Hook formulas for skew shapes and beyond

Greta Panova (University of Southern California, MSRI)

MSU, October 2021

Standard Young Tableaux

Irreducible representations of S_n:

Specht modules $\mathbb{S}_\lambda$, for all $\lambda \vdash n$.

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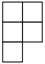

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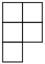
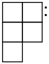


$1, 2, 3, 4, 5$ $<$ $\wedge$



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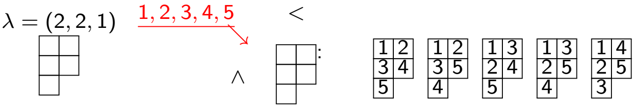

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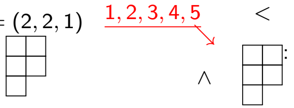
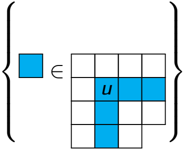


$$\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 5\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&2\\\hline 3&5\\\hline 4\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 5\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&3\\\hline 2&5\\\hline 4\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline 3\\\cline{1-1}\end{array}$$

Hook-length formula [Frame-Robinson-Thrall]:

$$\dim \mathbb{S}_\lambda = \#\{\text{SYTs of shape } \lambda\} = f^\lambda = \frac{|\lambda|!}{\prod_{u\in\lambda} h_u} = \frac{5!}{4*3*2*1*1}$$

Hook length of box $u = (i,j) \in \lambda$: $h_u = \lambda_i - j + \lambda'_j - i + 1 = \#$



Skew SYTs and SSYTs: formulas

Shape λ/μ, e.g. for $\lambda = (5,4,3), \mu = (3,1)$:

Skew SYT:

			2	8
	1	4	5	
3	6	7		

Skew SSYT:

			1	3
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Jacobi-Trudi[Feit 1953]:

$$f^{\lambda/\mu} = |\lambda/\mu|! \cdot \det \left[\frac{1}{(\lambda_i - \mu_j - i + j)!} \right]_{i,j=1}^{\ell(\lambda)} .$$

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No product formula:

$\lambda/\mu = \delta_{n+2}/\delta_n$:
$$\begin{array}{cccc} & & 3 & 7 \\ & 1 & 5 & \\ 2 & 4 & & \\ 6 & & & \end{array} \longleftrightarrow \quad 6 > 2 < 4 > 1 < 5 > 3 < 7 \qquad f^{\delta_{n+2}/\delta_n} = E_{2n+1}:$$

$$1 + E_1 x + E_2 \frac{x^2}{2!} + E_3 \frac{x^3}{3!} + E_4 \frac{x^4}{4!} + \ldots = \sec(x) + \tan(x).$$

Euler numbers: $2, 5, 16, 61 \ldots$.

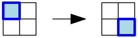
Hook-Length formula for skew shapes

Theorem (Naruse, SLC, September 2014)

$$f^{\lambda/\mu} = |\lambda/\mu|! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{u \in [\lambda] \setminus D} \frac{1}{h(u)},$$

where $\mathcal{E}(\lambda/\mu)$ *is the set of* excited diagrams *of* λ/μ.

Excited diagrams:

$\mathcal{E}(\lambda/\mu) = \{D \subset \lambda :$ obtained from μ via  $\}$

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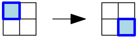
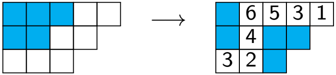
$\mathcal{E}(\lambda/\mu) = \{D \subset \lambda :$ obtained from μ via  $\}$



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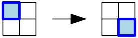
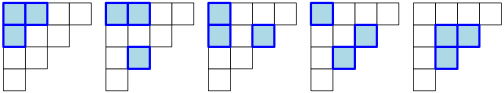
$\mathcal{E}(\lambda/\mu) = \{D \subset \lambda :$ obtained from μ via  $\}$



$$f^{(4321/21)} = 7! \left(\frac{1}{1^4 \cdot 3^3} + \frac{1}{1^3 \cdot 3^3 \cdot 5} + \frac{1}{1^3 \cdot 3^3 \cdot 5} + \frac{1}{1^2 \cdot 3^3 \cdot 5^2} + \frac{1}{1^2 \cdot 3^2 \cdot 5^2 \cdot 7} \right) = 61$$

Hook-Length formula for skew shapes

Theorem (Morales-Pak-P'16)

