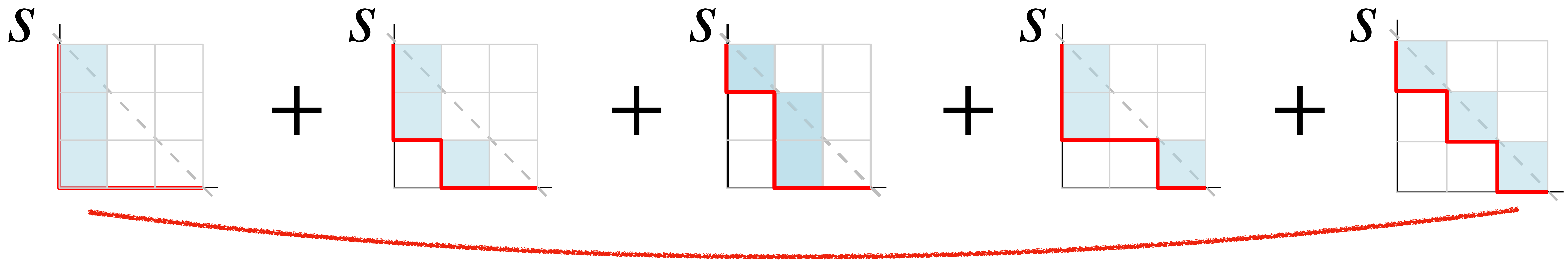


Parking function Frobenius $\Rightarrow \mathscr{P}_n(\mathbf{x}) := \sum_{\alpha \subseteq \lambda} s_{\alpha+1^n/\alpha}(\mathbf{x})$, where $\lambda =$ 

Classical Parking Function

Frobenius character of a parking function: $\mathscr{P}_n(\mathbf{x}) := \sum_{\alpha \subseteq \lambda} s_{\alpha + 1^n/\alpha}(\mathbf{x})$

$\mathscr{P}_3(\mathbf{x}) =$



Classical Dyck path!

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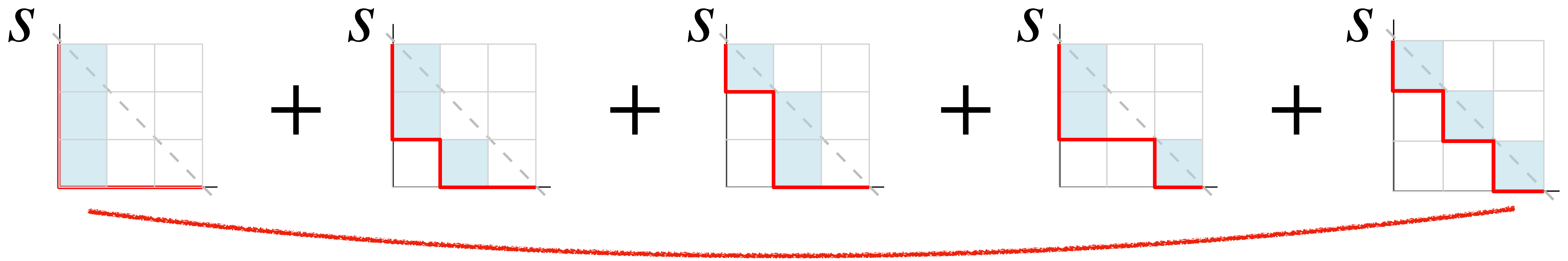
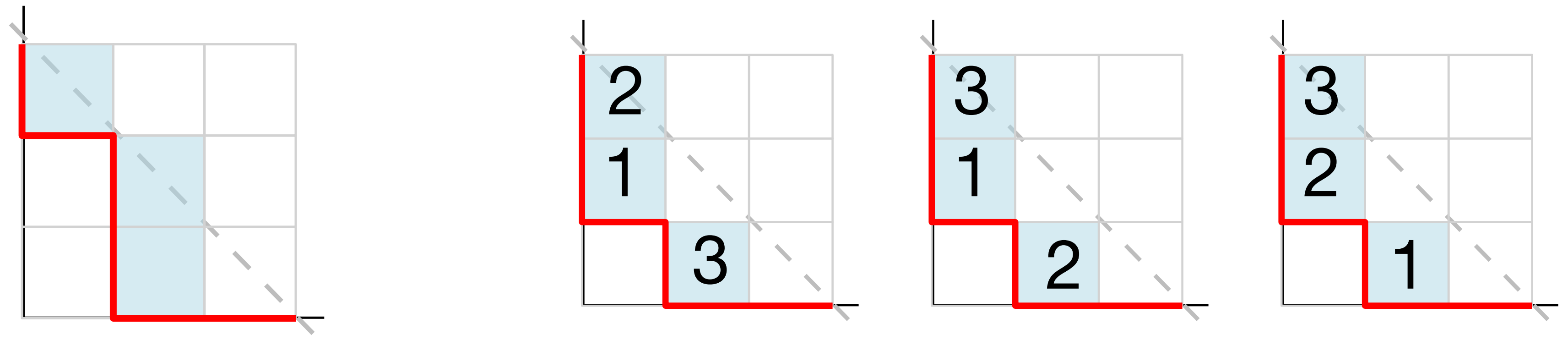


Classical Dyck path!



1,1,2   1,2,1   2,1,1

Classical Parking Function

Frobenius character of a parking function: $\mathscr{P}_n(\mathbf{x}) := \sum_{\alpha\subseteq\lambda} s_{\alpha+1^n/\alpha}(\mathbf{x})$

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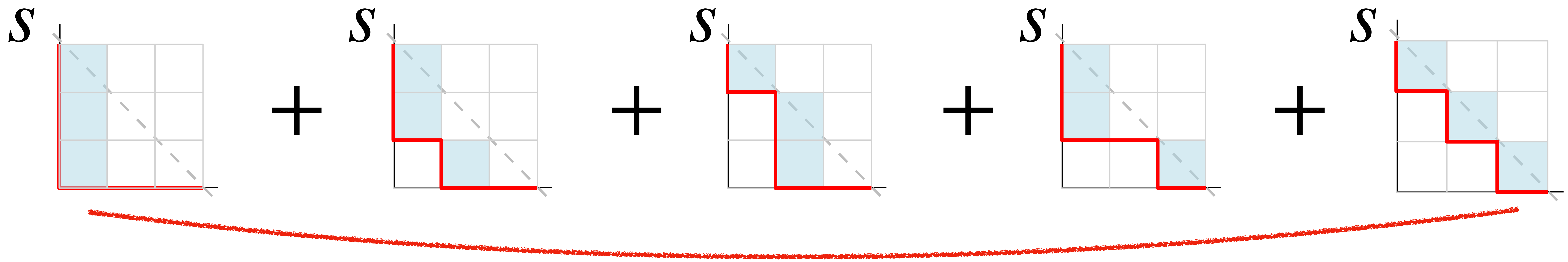
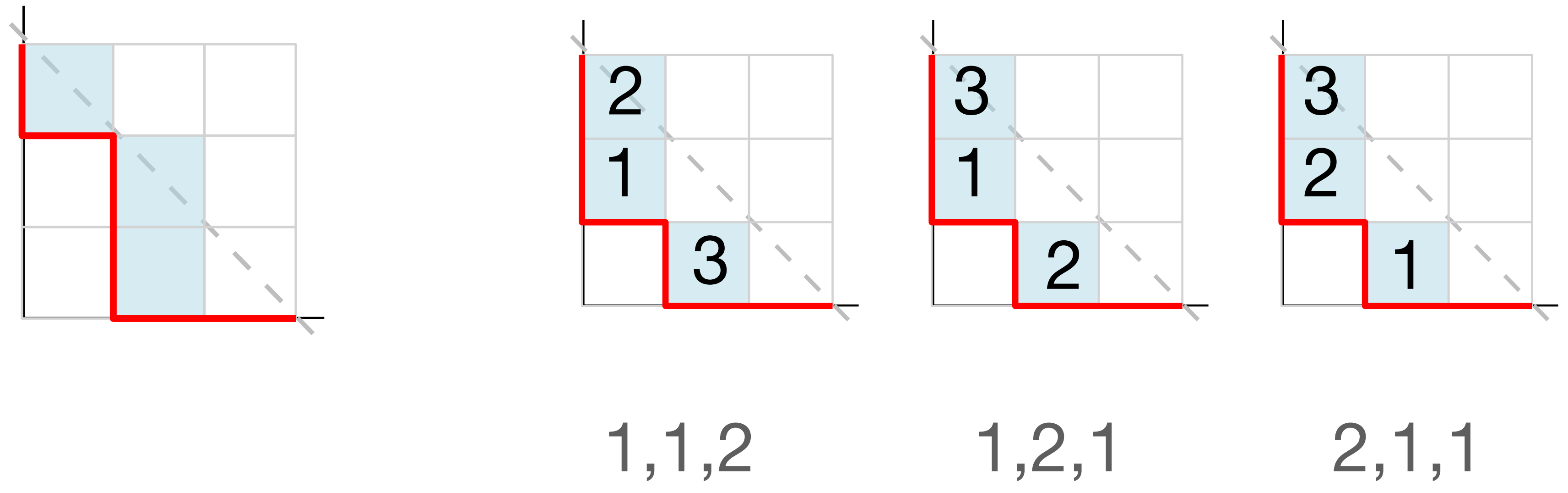


Classical Dyck path!

$$= e_3 + 3e_{21} + e_{111}$$

$$= \frac{1}{4}\binom{4}{3,1} e_3 + \frac{1}{4}\binom{4}{2,1,1} e_{21} + \frac{1}{4}\binom{4}{1,3} e_{111}$$

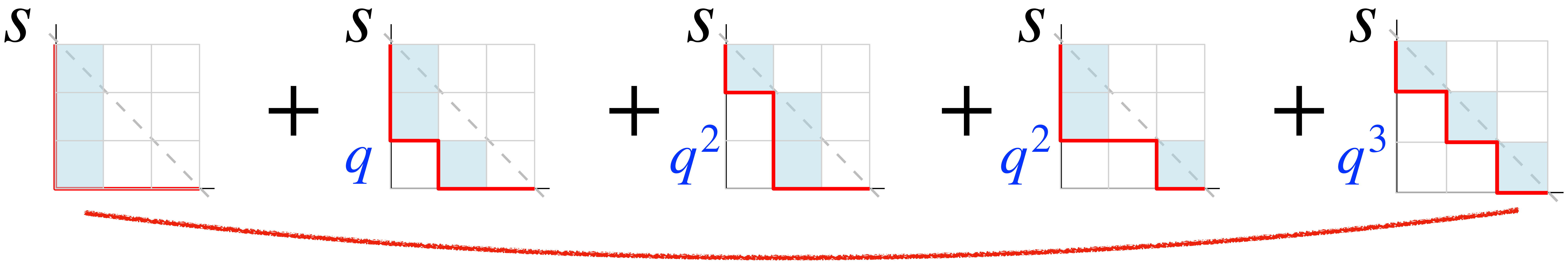
$$= \frac{1}{4}[t^3]\left(t^0 1 + t^1 e_1 + t^2 e_2 + t^3 e_3\right)^4$$



Classical Parking Function

Frobenius character of a parking function: $\mathscr{P}_n(\mathbf{x};q) := \sum_{\alpha \subseteq \lambda} q^{|\alpha|} s_{\alpha + 1^n/\alpha}(\mathbf{x})$

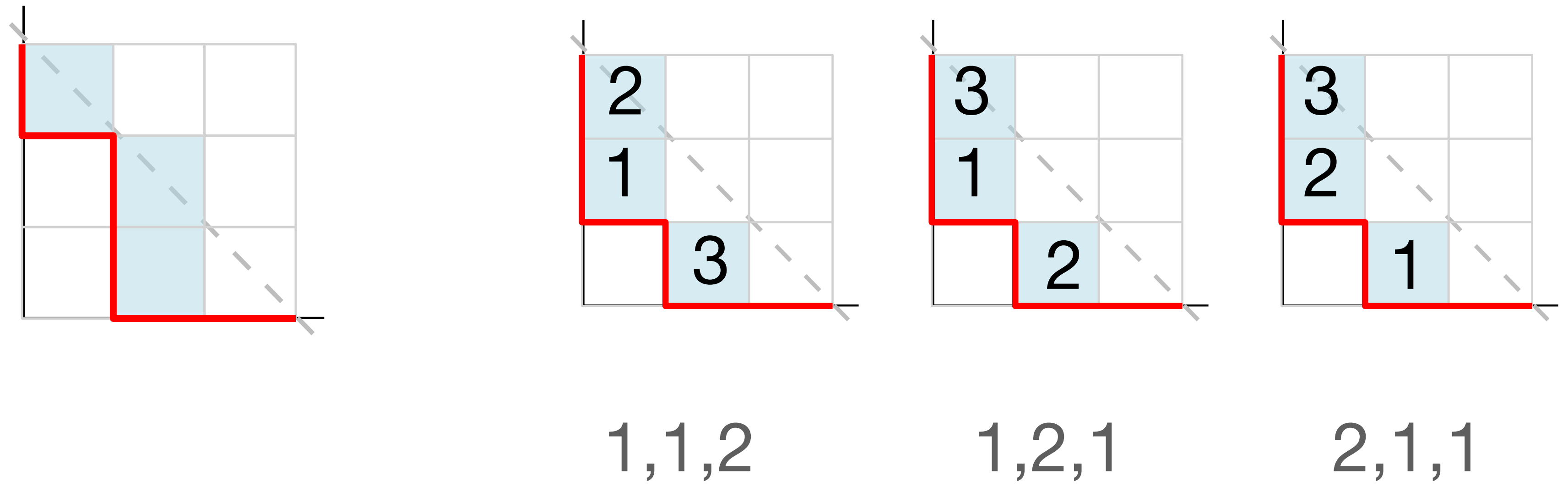
$\mathscr{P}_3(\mathbf{x};q) =$



Classical Dyck path!

$$= e_3 + \left(2q^2 + q\right) e_{21} + q^3 e_{111}$$

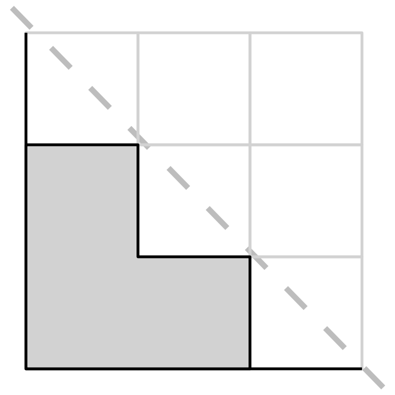
$$= q^3 s_3 + \left(2q^3 + 2q^2 + q\right) s_{21} + \left(q^3 + 2q^2 + q + 1\right) s_{111}$$



$\langle \mathscr{P}_3(\mathbf{x}), p_1^3(\mathbf{x}) \rangle = (3+1)^{3-1}$

$\langle \mathscr{P}_3(\mathbf{x}), e_3(\mathbf{x}) \rangle = 5$

We can generalize this concept by generalizing $\lambda =$



Diagonal Coinvariant ring

$$DR_n = \mathbb{C}[X,Y]/\left\langle \sum_{i=1}^{n} x_i^h y_i^k, \forall h+k>0 \right\rangle$$

The set of polynomials $f(X,Y) \in \mathbb{C}[X,Y]$ such that for all $h+k \geq 0$, we have $\sum_{1\leq j\leq n} \delta_{x_i}^h \delta_{y_i}^k f(X,Y) = 0$.

