Polynomials from Schubert Calculus via Diagrams

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Joint work with Tianyi Yu (UC San Diego) arXiv:2302.03643 & arXiv:2206.08993

NC STATE UNIVERSITY

Combinatorics and Graph Theory Seminar
Michigan State University
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Outline

Polynomials are power tools

- Schur polynomials
  Representation theory: characters of classical Lie algebras (and the symmetric group)
  Geometry: polynomial representatives of Grassmannian Schubert varieties

- Schubert polynomials
  Representation theory: characters of certain flagged Schur modules
  Geometry: polynomial representatives of Schubert varieties in the flag variety

Combinatorics of a Polynomial $f_\iota(\boldsymbol{x})$

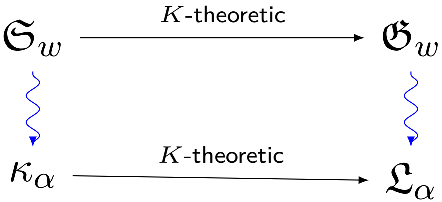
1. Is $f_\iota(\boldsymbol{x})$ the generating function over certain set of discrete objects $\mathcal{B}$?
2. Can we find special elements of $\mathcal{B}$ that capture important features of $f_\iota(\boldsymbol{x})$?
3. Can we find certain collections of indices such that $\{f_\iota(\boldsymbol{x}) \mid \iota \in I\}$ form a basis of the space that they span?

Ex. Schur polynomial $s_\lambda(x_1, \ldots, x_n)$

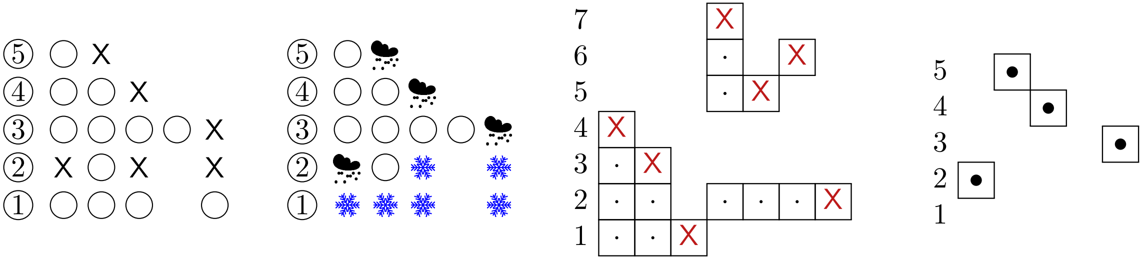
1. Semistandard Young tableaux of shape $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_\ell)$
2. There is a unique tableau with k's on k^{th} row. It is the only one with the property Yamanouchi. This gives solutions to many counting problems!
3. The Schur polynomials indexed by partitions of n form a basis of the ring of symmetric polynomials in n variables.

Four polynomials

- Schubert polynomial $\mathfrak{S}_w$
- Key polynomial κ_α
- Grothendieck polynomial $\mathfrak{G}_w$
- Lascoux polynomial $\mathfrak{L}_\alpha$



Many Diagrams



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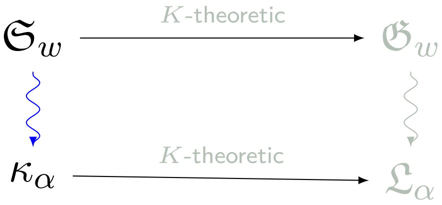


Schubert Polynomials

For a permutation $w \in S_n$,

$$\mathfrak{S}_w = \begin{cases} x_1^{n-1} x_2^{n-2} \dots x_{n-1}, & \text{if } w = [n, n-1, \dots, 1] \\ \partial_i(\mathfrak{S}_{s_i w}), & \text{if } w_i < w_{i+1} \end{cases}$$

$$\partial_i(f) = \frac{f - f^{s_i}}{x_i - x_{i+1}}$$

$$f^{s_i}(\dots, x_i, x_{i+1}, \dots) = f(\dots, x_{i+1}, x_i, \dots)$$

Ex.