For skew SSYTs, we have that

$$s_{\lambda/\mu}(1, q, q^2, \ldots) = \sum_{T \in SSYT(\lambda/\mu)} q^{|T|} = \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j)\in[\lambda]\setminus D} \left[\frac{q^{\lambda'_j - i}}{1 - q^{h(i,j)}} \right].$$

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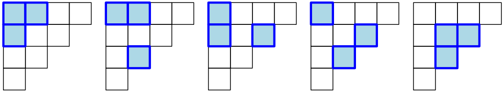

$$s_{\lambda/\mu}(1,q,q^2,\ldots) = \sum_{T\in SSYT(4321/21)} q^{|T|} = \frac{q^3}{(1-q)^4(1-q^3)^3} + 2\times\frac{q^5}{(1-q)^3(1-q^3)^3(1-q^5)} + \cdots$$

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Theorem (Morales-Pak-P'16)

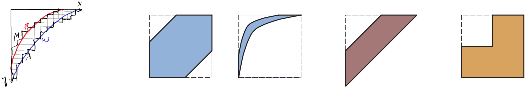
For (reverse) plane partitions of skew shape λ/μ:

$$\sum_{\pi \in RPP(\lambda/\mu)} q^{|\pi|} = \sum_{S \in PD(\lambda/\mu)} \prod_{u \in S} \left[\frac{q^{h(u)}}{1 - q^{h(u)}} \right].$$

where $PD(\lambda/\mu) := \{S \subset [\lambda] : S \subset [\lambda] \setminus D, \text{ for some } D \in \mathcal{E}(\lambda/\mu)\}$.

Applications

- Asymptotics of $f^{\lambda/\mu}$: $\log f^{\lambda^{(n)}/\mu^{(n)}} \sim \frac{1}{2} n \log n.$



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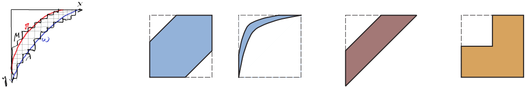
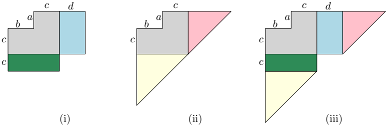


- Product formulas for special $f^{\lambda/\mu}$.



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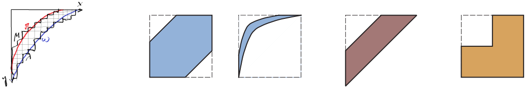

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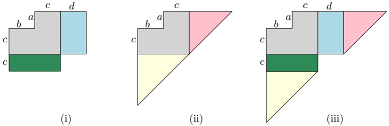

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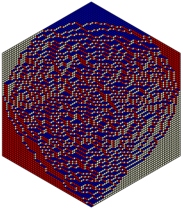
- Weighted lozenge tilings.



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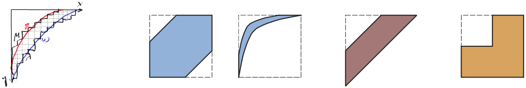

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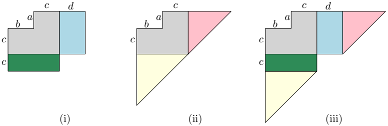

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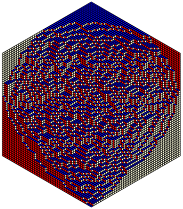


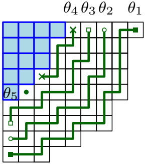
- Principle evaluations of Schubert polynomials and asymptotics.

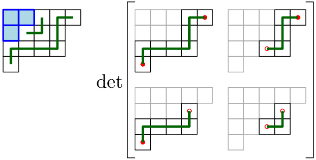
Non-intersecting lattice paths

Theorem[Lascoux-Pragacz, Hamel-Goulden] If $(\theta_1, \ldots, \theta_k)$ is a Lascoux–Pragacz decomposition (i.e. maximal outer border strip decomposition) of λ/μ, then

$$s_{\lambda/\mu} = \det \left[s_{\theta_i \# \theta_j} \right]_{i,j=1}^{k}.$$

where $s_\varnothing = 1$ and $s_{\theta_i \# \theta_j} = 0$ if the $\theta_i \# \theta_j$ is undefined.