S_n**-module:** permutation acts on the indices $x_i y_i \to x_{\sigma(i)} y_{\sigma(i)}$

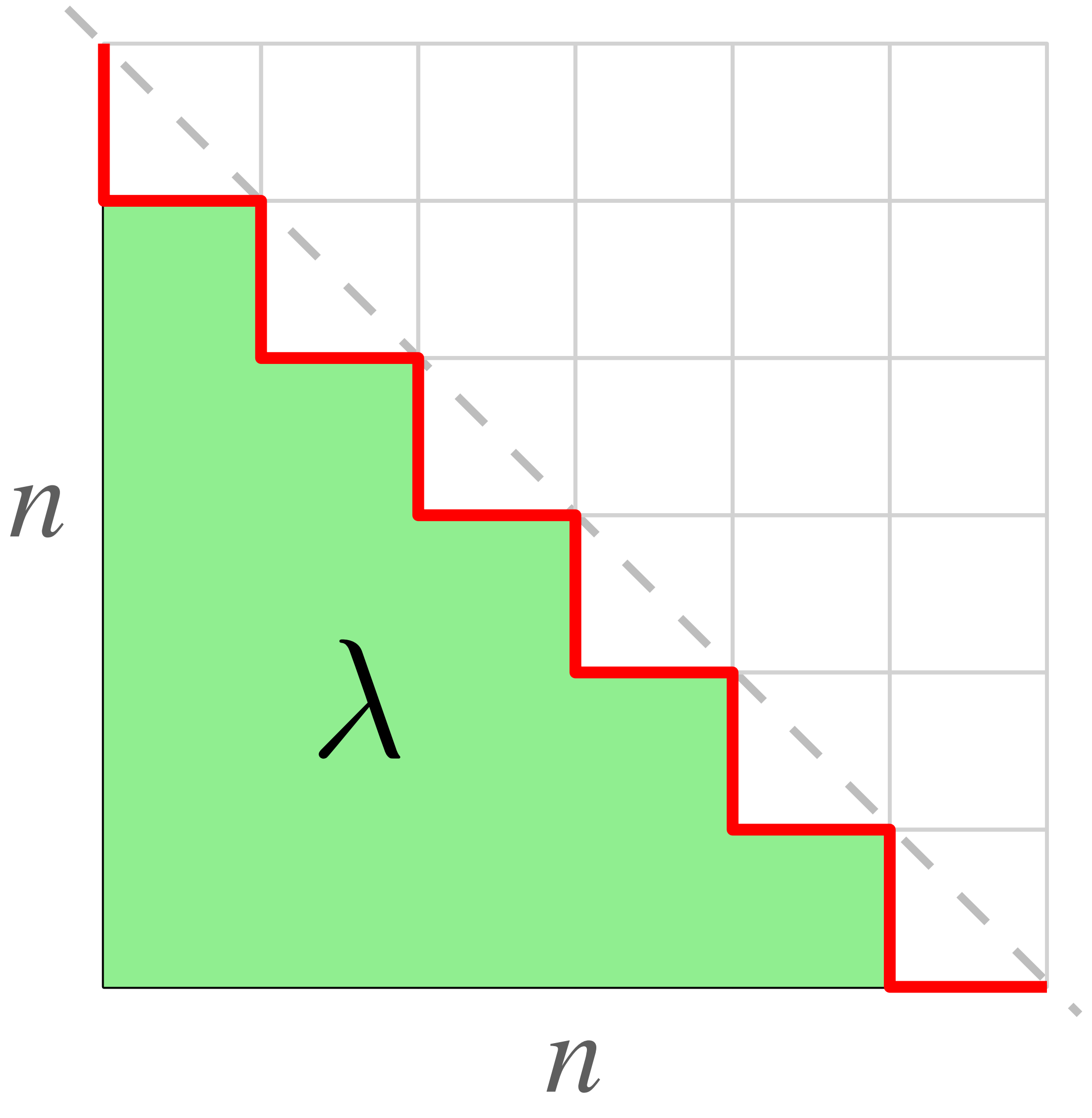
Shuffle Theorem (Haglund—Haiman—Loehr—Remmel—Ulyanov (2015) , H—Morse—Zabrocki (2012), Carlsson—Mellit (2018)):

$$\mathcal{F}(DR_n; q,t) = \sum_{P\in PF_n} q^{\text{coarea}(P)} t^{\text{dinv}(P)} x^P = \nabla e_n$$

$$\mathcal{F}(DR_n; 1,1) = \mathcal{P}_n(\mathbf{x}) \qquad \dim(DR_n) = (n+1)^{n-1}$$

λ-Dyck path/parking function enumeration

Classical Catalan

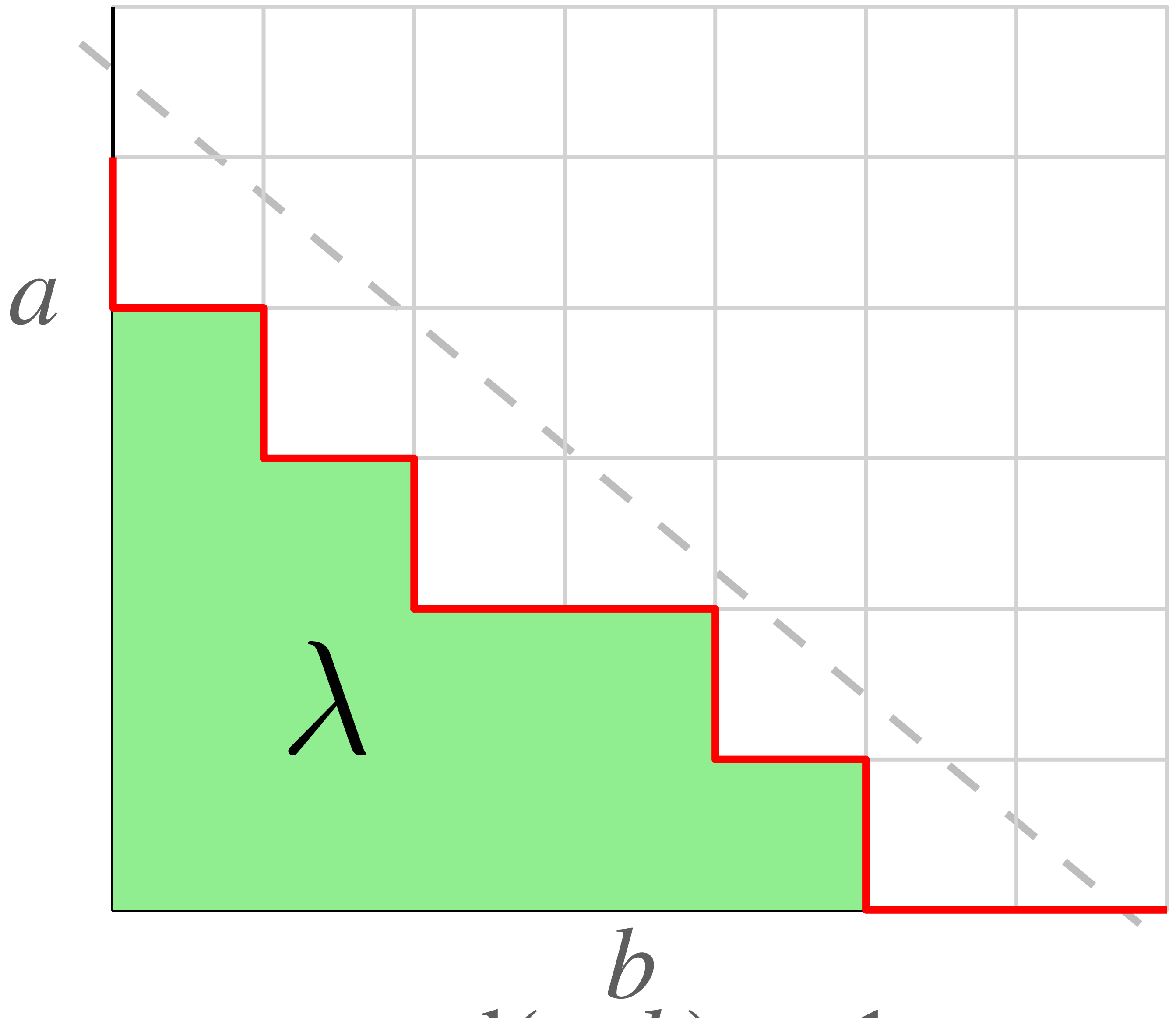


$\subseteq$

Rational Catalan

Bizley, 1954

Armstrong—Loehr—Warrington, 2014



$\gcd(a, b) = 1$

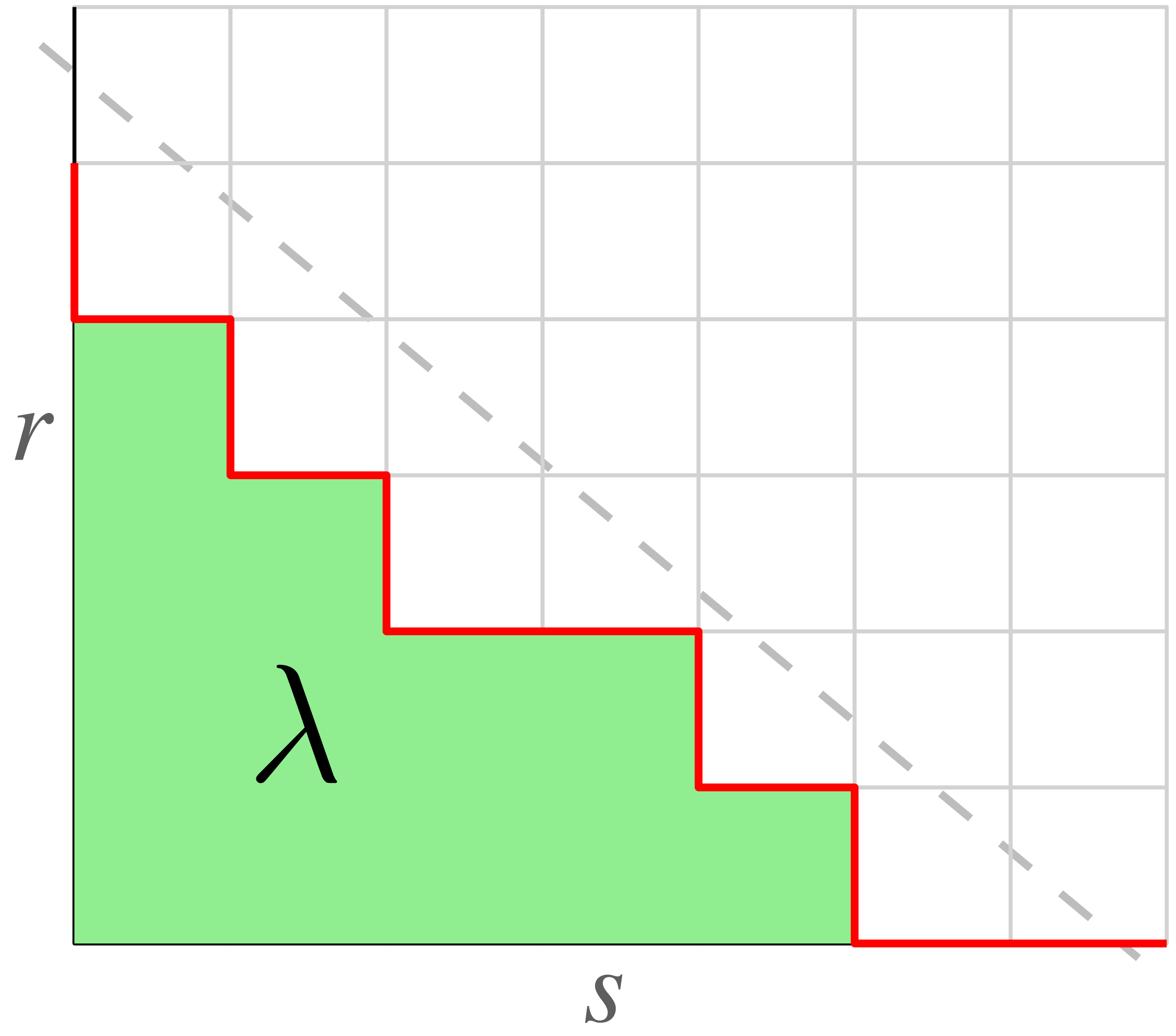
$\subseteq$

Triangular Catalan

Blasiak-Haiman-Morse-Pun-Seelinger, 2023

Bergeron—Mazin, 2023

Elizalde—Galván, 2023



$r, s \in \mathbb{R}_{\geq 0}$

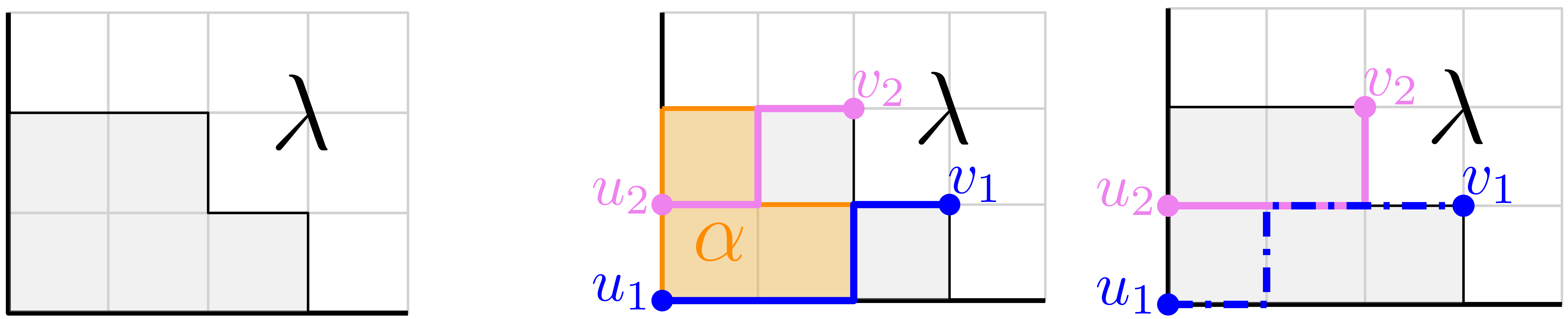
Today:

λ-Dyck path: sub-partitions α under any partition λ

λ-parking function: $\mathscr{P}_\lambda(\mathbf{x}) := \sum_{\alpha \subseteq \lambda} s_{\alpha+1^n/\alpha}(\mathbf{x})$

λ-Dyck path enumeration

Number of sub-partitions α under any partition λ



$\#(u_1 \to v_1) = \binom{4}{1}$ $\#(u_2 \to v_2) = \binom{3}{1}$

$\#(u_1 \to u_2)-$ consecutive $\uparrow$ = $\binom{3}{2}$

\# of $\alpha \subseteq \lambda = \binom{4}{1} \cdot \binom{3}{1} - \binom{3}{2}$

$$\sum_{\alpha \subseteq \alpha} q^{|\alpha|} = \det \begin{pmatrix} \binom{4}{1}_q & q\binom{3}{1}_q \\ 1 & \binom{3}{2}_q \end{pmatrix}$$

Lemma (Lindström—Gessel—Viennot, 1989)

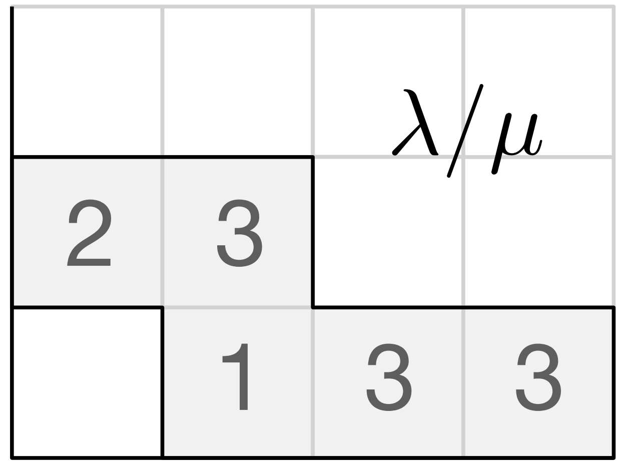
Let G be a directed acyclic graph and $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, \ldots, v_n\}$ be two distinct sets of vertices that are compatible. Then the number of non-intersecting paths from $u_i \to v_i$ is $\det \left(p(u_i, v_j)\right)_{1 \le i,j \le n}$, where $p(u_i, v_j)$ is the number of paths from $u_i \to v_j$.