$$\begin{aligned} \mathfrak{S}_{4321} &= x_1^3 x_2^2 x_3 \\ \mathfrak{S}_{4312} &= \partial_3(x_1^3 x_2^2 x_3) = \frac{x_1^3 x_2^2 x_3 - x_1^2 x_2^3 x_4}{x_3 - x_4} = x_1^3 x_2^2 \\ \mathfrak{S}_{2143} &= x_1 x_3 + x_1 x_2 + x_1^2 \end{aligned}$$

- Generalize the Schur polynomials
- Form a basis for the polynomial ring
- Comb. models: Kohnert diagrams 90; pipe dream Bergeron-Billey 93; bumpless pipe dream Lam-Lee-Shimozono 18; many more!

Kohnert Rule for $\mathfrak{S}_w$

Kohnert move:

- Select the rightmost Kohnert cell in any row
- Move it downward (jump allowed) to the first position available

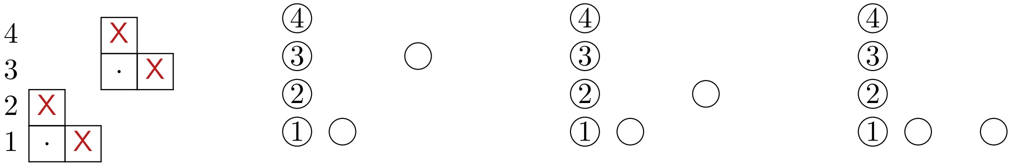
KD(w): The closure of the Rothe diagram $\{D(w)\}$ under all possible Kohnert moves.

Theorem (Kohnert)

The Schubert polynomials indexed by w, is given by

$$\mathfrak{S}_w = \sum_{D \in \mathsf{KD}(w)} \boldsymbol{x}^{\mathsf{wt}(D)}.$$

Ex. Let $w = 2143 \in S_4$, and we have computed $\mathfrak{S}_{2143} = x_1x_3 + x_1x_2 + x_1^2$.



Key Polynomials κ_α

For a weak composition α,

$$\kappa_\alpha = \begin{cases} x^\alpha, & \text{if } \alpha \text{ is weakly decreasing} \\ \pi_i(\kappa_{s_i\alpha}), & \text{if } \alpha_i < \alpha_{i+1}\,. \end{cases}$$

$$\pi_i(f) = \partial_i(x_i f)\,,$$

$$\partial_i(f) = \frac{f - f^{s_i}}{x_i - x_{i+1}}$$

Ex.

$$\begin{aligned} \kappa_{210} &= x_1^2 x_2\,, \\ \kappa_{120} &= \pi_1(\kappa_{210}) = \frac{x_1 \cdot x_1^2 x_2 - x_2 \cdot x_1 x_2^2}{x_1 - x_2} = x_1^2 x_2 + x_1 x_2^2 \end{aligned}$$

- Form a basis for the polynomial ring
- Generalize the Schur polynomials
- AKA the Demazure character
- Comb. models: Kohnert diagrams (90); Semi-skyline augmented fillings (Mason 09);

Kohnert Rule for κ_α

Key diagram D_α**:** A left-justified diagram with α_i Kohnert cells on row i

Kohnert move:

- Select the rightmost Kohnert cell in any row
- Move it downward (jump allowed) to the first position available

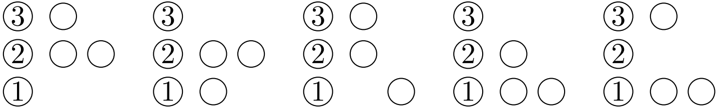
KD(α): The closure of $\{D_\alpha\}$ under all possible Kohnert moves.

Theorem (Kohnert)

The key polynomials indexed by α*, is given by*

$$\kappa_\alpha = \sum_{D \in \mathsf{KD}(\alpha)} \boldsymbol{x}^{\mathsf{wt}(D)}.$$

Ex. $\alpha = (0, 2, 1)$, we get $\kappa_\alpha = x_2^2 x_3 + x_1 x_2^2 + x_1 x_2 x_3 + x_1^2 x_2 + x_1^2 x_3$.