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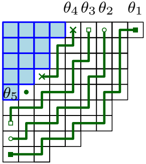

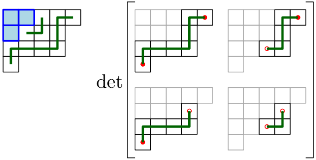

Can also switch to “inner border strip decomposition” [Kreiman].

NHLF: lattice paths

Lemma (MPP)

For a border strip $\theta = \lambda/\mu$ with end points (a, b) and (c, d) we have

$$s_\theta(1, q, q^2, \ldots,) = \sum_{\substack{\gamma:(a,b)\to(c,d),\\ \gamma\subseteq\lambda}} \prod_{(i,j)\in\gamma} \frac{q^{\lambda'_j - i}}{1 - q^{h(i,j)}}.$$

$$s(1, q, q^2, \ldots) = \underbrace{\frac{q^3}{(1-q^2)(1-q^1)(1-q^3)(1-q^1)(1-q^2)}}_{\gamma=(3,1),(3,2),(2,2),(2,3),(1,3):} + \frac{q^4}{(1-q)(1-q^2)^2(1-q^3)(1-q^4)}$$

$$+ \frac{q^1}{(1-q)(1-q^2)^2(1-q^3)(1-q^4)} + \frac{q^7}{(1-q)^2(1-q^3)(1-q^4)^2} + \frac{q^6}{(1-q)^2(1-q^5)(1-q^4)^2}$$

Proofs: induction on $|\lambda/\mu|$, or [multivariate] Chevalley formula for factorial Schurs.

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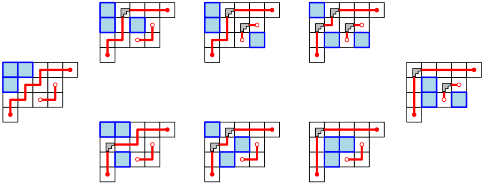

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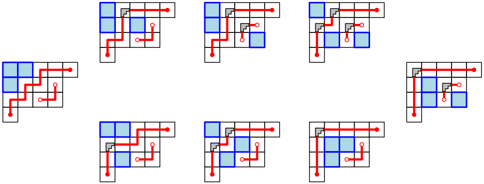

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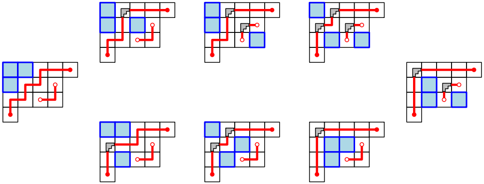

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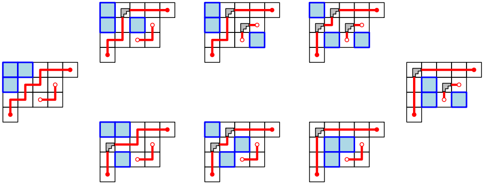

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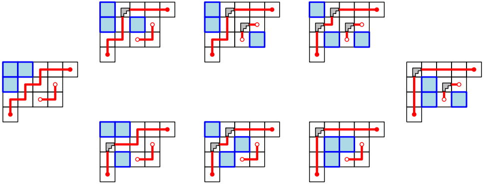

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Bijections

Hillman-Grassl map Φ: Reverse Plane Partitions of shape λ to Arrays of shape λ:

$$RRP \;\; P = \begin{array}{|c|c|c|}\hline 0&1&2\\\hline 1&1&3\\\hline 2\\\cline{1-1}\end{array} \to \begin{array}{|c|c|c|}\hline 0&1&2\\\hline 1&1&3\\\hline 1\\\cline{1-1}\end{array} \to \begin{array}{|c|c|c|}\hline 0&0&1\\\hline 0&0&3\\\hline 0\\\cline{1-1}\end{array} \to \begin{array}{|c|c|c|}\hline 0&0&1\\\hline 0&0&2\\\hline 0\\\cline{1-1}\end{array} \to \begin{array}{|c|c|c|}\hline 0&0&1\\\hline 0&0&1\\\hline 0\\\cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline 0&0&0\\\hline 0&0&0\\\hline 0\\\cline{1-1}\end{array}$$

$$\begin{array}{|c|c|c|}\hline 0&0&0\\\hline 0&0&0\\\hline 1\\\cline{1-1}\end{array} \to \begin{array}{|c|c|c|}\hline 1&0&0\\\hline 0&0&0\\\hline 1\\\cline{1-1}\end{array} \to \begin{array}{|c|c|c|}\hline 1&0&0\\\hline 0&0&1\\\hline 1\\\cline{1-1}\end{array} \to \begin{array}{|c|c|c|}\hline 1&0&0\\\hline 0&0&2\\\hline 1\\\cline{1-1}\end{array} \to \begin{array}{|c|c|c|}\hline 1&0&1\\\hline 0&0&2\\\hline 1\\\cline{1-1}\end{array} =: Array \;\; A = \Phi(P)$$