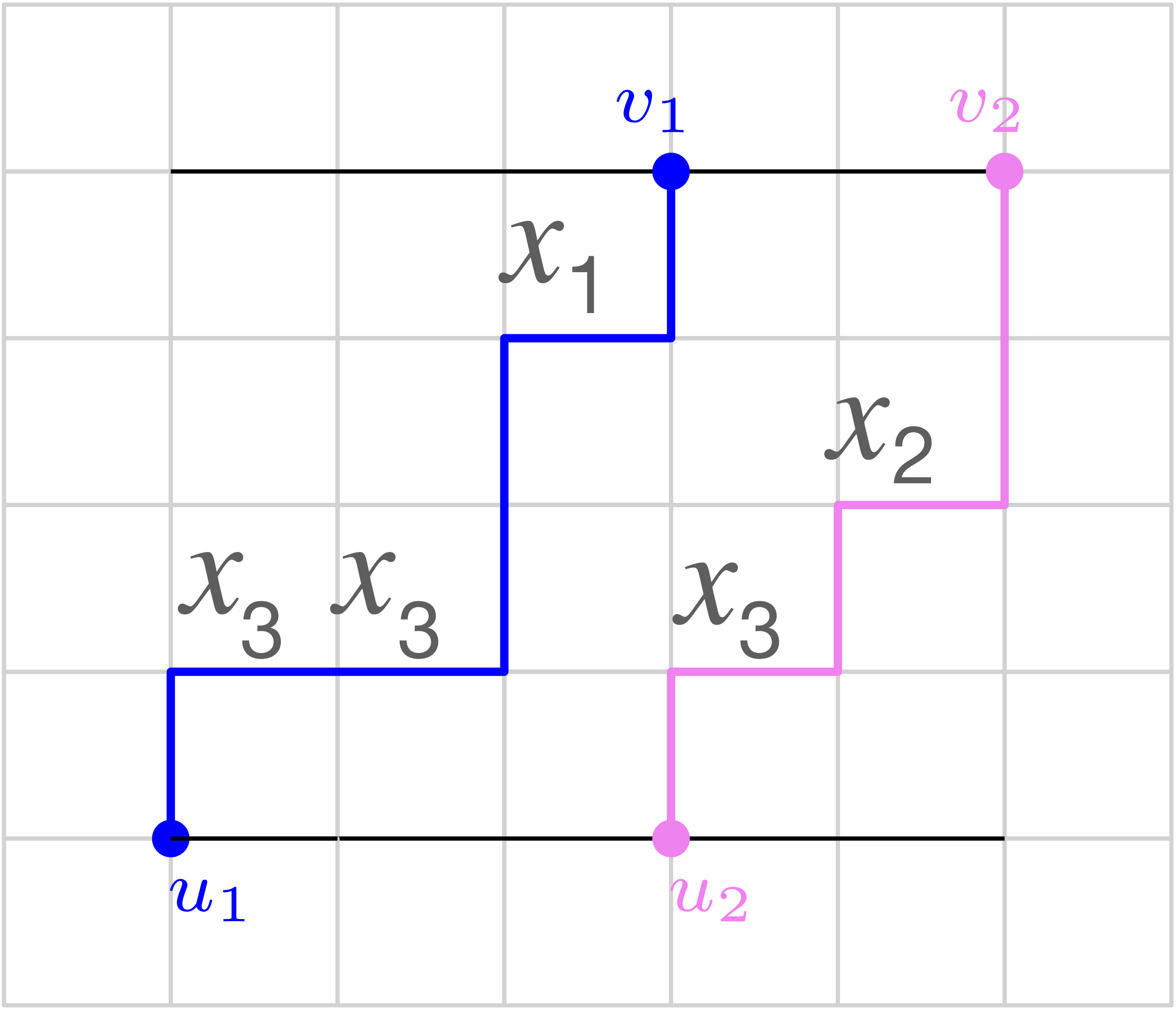
Theorem (Gessel—Viennot, Loehr 1989, 2009)

$$\sum_{\alpha \subseteq \lambda} q^{|\alpha|} = \det \left(q^{\binom{j-i+1}{2}} \binom{\lambda_j + 1}{j - i + 1}_q \right)_{1 \le i,j \le \ell(\lambda)}$$

λ-Parking function enumeration

$$\mathscr{P}_\lambda(\mathbf{x}) := \sum_{\alpha \subseteq \lambda} s_{\alpha + 1^n/\alpha}(\mathbf{x})$$





$\#(u_1 \to v_1) = h_3(\mathbf{x})$ $\#(u_2 \to v_2) = h_2(\mathbf{x})$

$\#(u_2 \to v_1) = h_5(\mathbf{x})$ $\#(u_1 \to v_2) = h_0(\mathbf{x})$

By LGV lemma:

$$s_{(4,2)/(1)}(\mathbf{x}) = \det \begin{pmatrix} h_3(\mathbf{x}) & h_1(\mathbf{x}) \\ h_5(\mathbf{x}) & h_2(\mathbf{x}) \end{pmatrix}$$

Jacobi-Trudi Identity (1841)

$$s_{\lambda/\mu}(\mathbf{x}) = \det \left(h_{\lambda_i - \mu_j - i + j}(\mathbf{x}) \right)_{1 \leq i,j \leq \ell(\lambda)}$$

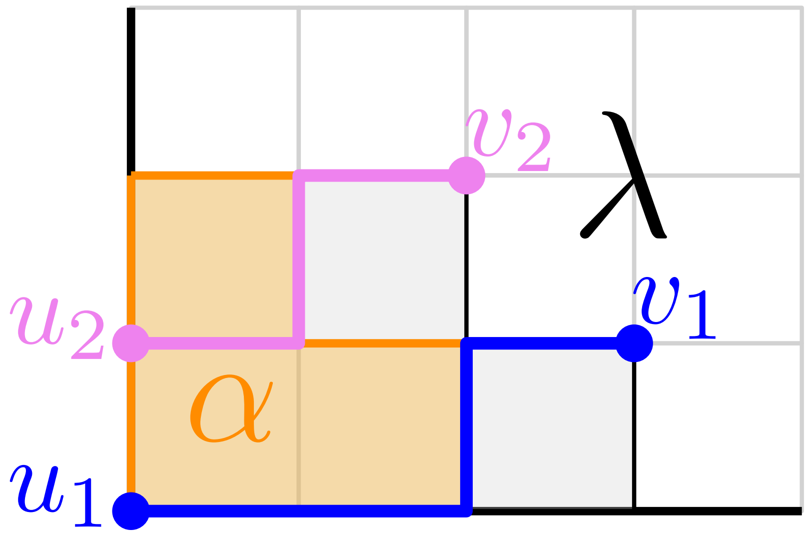
λ-Parking function enumeration

$$\mathscr{P}_\lambda(\mathbf{x}) := \sum_{\alpha \subseteq \lambda} s_{\alpha + 1^n/\alpha}(\mathbf{x})$$

?

λ-Dyck path enumeration

$$\sum_{\alpha \subseteq \lambda} q^{|\alpha|} = \det \left(q^{\binom{j-i+1}{2}} \binom{\lambda_j + 1}{j - i + 1}_q \right)_{1 \le i,j \le \ell(\lambda)}$$



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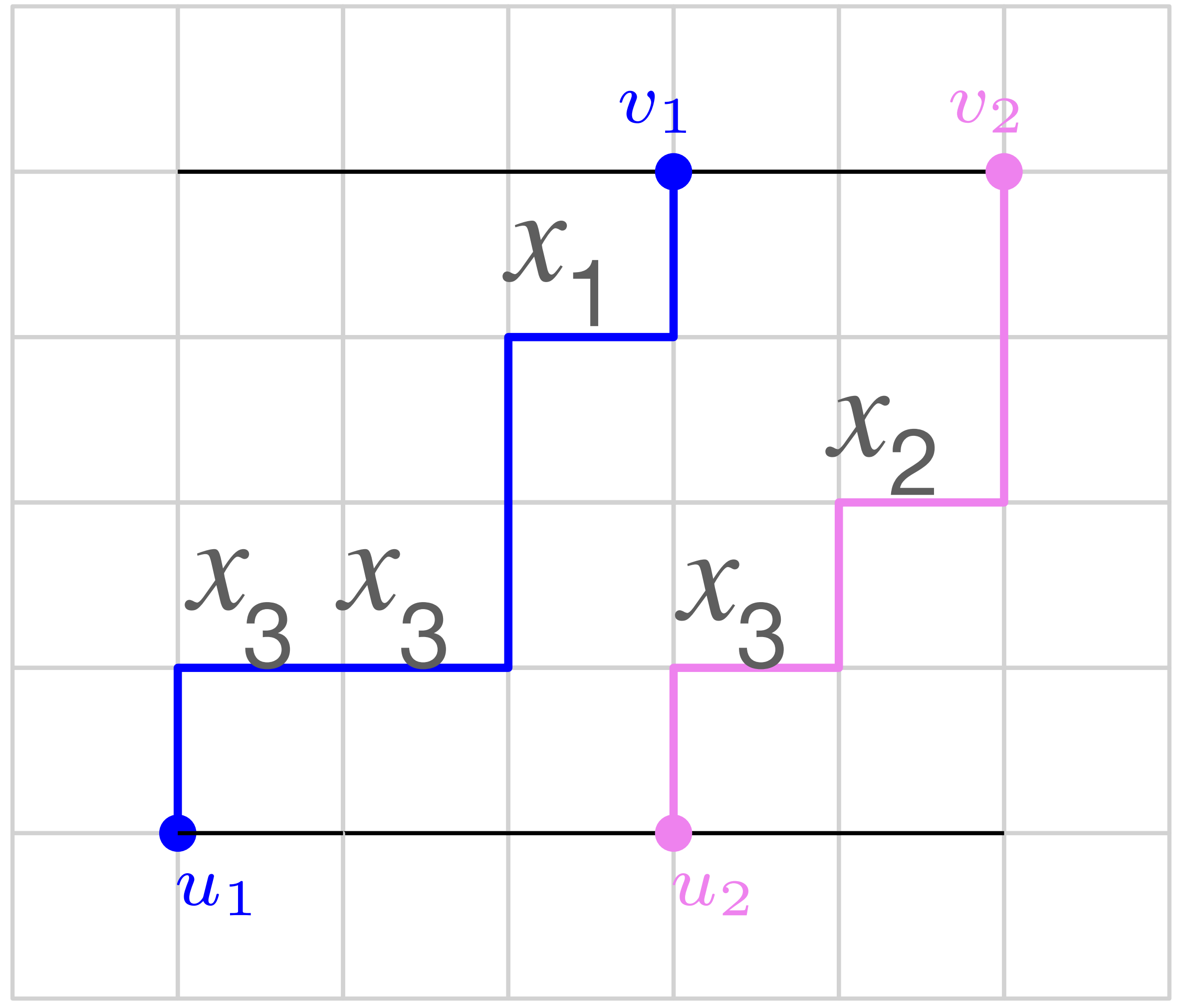


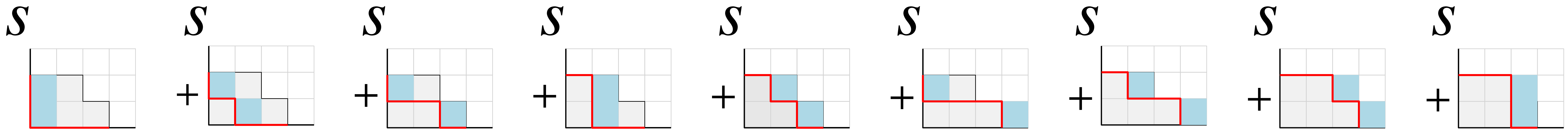
Lemma (Lindström — Gessel — Viennot, year)

$$\text{\# of non-intersecting paths from } u_i \to v_i = \det \left(p(u_i, v_j) \right)_{1 \le i,j \le n}.$$

λ-Parking function enumeration

$$\mathscr{P}_\lambda(q;\mathbf{x}) := \sum_{\alpha \subseteq \lambda} q^{\alpha} s_{\alpha + 1^n/\alpha}(\mathbf{x})$$

$\mathscr{P}_\lambda(\mathbf{x}) =$



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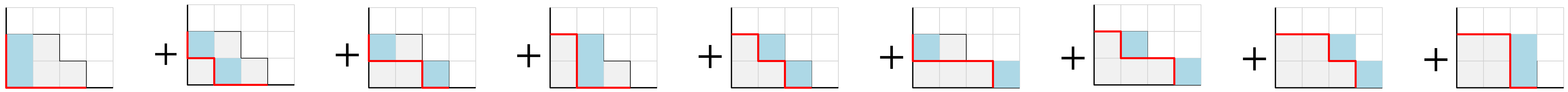
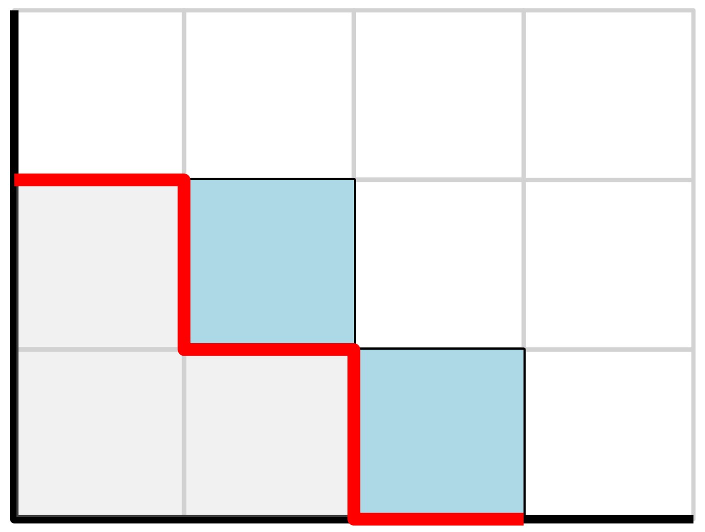
$\mathscr{P}_\lambda(\mathbf{x}) = s + s + s + s + s + s + s + s + s$