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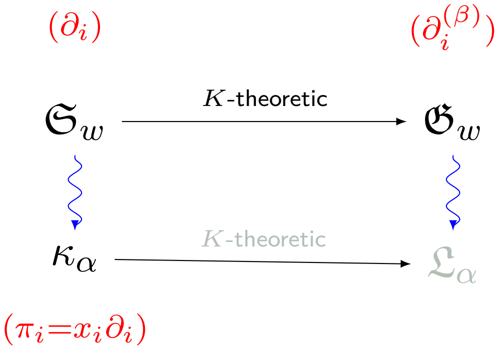

Grothendieck polynomials

For a permutation $w \in S_n$,

$$\mathfrak{G}_w = \begin{cases} x_1^{n-1}x_2^{n-2}\dots x_{n-1}, & \text{if } w=[n,n-1,\dots,1] \\ \partial_i^{(\beta)}(\mathfrak{G}_{s_iw}), & \text{if } w_i < w_{i+1}\,. \end{cases}$$

$$\partial_i^{(\beta)}(f) = \partial_i[(1+\beta x_{i+1})f]\,,$$
$$\partial_i(f) = \frac{f-f^{s_i}}{x_i - x_{i+1}}$$

Ex.

$$\begin{aligned} \mathfrak{G}_{4321} &= x_1^3x_2^2x_3 \\ \mathfrak{G}_{1423} &= (x_1^2+x_1x_2+x_2^2)+\beta(x_1^2x_2+x_1x_2^2) \\ \mathfrak{G}_{2143} &= (x_1x_2+x_1x_3+x_1^2)+\beta(x_1x_2x_3+x_1^2x_2+x_1^2x_3)+\beta^2x_1^2x_2x_3 \end{aligned}$$

- Represent the K-theory of Schubert varieties
- Constant terms are the Schubert polynomials (set $\beta = 0$)
- Combinatorial models: compatible sequence (Fomin-Greene 94); K-Kohnert diagrams (Conjectured Ross-Yong 13)

"Bottom" layers of Grothendieck polynomials: Schubert polynomials $\mathfrak{S}_w$

$$\mathfrak{G}_{2143} = (x_1x_3 + x_1x_2 + x_1^2) + \beta(x_1x_2x_3 + x_1^2x_2 + x_1^2x_3) + \beta^2x_1^2x_2x_3$$

Questions: what is the leading monomial for Schubert polynomial $\mathfrak{S}_w$?

Definition (Tail lexicographic order)

Compare monomials by first comparing the power of x_n, then $x_{n-1}, \ldots, x_1$. For example, $x_1x_2^3x_3^2 < x_1x_2^2x_3^3$.

Answer: Lehmer code (Billey-Jockusch-Stanley 93)

Definition (Lehmer code (aka inversion code))

Lehmer code of w is the sequence $L_1(w), \ldots, L_n(w)$ where $L_i(w) = \#\{j > i : w(i) > w(j)\}$

Ex. $w = 2143$, $L(w) = (1, 0, 1, 0)$, leading monomial x_1x_3

"Top" layers of Grothendieck polynomials: Castelnuovo-Mumford polynomials

$$\mathfrak{G}_{2143} = (x_1x_3 + x_1x_2 + x_1^2) + \beta(x_1x_2x_3 + x_1^2x_2 + x_1^2x_3) + \beta^2 x_1^2x_2x_3$$

Question: what is the leading monomial of $\widehat{\mathfrak{G}}_w$?

Answer: $x^{\mathsf{rajcode}(w)}$ (Pechenik-Speyer-Weigandt 21)

Definition (rajcode (Pechenik-Speyer-Weigandt 21))

Rajcode of w is the sequence $(\mathsf{rajcode}(w)_1, \mathsf{rajcode}(w)_2, \ldots, \mathsf{rajcode}(w)_n)$ where $\mathsf{rajcode}(w)_r$ the number of elements not in a longest increasing subsequence of $w(r), w(r+1), \ldots, w(n)$ that starts with $w(r)$.