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$$Weight(P) = |P| = 0 + 1 + 2 + 1 + 1 + 3 + 2 = 10 =$$

$$= \sum_{i,j} A_{i,j} hook(i,j) = 1 * 5 + 1 * 2 + 2 * 1 + 1 * 1 =: weight(A)$$

Bijections

Hillman-Grassl map Φ: Reverse Plane Partitions of shape λ to Arrays of shape λ:

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$$\begin{array}{ccc} 0&0&0\\ 0&0&0\\ 1&& \end{array} \to \begin{array}{ccc} 1&0&0\\ 0&0&0\\ 1&& \end{array} \to \begin{array}{ccc} 1&0&0\\ 0&0&1\\ 1&& \end{array} \to \begin{array}{ccc} 1&0&0\\ 0&0&2\\ 1&& \end{array} \to \begin{array}{ccc} 1&0&1\\ 0&0&2\\ 1&& \end{array} =: Array \;\; A = \Phi(P)$$

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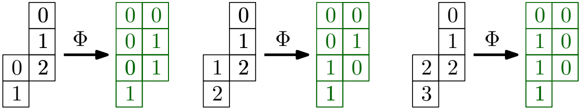
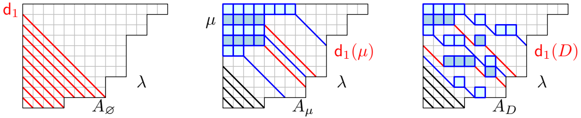
$$\sum_{P \in RPP(\lambda)} q^{|P|} = \sum_{A:Array(\lambda)} \prod_{(i,j)\in\lambda} q^{h(i,j)*A_{i,j}} = \prod_{(i,j)\in\lambda} \frac{1}{1-q^{h(i,j)})}$$

Bijections



Theorem (Morales-Pak-P)

The restricted Hillman-Grassl map is a bijection from the SSYTs of shape λ/μ to the excited arrays (diagrams in $\mathcal{E}(\lambda/\mu)$ with nonzero entries on the broken diagonals) .

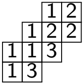


Hillman-Grassl on skew Reverse Plane Partitions

RPP: **weakly** increasing rows and columns:

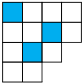
		1	2
	1	2	2
1	1	3	
1	3		

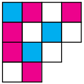
Hillman-Grassl on skew Reverse Plane Partitions

RPP: **weakly** increasing rows and columns: 

Skew RPPs $\Leftrightarrow$ arrays with support *"pleasant diagrams"*:

$$PD(\lambda/\mu) := \{S \subset [\lambda] : S \subset [\lambda] \setminus D, \text{ for some } D \in \mathcal{E}(\lambda/\mu)\}$$

Excited: 

Pleasant boxes : 

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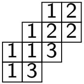

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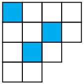
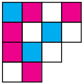
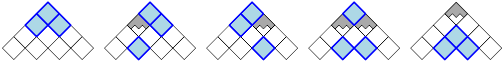
Excited: 

Pleasant boxes: 

Theorem (MPP)

The HG map is a bijection between skew RPPs of shape λ/μ and arrays with certain nonzero entries (at the "high peaks"):

$$\sum_{\pi \in RPP(\lambda/\mu)} q^{|\pi|} = \sum_{S \in PD(\lambda/\mu)} \prod_{u \in S} \left[\frac{q^{h(u)}}{1 - q^{h(u)}} \right].$$



2^7 2^6 2^6 2^5 2^6

Hillman-Grassl on skew Reverse Plane Partitions

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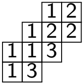

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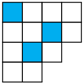

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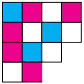

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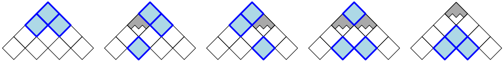
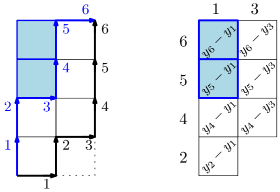


P-partitions or order polytope's volumes and $q \to 1$ limits: proof of original Naruse Hook-Length Formula for $f^{\lambda/\mu}$..