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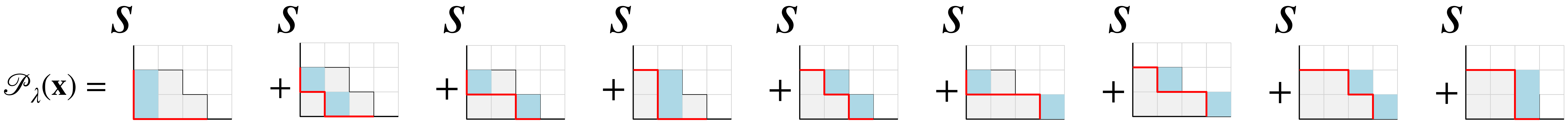

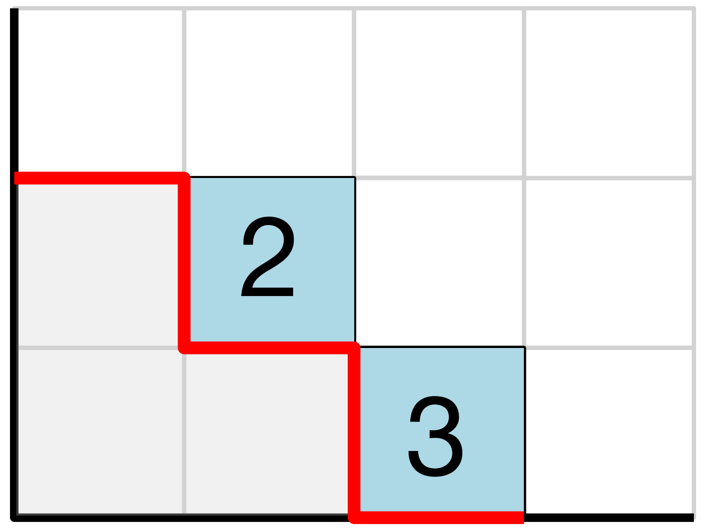

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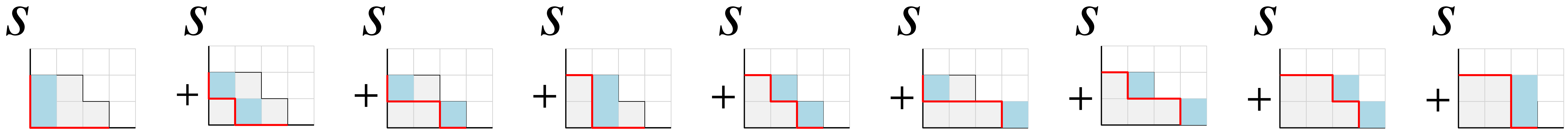
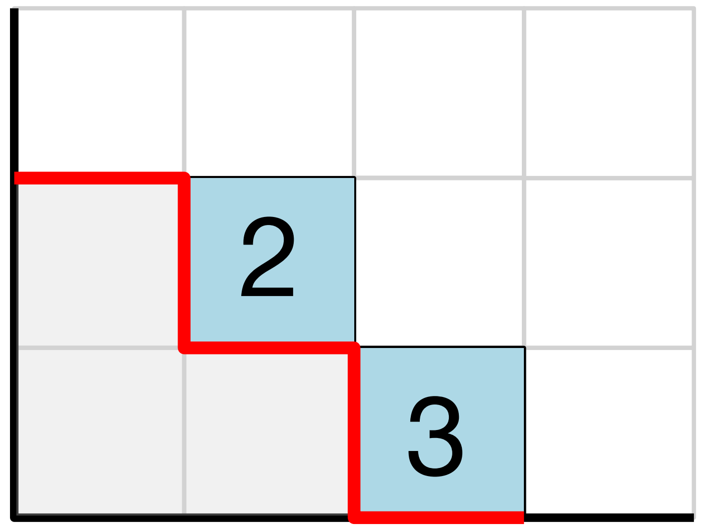
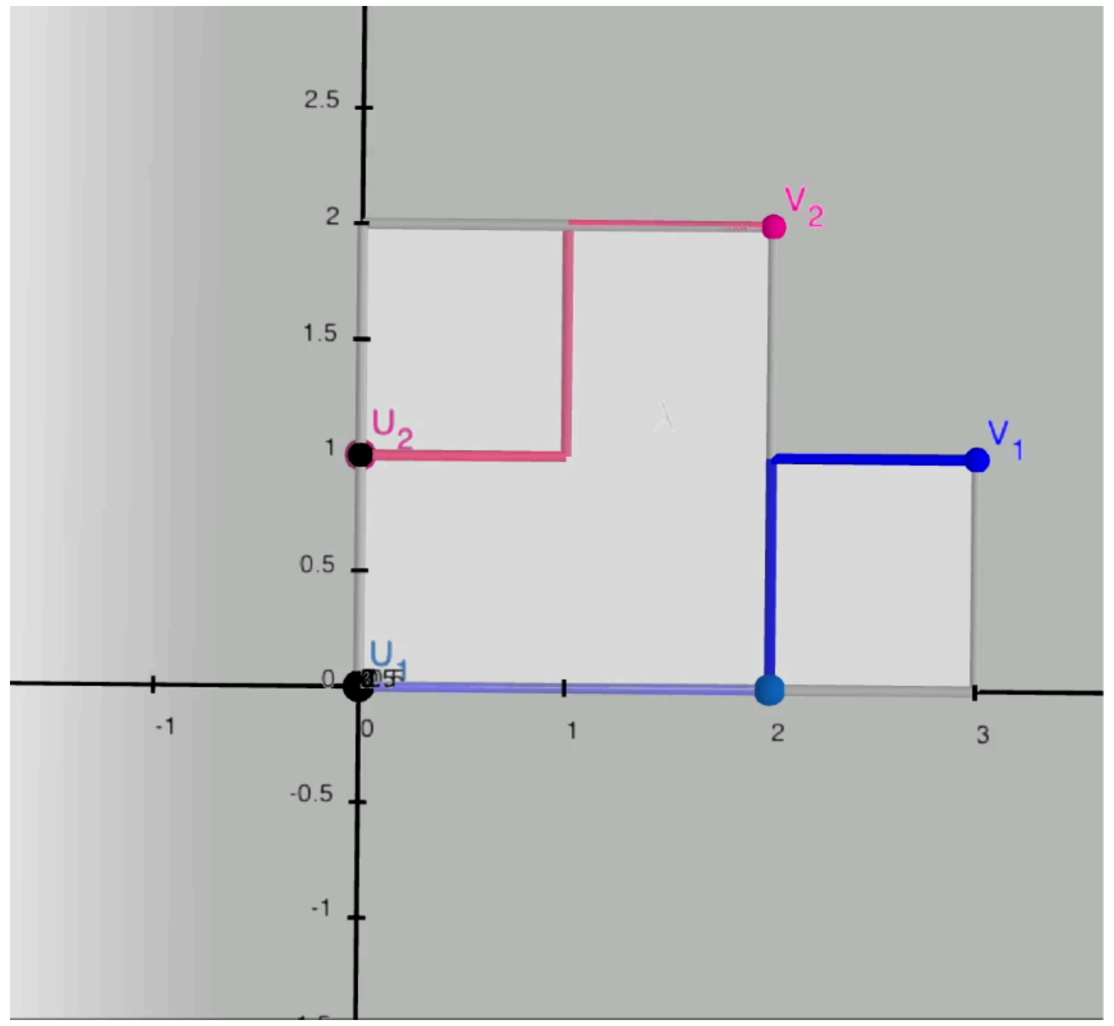
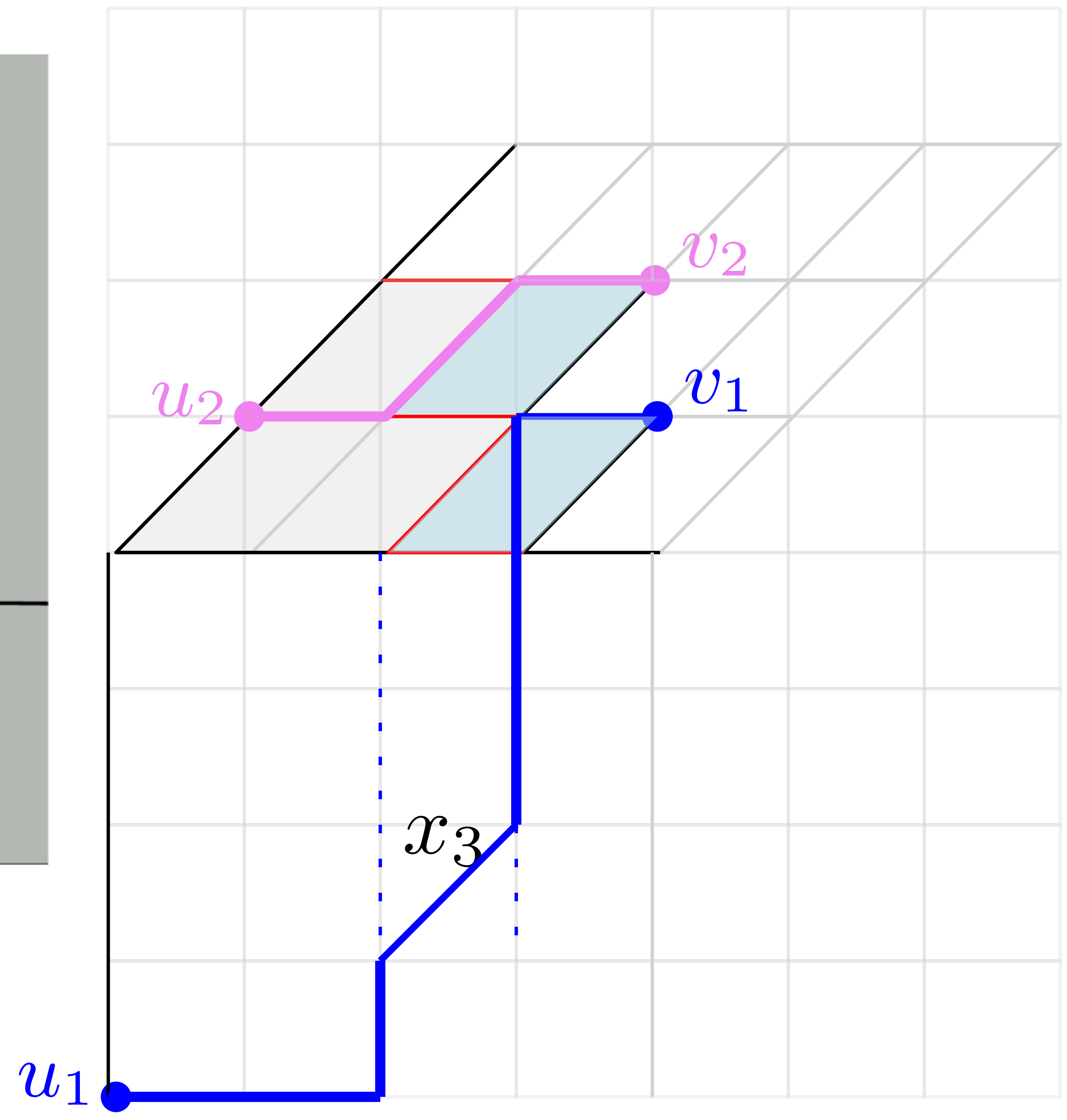
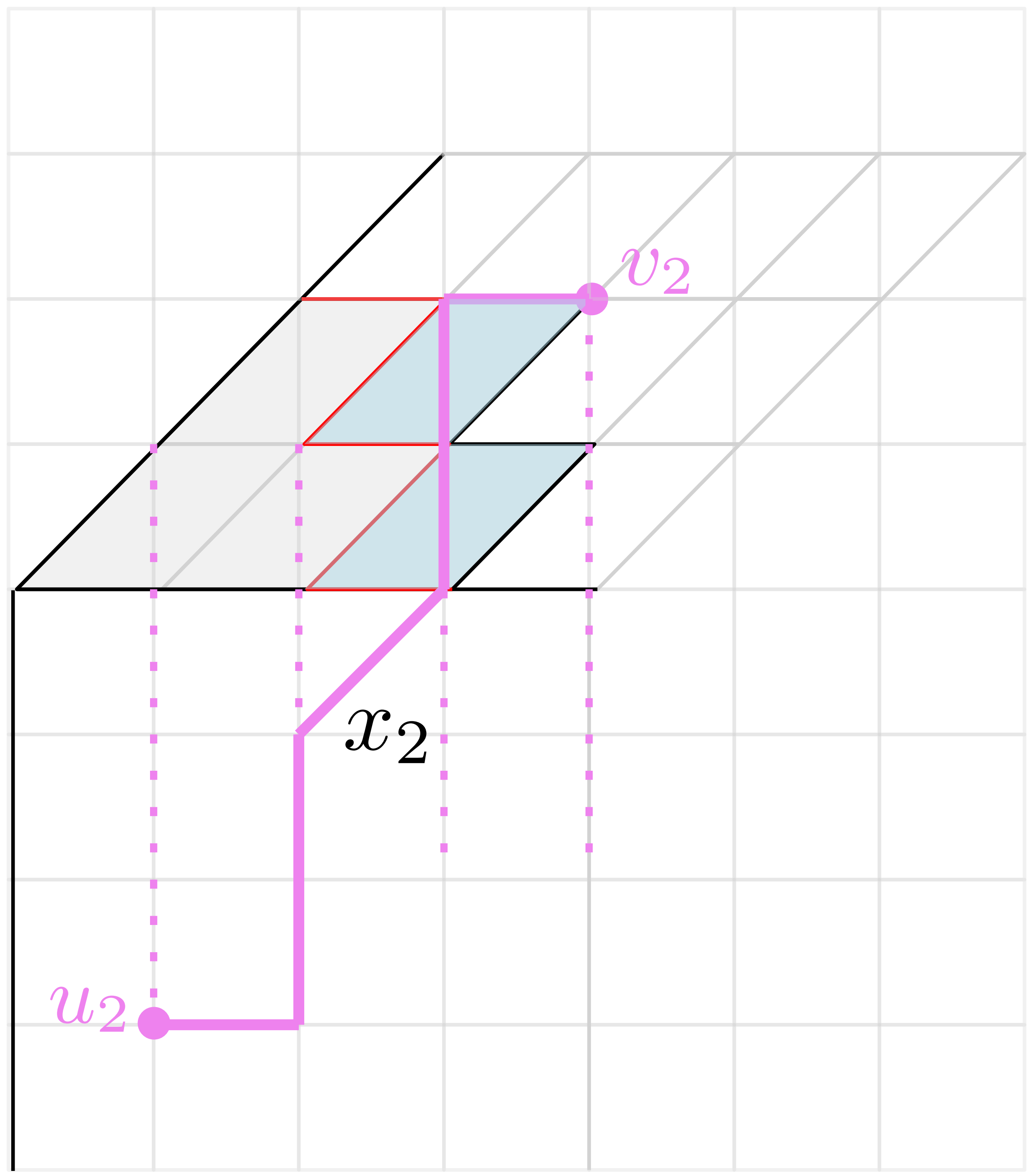
$\mathscr{P}_\lambda(\mathbf{x}) =$ 