Ex. $\mathsf{rajcode}(2143) = (2, 1, 1, 0)$

Result: permutations u and v have the same rajcode if and only if their Castelnuovo-Mumford polynomials are scaler multiple of each other (Pechenik-Speyer-Weigandt 21)

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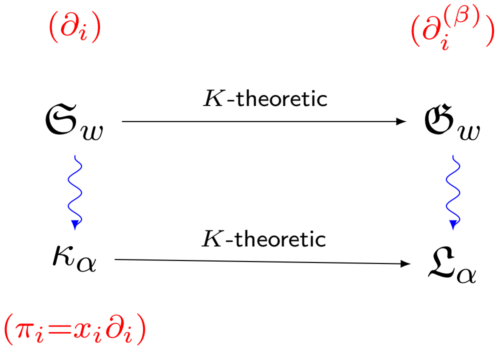


Lascoux Polynomials $\mathfrak{L}_\alpha$

For a weak composition α,

$$\mathfrak{L}_\alpha = \begin{cases} x^\alpha, & \text{if } \alpha \text{ is weakly decreasing} \\ \pi_i^{(\beta)}(\mathfrak{L}_{s_i\alpha}), & \text{if } \alpha_i < \alpha_{i+1} . \end{cases}$$

$$\pi_i^{(\beta)}(f) = \pi_i[(1+\beta x_{i+1})f] ,$$
$$\pi_i(f) = \partial_i(x_i f)$$

Ex.

$$\begin{aligned} \mathfrak{L}_{210} &= x_1^2 x_2 , \\ \mathfrak{L}_{120} &= \pi_1^{(\beta)}(\mathfrak{L}_{210}) = \pi_1(x_1^2x_2) + \beta\partial_i(x_1x_2 \cdot x_1^2x_2) = (x_1^2x_2 + x_1x_2^2) + \beta x_1^2 x_2^2 \end{aligned}$$

- Recover the key polynomials when set $\beta = 0$
- Grothendieck polynomials expand positively into Lascoux polynomials (Shimozono-Yu 21)
- Comb. models: Tableaux (Buciumas-Scrimshaw-Weber 20);
  K-Kohnert diagram (P.-Yu 22)

Ross & Yong's Conjecture

K-**Kohnert move**:

- Select the rightmost cell $\bullet$ in a row
- Move this cell downward to the first position available
- Can jump over other $\bullet$, but cannot jump over any X
- After the move, it may or may not leave a ghost cell (X) at the original position

KKD(α)**:** the closure of $\{D_\alpha\}$ under all possible K-Kohnert moves.

Theorem (P.-Yu 22)

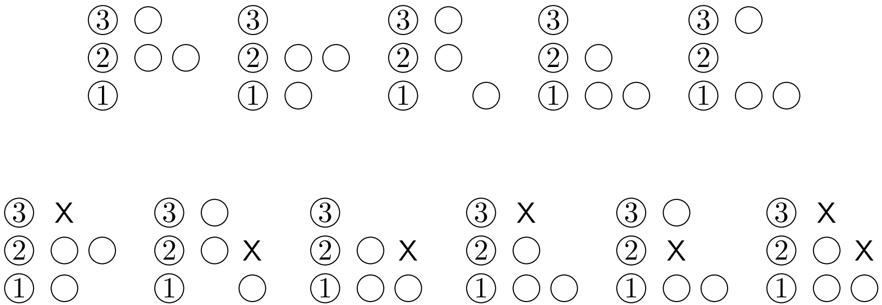
The Lascoux polynomials indexed by α, is given by

$$\mathfrak{L}_\alpha^{(\beta)} = \sum_{D \in \mathsf{KKD}(\alpha)} \beta^{\mathsf{ex}(D)} \boldsymbol{x}^{\mathsf{wt}(D)},$$

ex(D) : *number of* X *in* D.