Algebraic proof for SSYTs:



$v = 245613$, $w = 361245$

[Ikeda-Naruse, Kreiman]:
$w \preceq v$ –Grassmannian permutations
Schubert class X_w localization at v:

$$[X_w]\big|_v = \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in D} \left(y_{v(d+j)} - y_{v(d-i+1)}\right).$$

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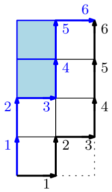

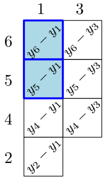

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Factorial Schur functions:

$$s_\mu^{(d)}(\mathbf{x}|\mathbf{a}) := \frac{\det\left[(x_j - a_1)\cdots(x_j - a_{\mu_i+d-i})\right]_{i,j=1}^{d}}{\prod_{1\le i<j\le d}(x_i - x_j)},$$

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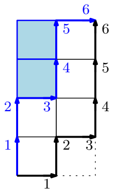

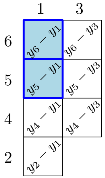

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[Knutson-Tao, Lakshmibai–Raghavan–Sankaran] Schubert class at a point:

$$[X_w]\big|_v = (-1)^{\ell(w)} s_\mu^{(d)}\left(y_{v(1)},\ldots,y_{v(d)}|y_1,\ldots,y_{n-1}\right),$$

where $v \to \lambda$ and $w \to \nu$

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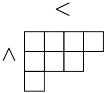
$$= \frac{\det[\prod_{r=1}^{\mu_j+d-j}(q^{\lambda_i+d+1-i}-q^r)]_{i,j=1}^d}{\prod_{i<j}(q^{\lambda+d+1-i}-q^{\lambda_j+d+1-j})} = \ldots[simplifications]\ldots$$

$$= (factor)\det\Big[\underbrace{\frac{1}{(1-q)(1-q^2)\cdots(1-q^{\lambda_i-i-\mu_j+j})}}_{h_{\lambda_i-i-\mu_j+j}(1,q,\ldots)}\Big] \underbrace{=}_{\text{Jacobi-Trudi}} s_{\lambda/\mu}(1,q,\ldots)$$

Increasing Tableaux



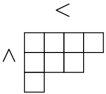
Increasing Tableau

1	2	6	8
2	3	8	
6			

Standard Increasing Tableau

1	2	3	5
2	3	4	
5			

Increasing Tableaux



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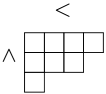
$m(T) := \max\{T(i,j)\}, \quad [T_{<k}] = \{(i,j) : T(i,j) < k\}$

Increasing Tableaux



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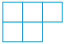
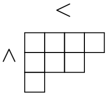
$m(T) := \max\{T(i,j)\}$, $[T_{<k}] = \{(i,j) : T(i,j) < k\}$ e.g. $[T_{<4}] =$ 

Increasing Tableaux



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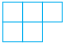
$m(T) := \max\{T(i,j)\}$, $[T_{<k}] = \{(i,j) : T(i,j) < k\}$ e.g. $[T_{<4}] =$ 

Theorem (Morales-Pak-Panova'21+)

Fix $d \geq 1$, $\beta \in \mathbb{R}$. For every $\lambda \vdash n$ with $\ell(\lambda) \leq d$, we have:

$$\sum_{T \in \mathrm{SIT}(\lambda)} \prod_{k=1}^{m(T)} \left(\left[\prod_{i=1}^{d} \frac{1+\beta\left([T_{<k}]_i\right)+d-i+1)}{1+\beta\left(\lambda_i+d-i+1\right)} \right] - 1 \right)^{-1} \tag{K-HLF}$$

$$= \frac{1}{(-\beta)^n} \prod_{i=1}^{\ell(\lambda)} \left(1+\beta(\lambda_i+d-i+1)\right)^{\lambda_i} \prod_{(i,j)\in\lambda} \frac{1}{h(i,j)}.$$

Theorem (**Multivariate K-HLF**, Morales-Pak-Panova'21+)