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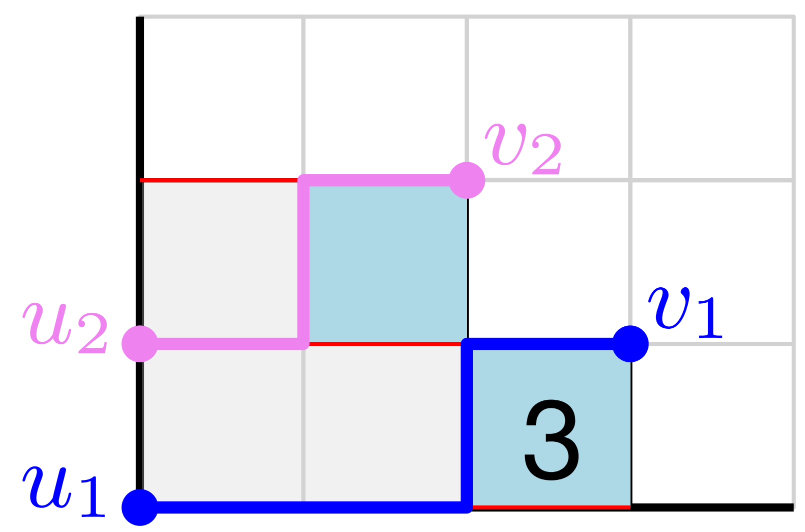
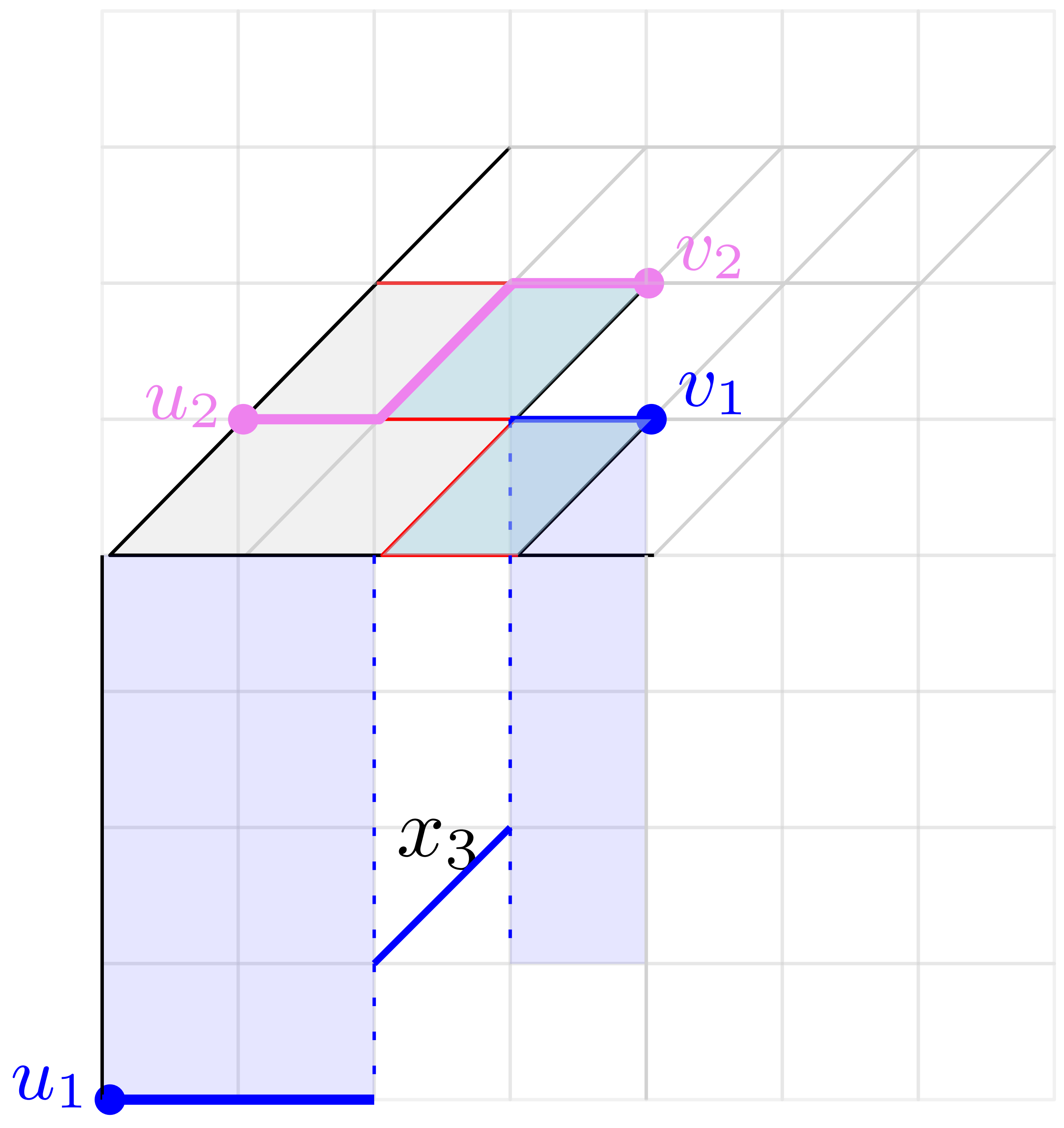


$$\#(u_1 \to v_1) = \binom{4}{1}_q \cdot h_1(\mathbf{x})$$

$$\#(u_2 \to v_2) = \binom{3}{1}_q \cdot h_1(\mathbf{x})$$



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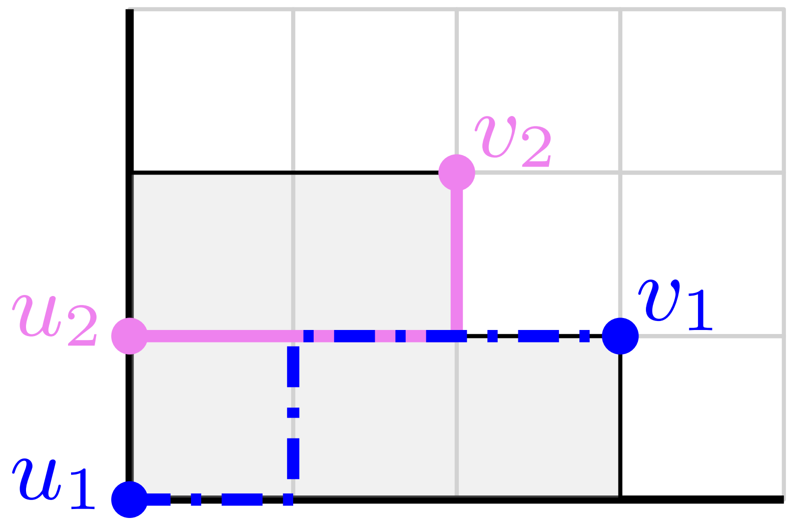

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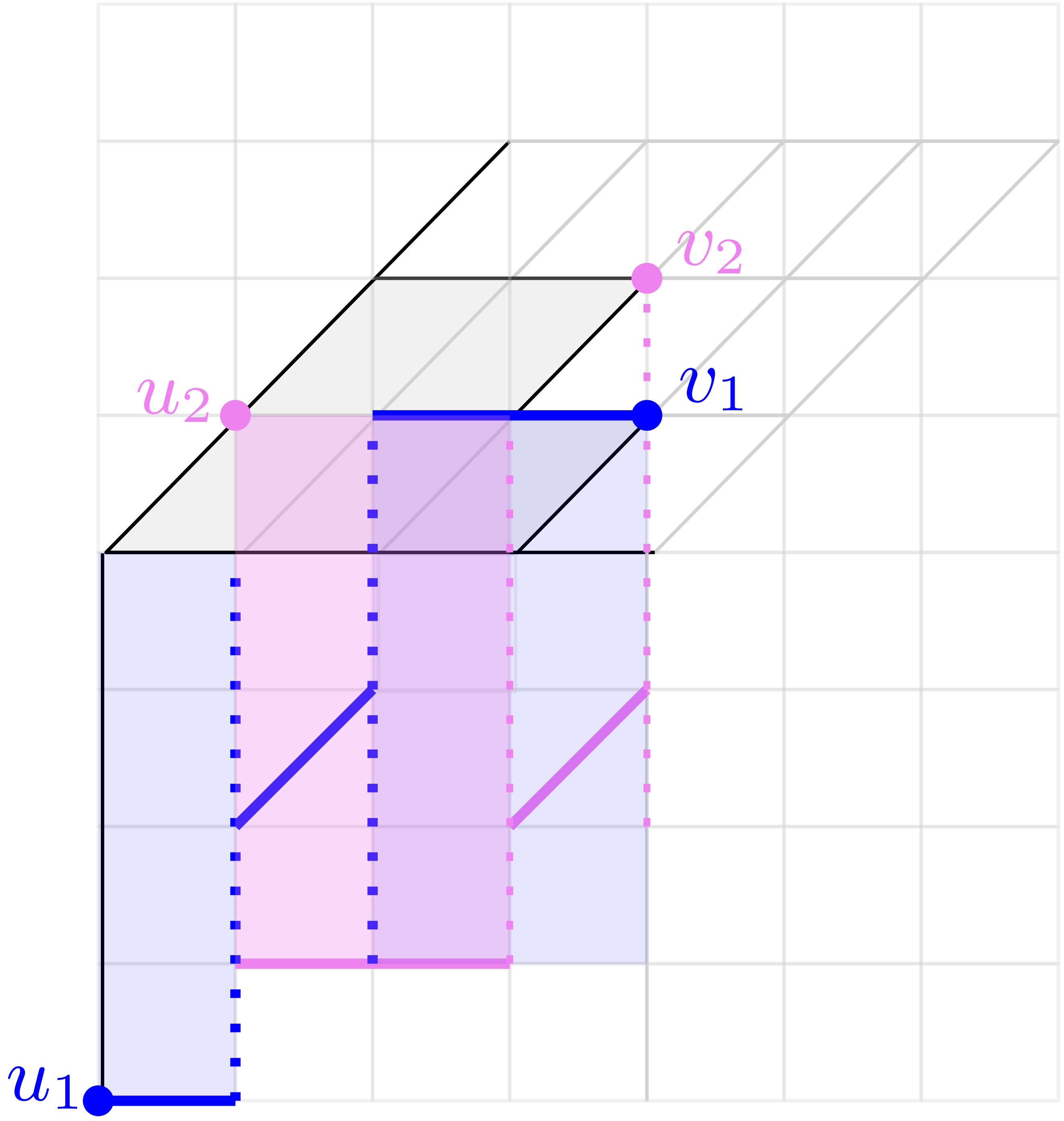

$$\#(u_1 \to v_2) = q \cdot \binom{3}{2}_q \cdot h_1^2(\mathbf{x})$$

λ-Parking function enumeration

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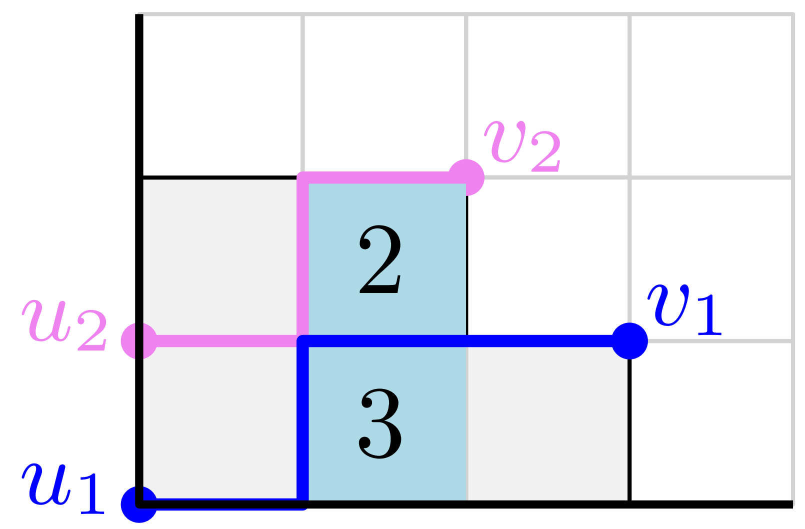
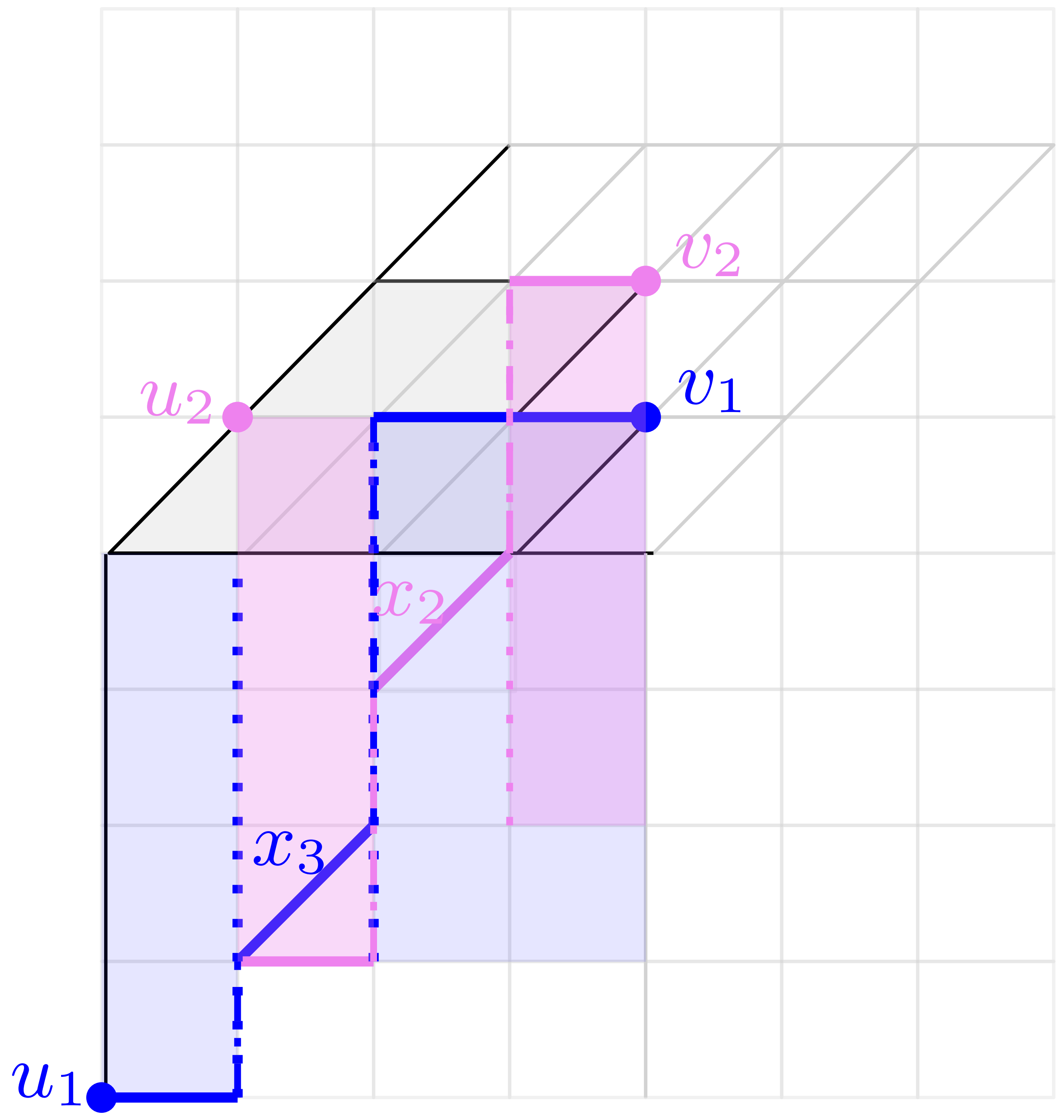