An Example on Ross & Yong's Conjecture/Theorem

Ex. $\alpha = (0, 2, 1)$



$$\mathfrak{L}_\alpha^{(\beta)} = (x_2^2x_3 + x_1x_2^2 + x_1x_2x_3 + x_1^2x_2 + x_1^2x_3) + \beta(2x_1x_2^2x_3 + x_1^2x_2^2 + 2x_1^2x_2x_3) + \beta^2x_1^2x_2^2x_3$$

"Bottom" and "top" layers of Lascoux polynomials

$$\mathfrak{L}_{021}^{(\beta)} = (x_2^2x_3 + x_1x_2^2 + x_1x_2x_3 + x_1^2x_2 + x_1^2x_3) + \beta(2x_1x_2^2x_3 + x_1^2x_2^2 + 2x_1^2x_2x_3) + \beta^2x_1^2x_2^2x_3$$

Question: what is the leading monomial for κ_α?

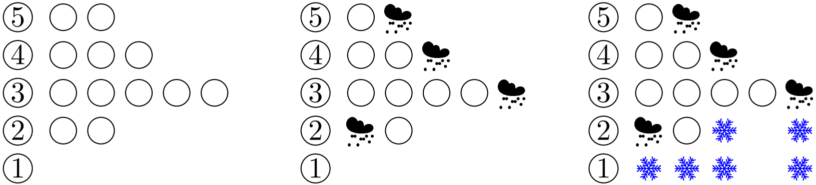
Answer: x^α

Question: What about the leading monomial for top layers of Lascoux polynomials $\widehat{\mathfrak{L}}_\alpha$?

Answer: Snow diagrams!

Snow diagrams and rajcode for weak compositions

Ex. Let $\alpha = (0, 2, 5, 3, 2)$.



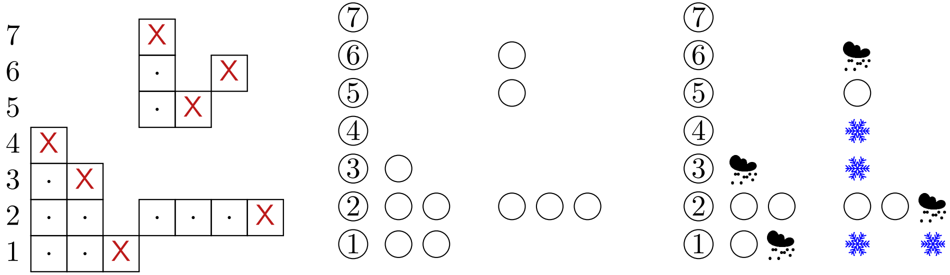
$$\text{rajcode}(0, 2, 5, 3, 2) = (4, 4, 5, 3, 2)\,, \quad x_1^4 x_2^4 x_3^5 x_4^3 x_5^2$$

Theorem (P.-Yu 23)

- *The leading monomial for top Lascoux* $\widehat{\mathfrak{L}}_\alpha$ *is* $x^{\text{rajcode}(\alpha)}$.
- *Two weak compositions* α *and* β *have the same* rajcode *if and only if their top Lascoux polynomials are scaler multiple of each other.*

Snow diagrams for permutations

Ex. Let $w = 3721564 \in S_7$. Compute its Rothe diagram.

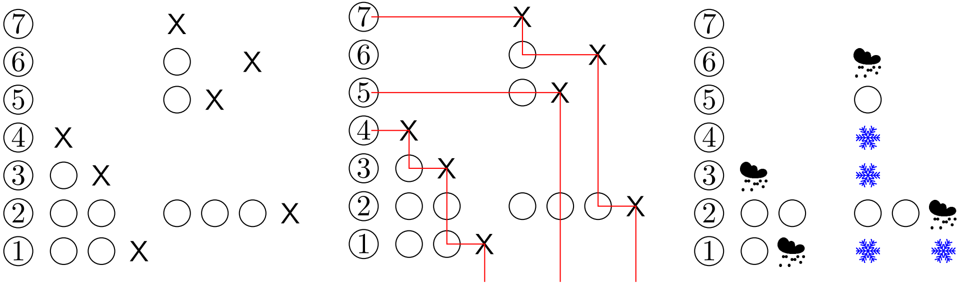


rajcode$(w) = (4, 5, 2, 1, 1, 1, 0)$.