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$$= \frac{1}{\beta^n} \prod_{i=1}^{d} \left(1 + \beta y_{\lambda_i + d - i + 1}\right)^{\lambda_i} \prod_{(i,j) \in \lambda} \frac{1}{y_{d+j-\lambda'_j} - y_{\lambda_i + d - i + 1}} .$$

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Corollary:

$$\sum_{T \in \mathrm{SIT}(\lambda)} q^{\sum T(i,j)} \prod_{k=1}^{m(T)} \frac{1}{1 - q^{|[T_{\geq k}]|}} = q^{\sum_{(i,j)\in\lambda} i+j-1} \prod_{(i,j) \in \lambda} \frac{1}{1 - q^{h(i,j)}} .$$

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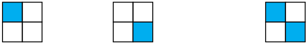
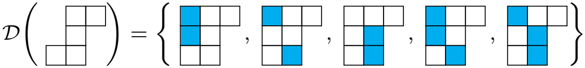
$$\underset{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \\ \cline{1-1}\end{array}}{q^5 \frac{1}{(1-q^3)(1-q^2)}} + \underset{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}}{q^6 \frac{1}{(1-q^3)(1-q^2)(1-q)}} + \underset{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}}{q^6 \frac{1}{(1-q^3)(1-q^2)(1-q)}}$$

$$= q^5 \frac{1}{(1-q^3)(1-q)^2}$$

Generalized Excited Diagrams $\mathcal{D}(\lambda/\mu)$

Type I move Type II move





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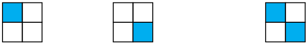
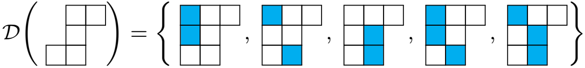




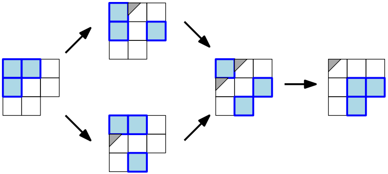
Proposition (Naruse-Okada)

$$\mathcal{D}(\lambda/\mu) \;=\; \bigcup_{D\in\mathcal{E}(\lambda/\mu)} \{D \cup S \;:\; S \subseteq \pi(D)\},$$

$$\left|\mathcal{D}(\lambda/\mu)\right| \;=\; \sum_{D\in\mathcal{E}(\lambda/\mu)} 2^{|\pi(D)|}.$$

$\mathcal{E}(\lambda/\mu) = \{$$\}$

$\pi(D) = \{$$\}$



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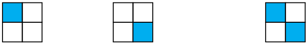
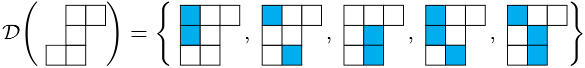
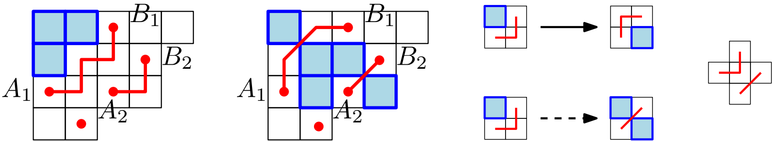




Non-intersecting Delannoy paths[MPP] with forbidden configuration:



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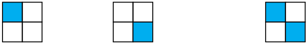

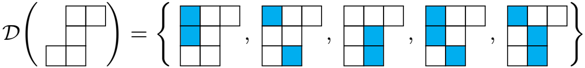

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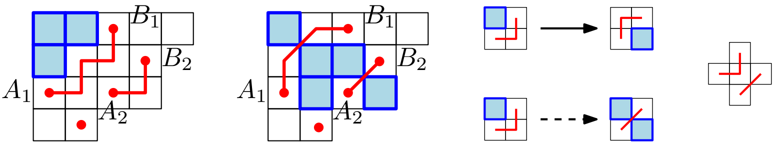
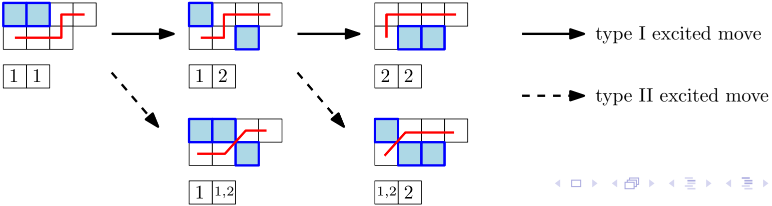