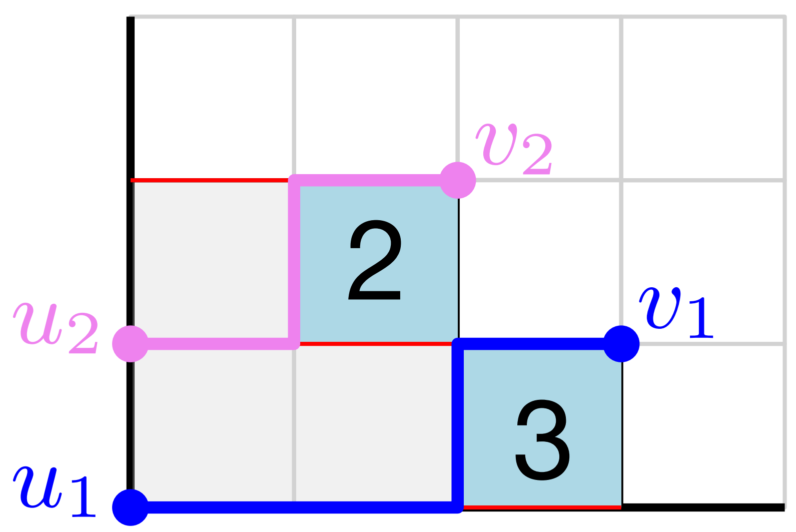
$$\#(u_1 \to v_1) = \binom{4}{1}_q \cdot h_1(\mathbf{x}) \qquad \#(u_2 \to v_2) = \binom{3}{1}_q \cdot h_1(\mathbf{x})$$



$$\#(u_1 \to v_2) = q \cdot \binom{3}{2}_q \cdot h_1^2(\mathbf{x}) \quad + \quad \binom{3}{1}_{q^2} \cdot h_2(\mathbf{x}) \qquad \#(u_2 \to v_1) = 1$$

λ-Parking function enumeration

$$\mathscr{P}_\lambda(q;\mathbf{x}) := \sum_{\alpha\subseteq\lambda} q^{|\alpha|} s_{\alpha+1^n/\alpha}(\mathbf{x})$$

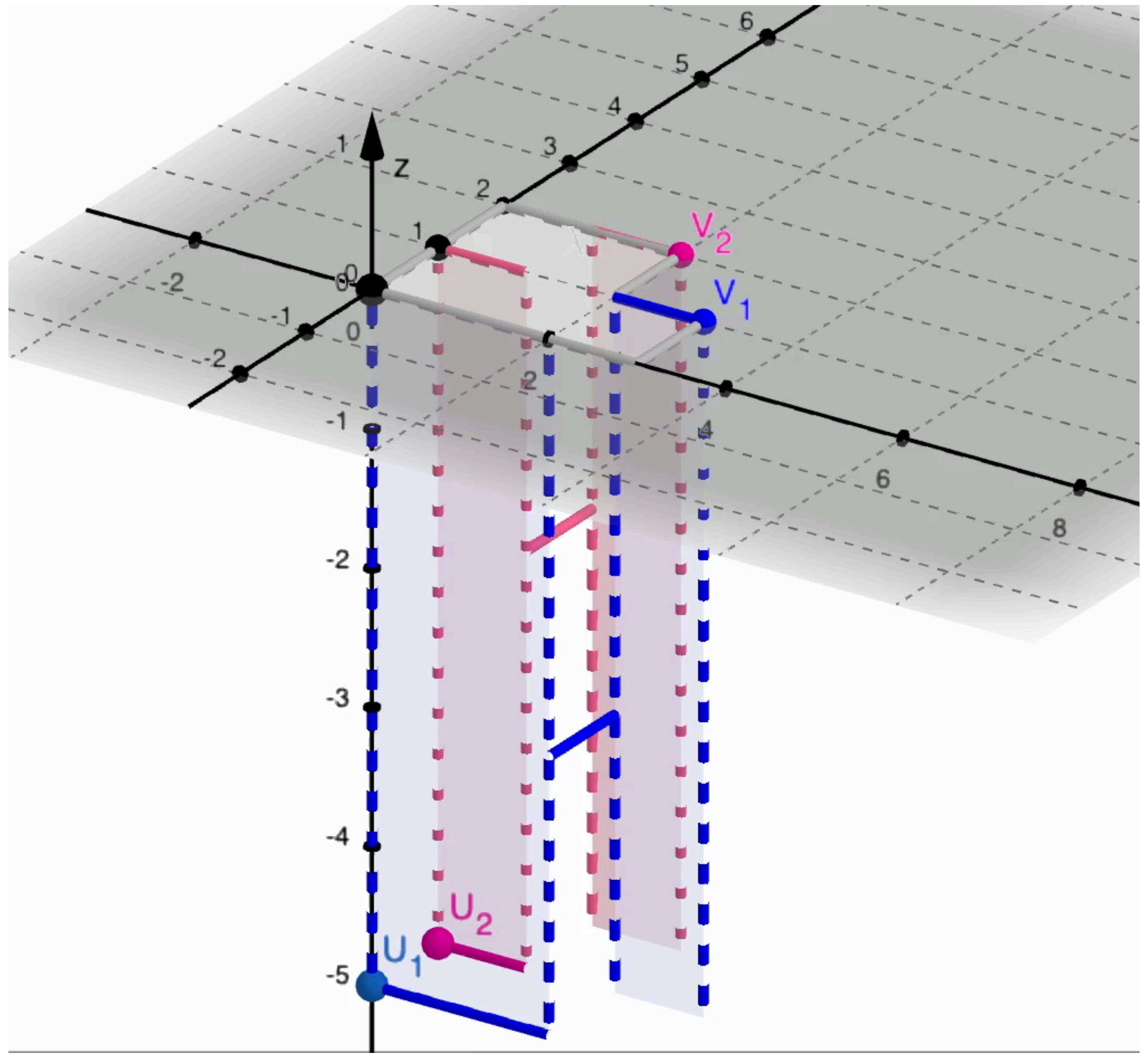


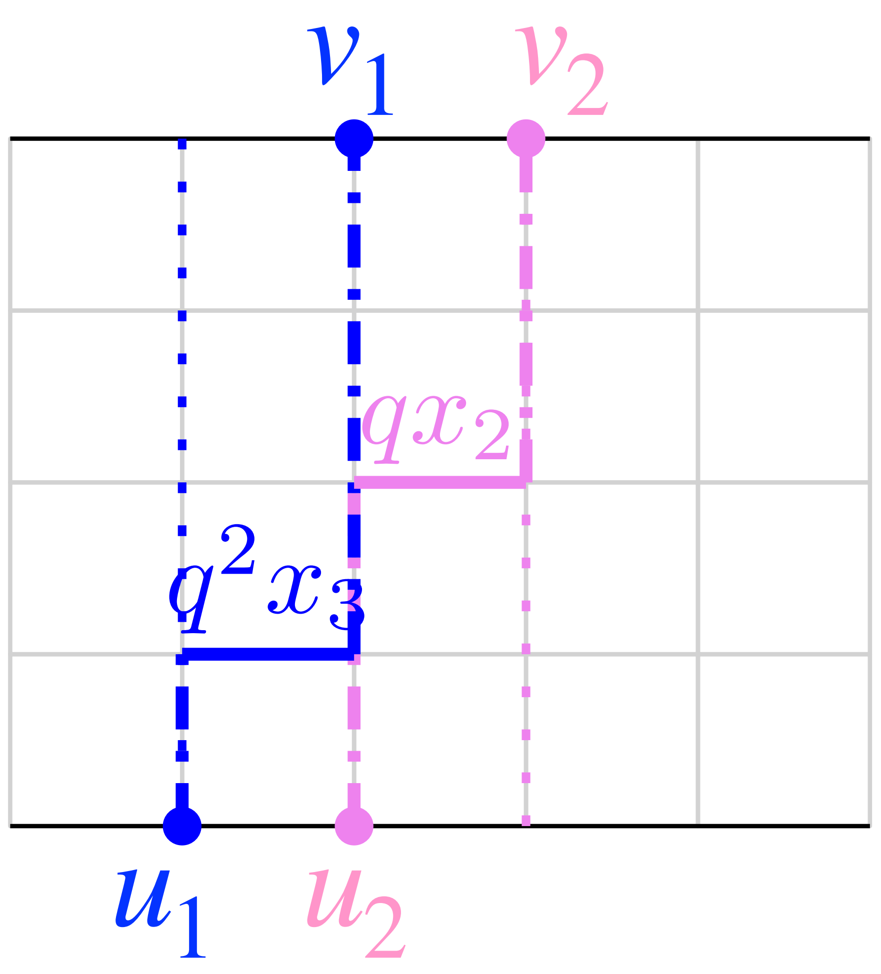
$$\#(u_1 \to v_1) = \binom{4}{1}_q \cdot h_1(\mathbf{x}) = h_1[(1+q+q^2+q^3)\cdot\mathbf{x}]$$

$$\#(u_2 \to v_2) = \binom{3}{1}_q \cdot h_1(\mathbf{x}) = h_1[(1+q+q^2)\cdot\mathbf{x}]$$

$$\#(u_1 \to v_2) = q\cdot\binom{3}{2}_q \cdot h_1^2(\mathbf{x}) \quad + \quad \binom{3}{1}_{q^2} \cdot h_2(\mathbf{x})$$

$$\#(u_2 \to v_1) = 1$$





λ-Parking function enumeration

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$$\#(u_1 \to v_1) = \binom{4}{1}_q \cdot h_1(\mathbf{x}) = h_1[(1+q+q^2+q^3)\cdot\mathbf{x}]$$

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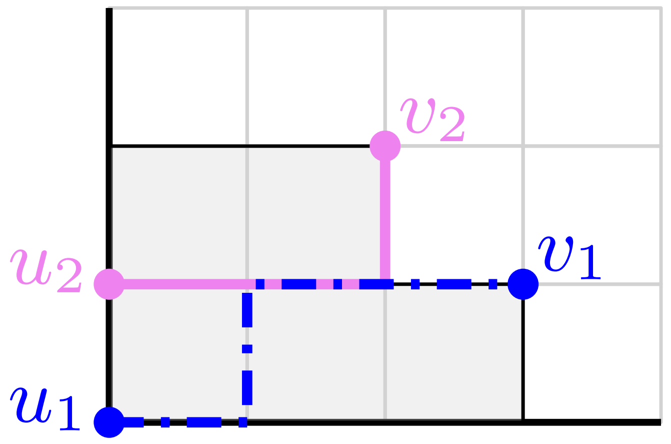
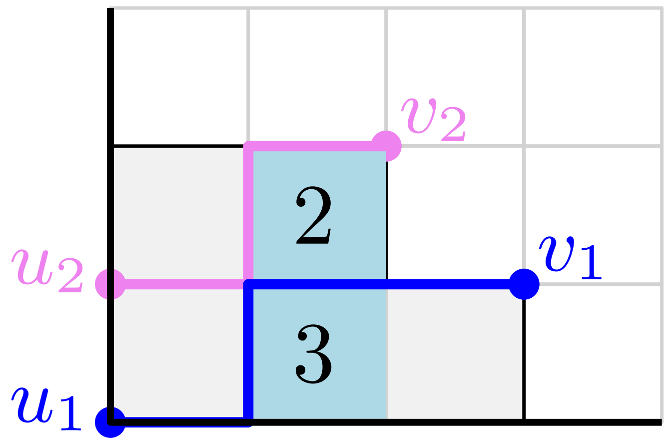
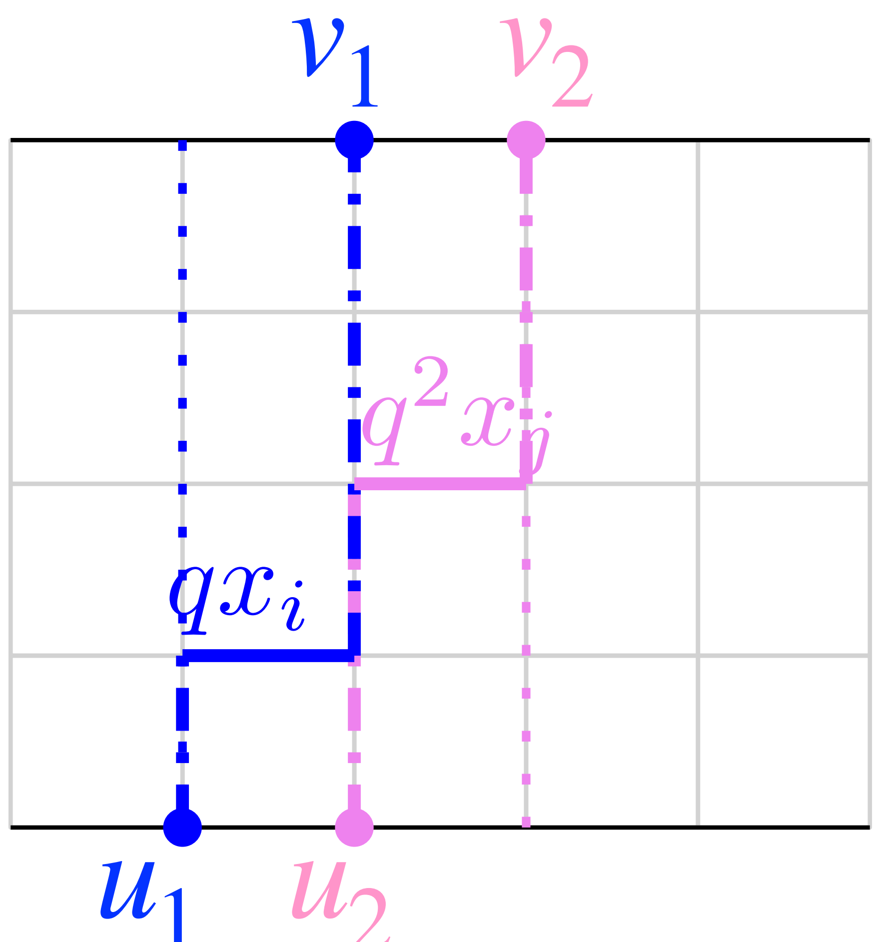


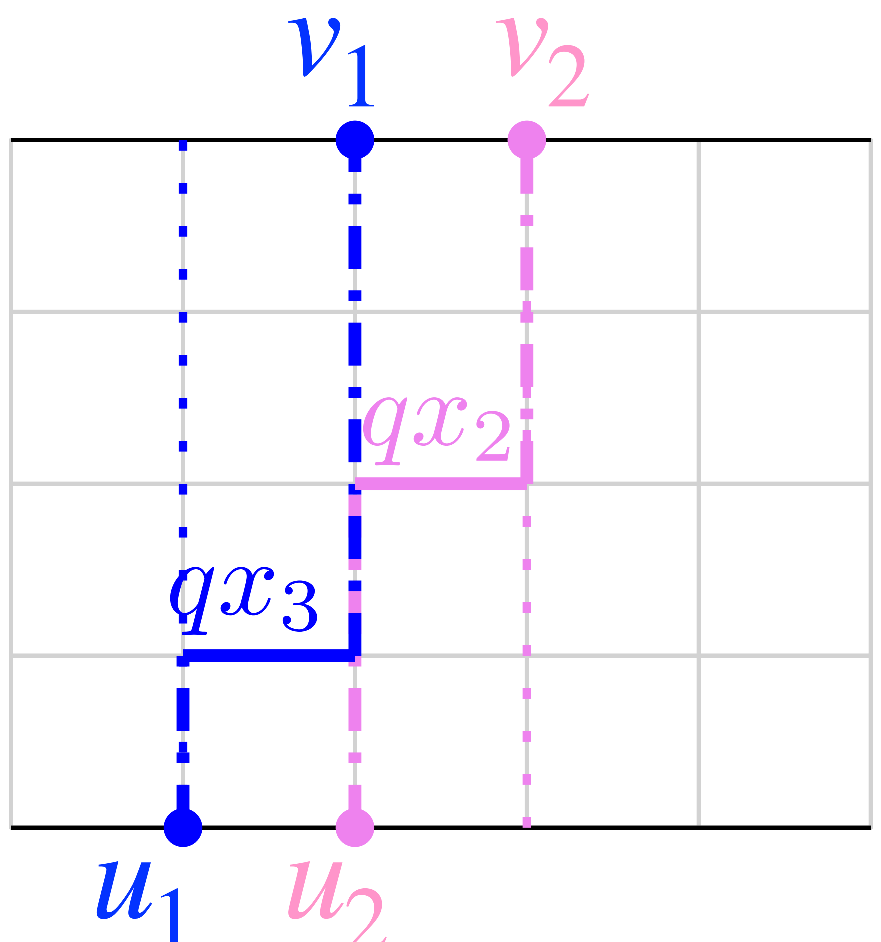


$$\#(u_1 \to v_2) = q \cdot \binom{3}{2}_q \cdot h_1^2(\mathbf{x}) \quad + \quad \binom{3}{1}_q \cdot h_2(\mathbf{x})$$

$$= h_2[(1+q+q^2)\cdot\mathbf{x}]$$

$$\#(u_2 \to v_1) = 1$$





$q^a x_i \leq$	$q^b x_j$

where $q^a x_i \leq q^b x_j$ if $a < b$ or $a = b$ and $i \geq j$

$qx_3 < q^2 x_2$

$qx_2 < qx_3$

λ-Parking function enumeration

$$\mathscr{P}_\lambda(q;\mathbf{x}) := \sum_{\alpha\subseteq\lambda} q^{|\alpha|} s_{\alpha+1^n/\alpha}(\mathbf{x})$$

$$\#(u_1 \to v_1) = \binom{4}{1}_q \cdot h_1(\mathbf{x}) = h_1[(1+q+q^2+q^3)\cdot\mathbf{x}]$$

$$\#(u_2 \to v_2) = \binom{3}{1}_q \cdot h_1(\mathbf{x}) = h_1[(1+q+q^2)\cdot\mathbf{x}]$$

$$\#(u_1 \to v_2) = q\cdot\binom{3}{2}_q \cdot h_1^2(\mathbf{x}) \quad + \quad \binom{3}{1}_{q^2} \cdot h_2(\mathbf{x}) = h_2[(1+q+q^2)\cdot\mathbf{x}]$$

$$\#(u_2 \to v_1) = 1$$

By LGV Lemma, we have:

$$\mathscr{P}_\lambda(q;\mathbf{x}) = \det\begin{pmatrix} h_1[(1+q+q^2+q^3)\cdot\mathbf{x}] & h_2[(1+q+q^2)\cdot\mathbf{x}] \\ 1 & h_1[(1+q+q^2)\cdot\mathbf{x}]\end{pmatrix}$$

$$\mathscr{P}_{32}(q;\mathbf{x}) = \left(q^5+2q^4+2q^3+2q^2+q+1\right) h_{11}(\mathbf{x}) - \left(q^4+q^2+1\right) h_2(\mathbf{x})$$

$$= \left(q^5+q^4+2q^3+q^2+q\right) s_2(\mathbf{x}) + \left(q^5+2q^4+2q^3+2q^2+q+1\right) s_{11}(\mathbf{x})$$

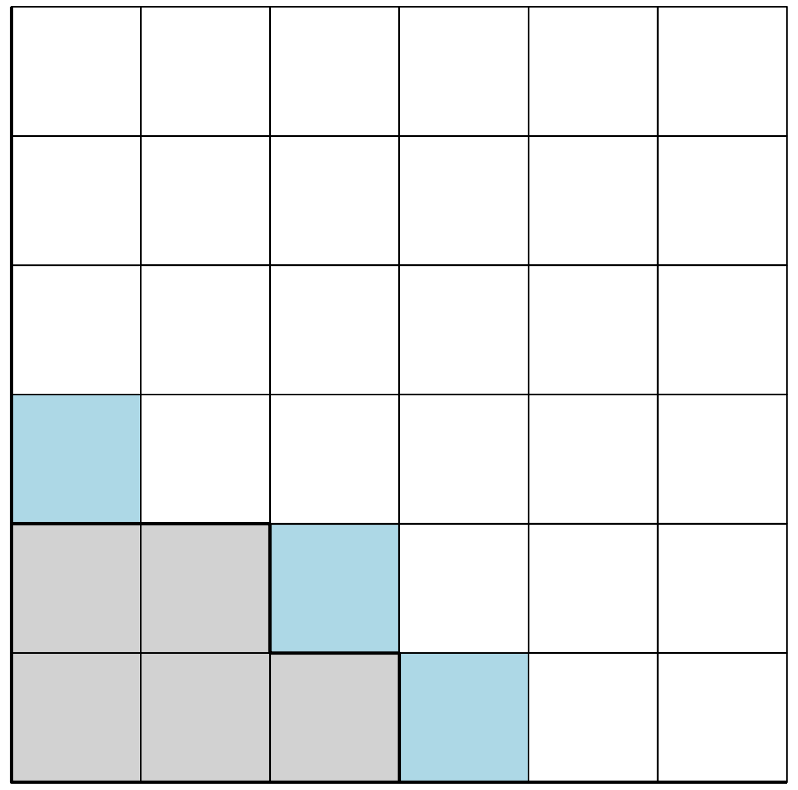
(Bergeron — Lanciault — P. 2023+)

$$\mathscr{P}_\lambda(q;\mathbf{x}) = \det\left(h_{j-i+1}[(1+\cdots+q^{\lambda_j})\cdot\mathbf{x}]\right)_{1\le,i,j\le\ell(\lambda)+1}$$

Observations

- Stabilization as n grows
- For $\lambda > \lambda'$ in dominance relation, $\mathscr{P}_{\lambda'}(q;\mathbf{x}) - \mathscr{P}_{\lambda}(q;\mathbf{x})$ is Schur positive
- (q,t)-Parking function?
- The positivity of the predicted $(q,t)-$polynomials is related to the concavity of λ

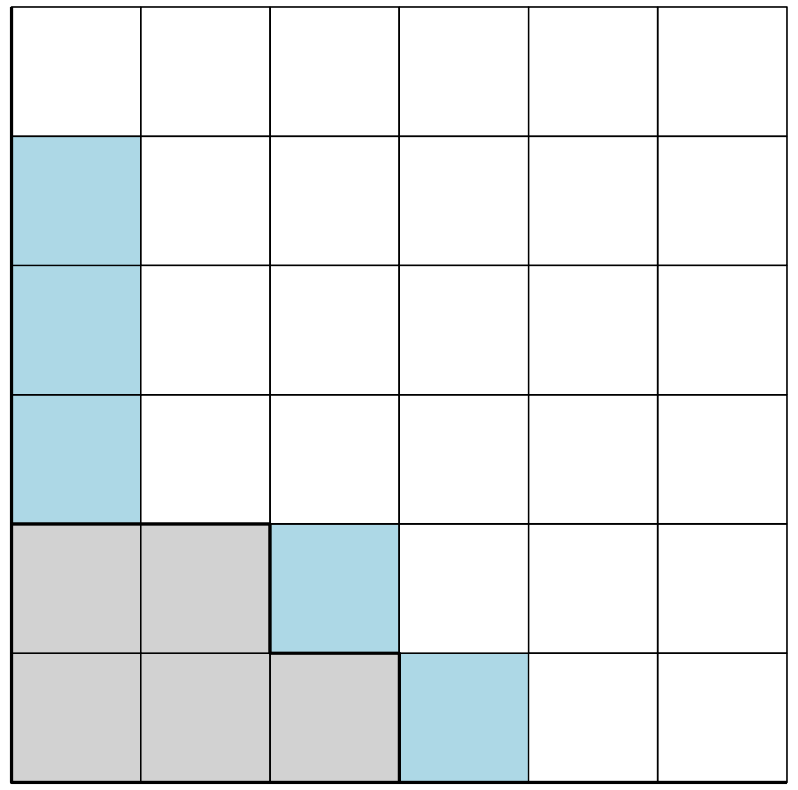
Stabilization



$$\mathscr{P}_{32}(\mathbf{x};q) = e_3 + \left(q^4+q^3+2q^2+q\right)e_{21} + \left(q^5+q^4+q^3\right)e_{111}$$

$$= \left(q^5+q^4+q^3\right)s_3 + \left(2q^5+3q^4+3q^3+2q^2+q\right)s_{21}$$

$$+\left(q^5+2q^4+2q^3+2q^2+q+1\right)s_{111}$$

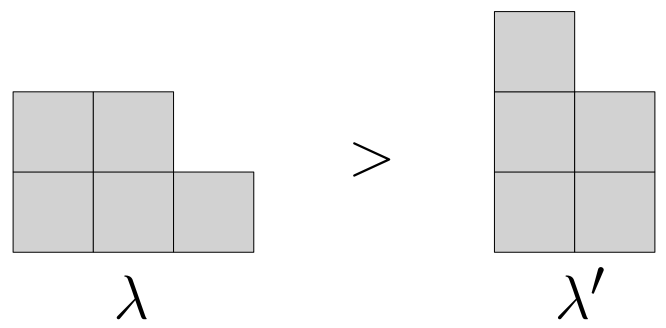


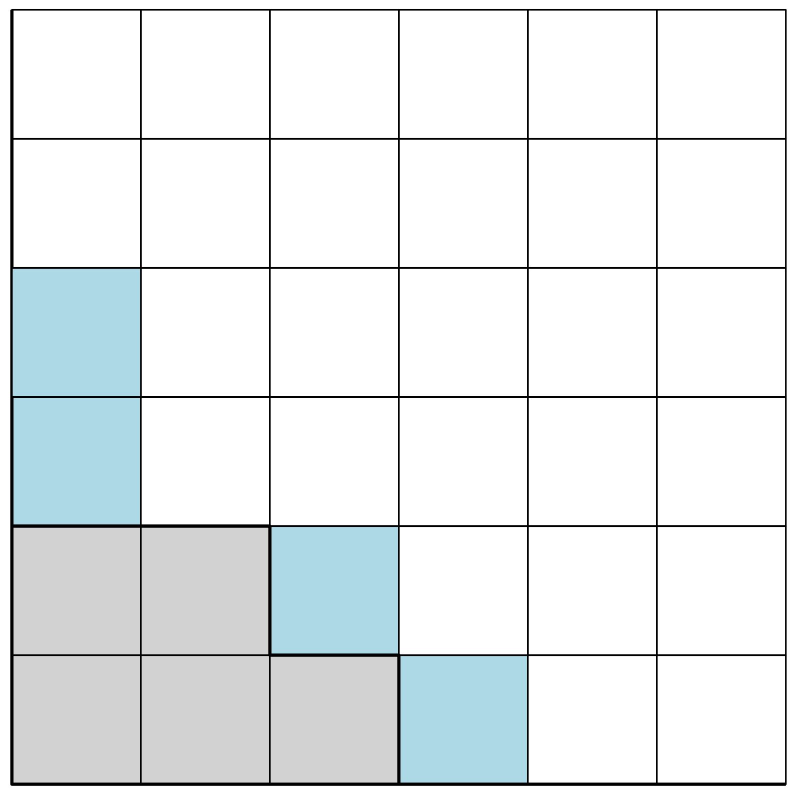
$$\mathscr{P}_{(32,0^{n-2})}(\mathbf{x};q) = \left((q^5+q^4+q^3)e_{11} + (q^4+q^2)e_2\right)e_{n-2} + \left((q^3+q^2+q)e_1\right)e_{n-1} + e_n$$

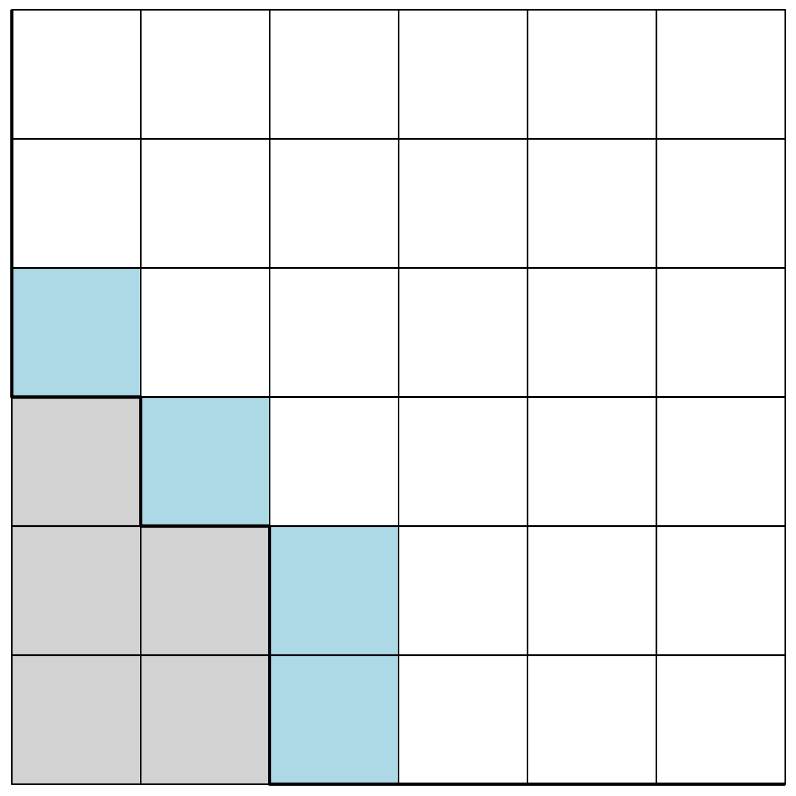
$$\mathscr{P}_{(32,0^3)} = \left(q^5+q^4+q^3\right)s_{311} + \left(q^5+2q^4+q^3+q^2\right)s_{221}$$

$$+\left(2q^5+3q^4+3q^3+2q^2+q\right)s_{2111} + \left(q^5+2q^4+2q^3+2q^2+q+1\right)s_{11111}$$