Flagged set-valued tableaux:



Skew K-HLF

Theorem (Morales-Pak-Panova'21+)

Fix $d \geq 1$, $\beta \in \mathbb{R}$. For every $\mu \subset \lambda$ with $\ell(\lambda) \leq d$, we have:

$$\sum_{T \in \mathrm{SIT}(\lambda/\mu)} \prod_{k=1}^{m(T)} \left(\left[\prod_{i=1}^{d} \frac{1+\beta\big([T_{<k}]_i + d - i + 1\big)}{1+\beta(\lambda_i + d - i + 1)} \right] - 1 \right)^{-1} = \sum_{D \in \mathcal{D}(\lambda/\mu)} (-\beta)^{|D| - |\lambda|} \prod_{(i,j) \in \lambda \setminus D} \frac{\beta(\lambda_i + d - i + 1) + 1}{h(i,j)} . \qquad \text{(K-NHLF)}$$

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Factorial Grothendieck polynomials

(double Grothendieck polynomials for Grassmannian permutations)

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$$x \oplus y \;:=\; x + y + \beta xy\,, \qquad x \ominus y \;:=\; \frac{(x-y)}{(1+\beta y)}\,, \qquad \ominus x \;:=\; \frac{-x}{1+\beta x}\,,$$

$$\text{and} \qquad [x\,|\,\mathbf{y}]^k \;:=\; (x \oplus y_1)(x \oplus y_2)\,\cdots\,(x \oplus y_k)\,,$$

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[McNamara]: The **Factorial Grothendieck polynomials** are given by:

$$G_\mu(x_1, \ldots, x_d \,|\, \mathbf{y}) = \det\Big([x_i \,|\, \mathbf{y}]^{\mu_j + d - j} (1 + \beta x_i)^{j-1} \Big)_{i,j=1}^{d} \prod_{1 \le i < j \le d} \frac{1}{(x_i - x_j)}.$$

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Vanishing property:
evaluation at $\mathbf{y}_\lambda := (\ominus\, y_{\lambda_1+d}, \ominus\, y_{\lambda_2+d-1}, \ldots, \ominus\, y_{\lambda_d+1})$ for $\ell(\lambda) \le d$,

$$G_\mu(\mathbf{y}_\lambda\,|\,\mathbf{y}) \;=\; \begin{cases} 0 & \text{if } \mu \not\subseteq \lambda, \\ \prod_{(i,j)\in\lambda}(y_{d+j-\lambda'_j} \ominus y_{\lambda_i+d-i+1}) & \text{if } \mu = \lambda. \end{cases}$$

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Pieri rule:

$$G_\mu(\mathbf{x}\,|\,\mathbf{y})\big(1+\beta G_1(\mathbf{x}\,|\,\mathbf{y})\big) \;=\; \big(1+\beta G_1(\mathbf{y}_\mu\,|\,\mathbf{y})\big) \sum_{\nu\mapsto\mu} \beta^{|\nu/\mu|}\, G_\nu(\mathbf{x}\,|\,\mathbf{y})\,.$$

Proof of K-HLF

$\mathbf{x} = \mathbf{y}_\lambda$

$$G_\mu(\mathbf{y}_\lambda \mid \mathbf{y}) \big(wt(\lambda/\mu) - 1\big) \; = \sum_{\nu \mapsto \mu} \beta^{|\nu/\mu|} \, G_\nu(\mathbf{y}_\lambda \mid \mathbf{y}) \, ,$$

where

$$wt(\lambda/\mu) \; := \; \prod_{i=1}^{d} \frac{1 + \beta y_{\mu_i + d - i + 1}}{1 + \beta y_{\lambda_i + d - i + 1}} \, .$$

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$T \in \mathrm{SIT}(\lambda/\mu)$ is a chain of shapes

$$\lambda = \nu(T_{\le k}) \mapsto \nu(T_{\le k-1}) \mapsto \ldots \mapsto \nu(T_{\le 1}) \mapsto \nu(T_{\le 0}) = \mu.$$

$$\sum_{T\in\mathrm{SIT}(\lambda)} \prod_{k=0}^{m(T)-1} \frac{\beta^{|[T_{\le k+1}]| - |[T_{\le k}]|}}{wt(\lambda/\nu^{(k)}) - 1} = \frac{G_\lambda(\mathbf{y}_\lambda \mid \mathbf{y})}{G_\emptyset(\mathbf{y}_\lambda \mid \mathbf{y})}.$$