Dominant partition





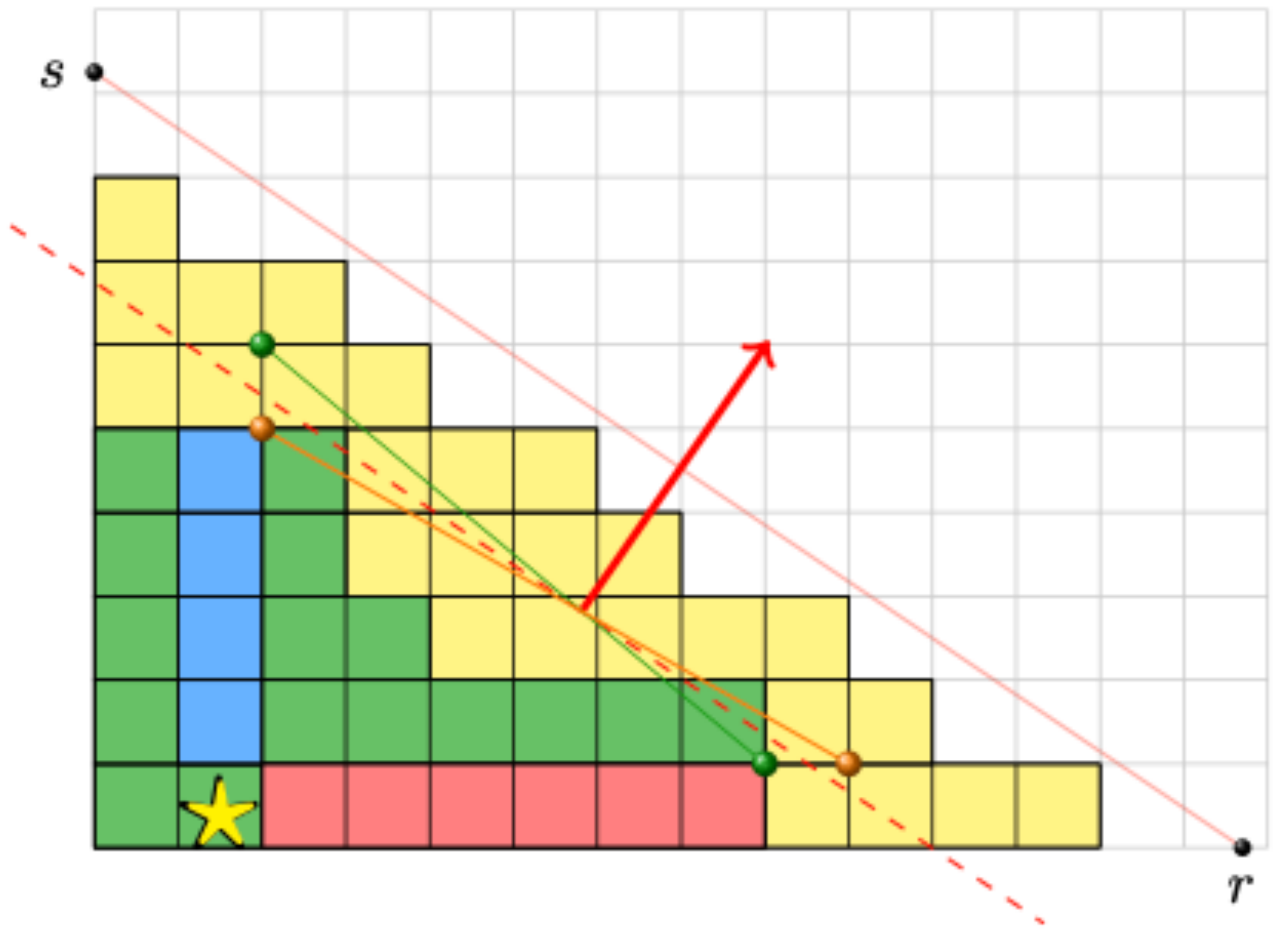


$$\mathscr{P}_{(32,0^2)}(\mathbf{x};q) - \mathscr{P}_{(221,0)}(\mathbf{x};q) = 0$$

$$\mathscr{P}_{(32,0^3)}(\mathbf{x};q) - \mathscr{P}_{(221,0^2)}(\mathbf{x};q) = \left(q^5 + q^4\right) s_{32} + \left(q^5 + q^4 + q^3\right) s_{221}$$

Dinv & Sim

$$\mathscr{P}(\mathbf{x}; q, t) = \sum_{P \in PF_n} q^{\mathrm{coarea}(P)} t^{\mathrm{dinv}(P)} x^P$$



Definition. Similar cells (Bergeron — Mazin, 2022)

Let τ be a triangular partition.

$$\mathrm{Sim}_\tau(\alpha) := \{c \in \alpha \,|\, t'(c, \alpha) \leq t_\tau < t''(c, \alpha)\}$$

Counting τ-Dyck path in q, t:

$$\mathscr{A}_\tau(q, t) := \sum_{\alpha \subseteq \tau} q^{\mathrm{coarea}(\alpha)} t^{\mathrm{sim}_\tau(\alpha)}$$

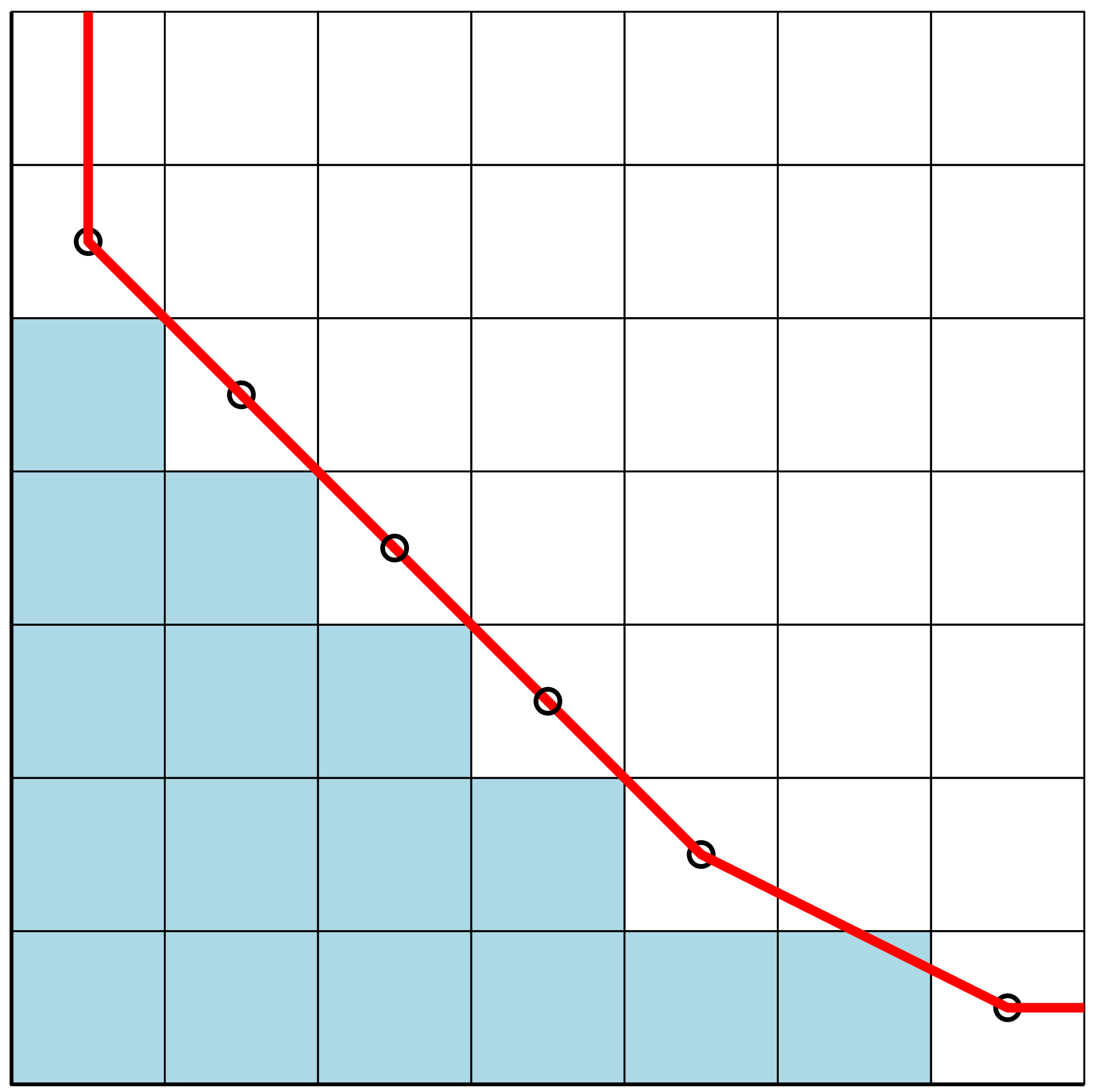
$\mathscr{A}_\tau(q, t)$ is symmetric in q, t.

$(1, s_1)$			
$(2, s_2)$			
$(21, s_{11} + s_3)$	$(3, s_3)$		
$(31, s_{21} + s_4)$	$(4, s_4)$		
$(32, s_{31} + s_5)$	$(41, s_{31} + s_5)$	$(5, s_5)$	
$(321, s_{31} + s_{41} + s_6)$	$(42, s_{22} + s_{41} + s_6)$	$(51, s_{41} + s_6)$	$(6, s_6)$

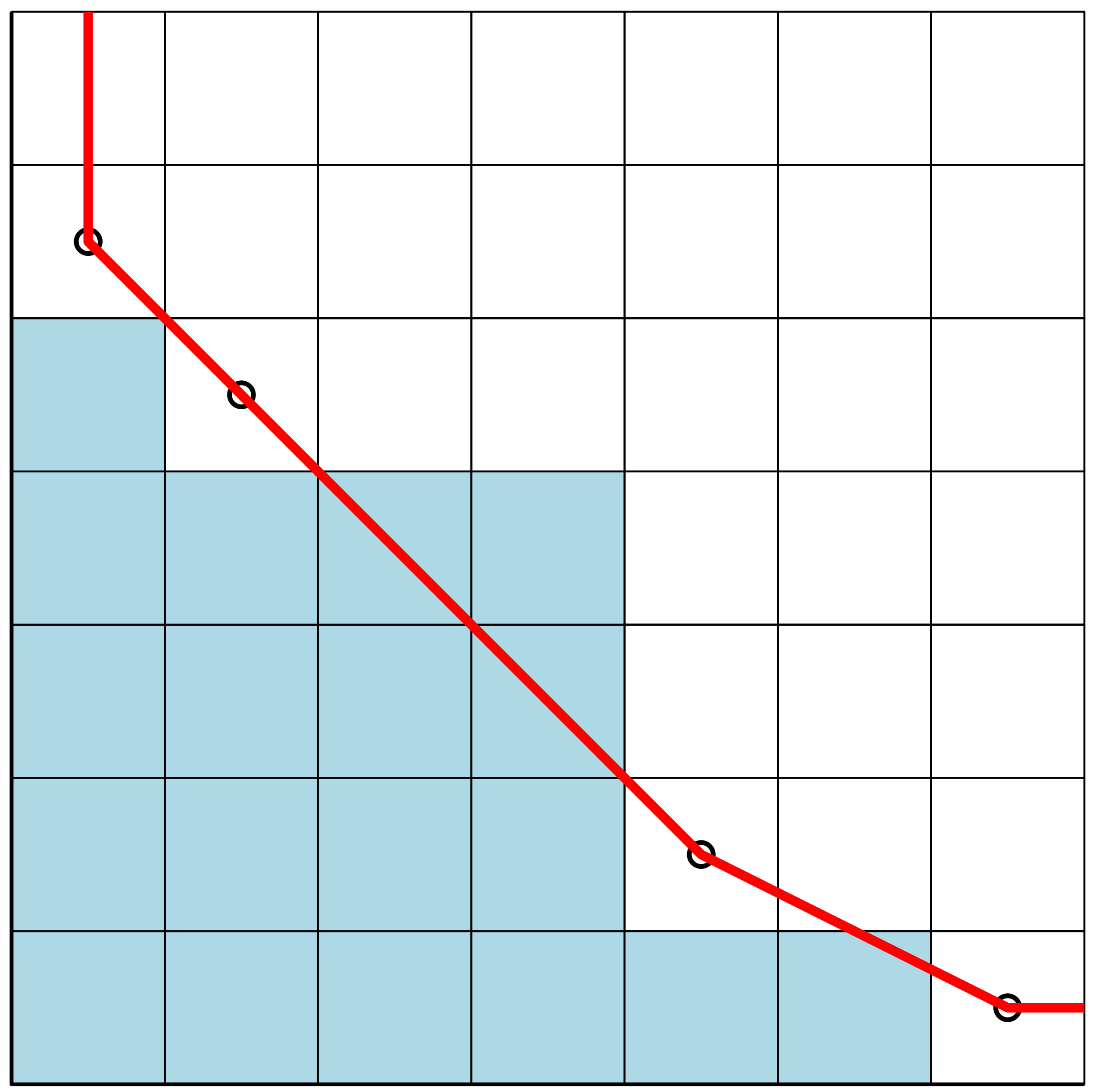
*Image and data from F. Bergeron and M. Mazin, Combinatorics of triangular partitions, Enumer. Comb. Appl. {\bf 3} (2023), no.~1, Paper No. S2R1, 20 pp.; MR4528166

Concave partitions

A partition is concave if no cell of its diagram lies in the convex-hull of its complement



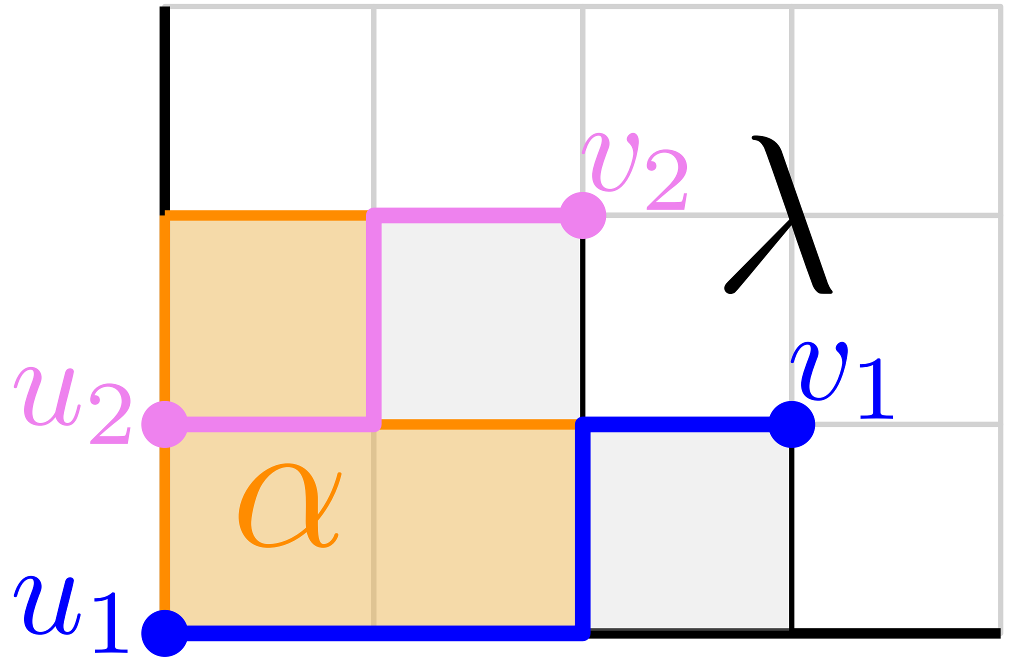
Concave



NOT Concave

coefficients $\in \mathbb{N}(q,t)$

$\big(s_4(q,t) + s_{3,1}(q,t) + s_1(q,t) - s_3(q,t) - s_2(q,t) - s_{1,1}(q,t)\big)\, s_6(\mathbf{x}) + \cdots$



Thank you!