Theorem (Multivariate K-HLF, Morales-Pak-Panova'21+)

Fix $d \ge 1$. For every $\lambda \vdash n$ with $\ell(\lambda) \le d$ we have:

$$\sum_{T\in\mathrm{SIT}(\lambda)} \prod_{k=1}^{m(T)} \left(\left[\prod_{i=1}^{d} \frac{1+\beta y_{[T_{<k}]_i+d-i+1}}{1+\beta y_{\lambda_i+d-i+1}} \right] - 1 \right)^{-1} = \frac{1}{\beta^n} \frac{\prod_{i=1}^{d} \left(1+\beta y_{\lambda_i+d-i+1}\right)^{\lambda_i}}{\prod_{(i,j)\in\lambda} \left(y_{d+j-\lambda'_j} - y_{\lambda_i+d-i+1}\right)}.$$

Skew K-HLF

Evaluations of factorial Grothendieck polynomials $G_\mu(\mathbf{y}_\lambda|\mathbf{y})$ for $\mu \subset \lambda$
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[Graham-Kreiman]: Structure constants

$$K_{\mu\lambda}^{\lambda} = \sum_{D \in \mathcal{D}(\lambda/\mu)} (-1)^{|D|-|\mu|} \prod_{(i,j)\in D} \frac{y_{d+j-\lambda'_j} - y_{\lambda_i+d+1-i}}{1 - y_{\lambda_i+d+1-i}}.$$

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[Lenart–Postnikov] *Equivariant K-theory Chevalley formula*:

$$K^\lambda_{\mu\lambda}\left(\frac{K^\lambda_{1\lambda} - 1 + wt'(\mu)}{wt'(\mu)}\right) \;=\; \sum_{\nu\mapsto\mu} (-1)^{|\nu/\mu|-1}\, K^\lambda_{\nu\lambda}\,,$$

where $wt'(\mu) \;:=\; \prod_{(i,j)\in\mu} \frac{1-y_{i+j-1}}{1-y_{i+j}}$.

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where $wt'(\mu) \; := \prod_{(i,j)\in\mu} \frac{1-y_{i+j-1}}{1-y_{i+j}}$.

Compare with Chevalley formula for factorial Grothendiecks at $\beta = -1$:

$$G_\mu(\mathbf{y}_\lambda \mid \mathbf{y})\left(\frac{G_1(\mathbf{y}_\lambda|\mathbf{y}) \; - \; G_1(\mathbf{y}_\mu \mid \mathbf{y})}{1+\beta G_1(\mathbf{y}_\mu \mid \mathbf{y})}\right) \; = \sum_{\nu\supsetneq\mu} \beta^{|\nu/\mu|-1} \, G_\nu(\mathbf{y}_\lambda \mid \mathbf{y}) \, .$$

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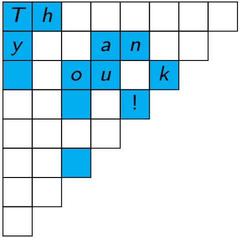
General β: substitute $y_i \leftarrow -\beta y_i$

Theorem (Skew K-HLF, Morales-Pak-Panova'21+)

Fix $d \geq 1$, $\beta \in \mathbb{R}$. For every $\mu \subset \lambda$ with $\ell(\lambda) \leq d$, we have:

$$\sum_{T\in\mathrm{SIT}(\lambda/\mu)} \prod_{k=1}^{m(T)} \left(\left[\prod_{i=1}^{d} \frac{1 + \beta y_{\nu_i(T_{<k})+d-i+1}}{1 + \beta y_{\lambda_i+d-i+1}}\right] - 1\right)^{-1} = \sum_{D\in\mathcal{D}(\lambda/\mu)} \beta^{|D|-|\lambda|} \prod_{(i,j)\in\lambda\setminus D} \frac{\beta y_{\lambda_i+d-i+1} + 1}{y_{d+j-\lambda'_j} - y_{\lambda_i+d+1-i}}.$$



$$\left|\mathcal{D}(\delta_{n+2k}/\delta_n)\right| = 2^{-\binom{k}{2}} \det\left[s_{n-2+i+j}\right]_{i,j=1}^{k}$$