Theorem (P.-Yu 23)

The snow diagram of the Rothe diagram of a permutation w gives the rajcode(w).

Shadow diagrams and dark clouds



Theorem (P.-Yu 23)

The turning points of the shadow diagram are exactly the dark clouds in the snow diagram.

Filtered Vector Space

Definition

Let $\widehat{V}_n := \mathbb{Q}\text{-span}\{\widehat{\mathfrak{G}}_w : w \in S_n\}$ and $\widehat{V} := \bigcup_{n\geq 1} \widehat{V}_n = \mathbb{Q}\text{-span}\{\widehat{\mathfrak{G}}_w : w \in S_+\}$.

$$\widehat{V}_1 \subset \widehat{V}_2 \subset \cdots \subset \widehat{V}_n \subset \dots$$

Theorem (Pechenik-Speyer-Weigandt 21)

- *There is a basis for $\widehat{V}_n$ by $\widehat{\mathfrak{G}}_w$ by inverse fireworks permutations.*
- *The dimension of $\widehat{V}_n$ is the n-th Bell number B_n.*

Ex. $B_3 = 5$, inverse fireworks permutations in S_3 are $123, 213, 132, 312, 321$.

$$\mathfrak{G}_e^{(\beta)} = 1, \qquad \mathfrak{G}_{213}^{(\beta)} = x_1, \quad \mathfrak{G}_{132}^{(\beta)} = (x_1 + x_2) + \beta x_1 x_2,$$

$$\mathfrak{G}_{231}^{(\beta)} = x_1 x_2, \qquad \mathfrak{G}_{312}^{(\beta)} = x_1^2, \quad \mathfrak{G}_{321}^{(\beta)} = x_1^2 x_2$$

Snowy weak compositions

Let C_n be the set of weak compositions within the staircase shape $(n-1, n-2, \ldots, 1)$.

Definition (P.-Yu 23)

A weak composition is snowy if its positive entries are distinct.

Theorem (P.-Yu 23)

Any $\alpha \in C_n$ have the same rajcode *with exactly one snowy weak composition in C_n.*

Ex. Snowy weak compositions in C_3 are $000, 100, 010, 200, 210$.

$$\mathfrak{L}_0^{(\beta)} = 1\,, \qquad \mathfrak{L}_{100}^{(\beta)} = x_1\,, \quad \mathfrak{L}_{010}^{(\beta)} = (x_1 + x_2) + \beta x_1 x_2$$

$$\mathfrak{L}_{110}^{(\beta)} = x_1 x_2\,, \qquad \mathfrak{L}_{200}^{(\beta)} = x_1^2\,, \quad \mathfrak{L}_{210}^{(\beta)} = x_1^2 x_2$$

Hilbert series for $\widehat{V}_n$

Theorem (P.-Yu 23)

$\{\widehat{\mathfrak{L}}_\alpha \mid \alpha \in C_n$ and α is snowy$\}$ form a basis for $\widehat{V}_n$.

Definition

Suppose A is a polynomial vector space with a basis $\mathfrak{B}$ consisting of homogeneous polynomials. Then

$$\mathrm{Hilb}(A;q) := \sum_{f \in \mathfrak{B}} q^{\deg(f)}.$$

Corollary

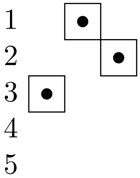
$$\mathrm{Hilb}(\widehat{V}_n;q) = \sum_{\alpha \in C_n,\ \textit{snowy}} q^{\mathsf{raj}(\alpha)}$$

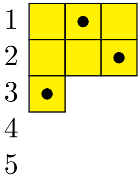
Non-attacking Rook diagrams

Definition

A non-attacking rook diagram is a diagram with at most one tile in each column or row. Let Rook_n be the set of non-attacking rook diagrams within the staircase shape $(n-1, n-2, \ldots, 1)$. There is a statistic on it: $\mathsf{NW}(\cdot)$ on them.

Ex. Let $R \in \mathsf{Rook}_5$.
Then $\mathsf{NW}(R) = 7$.





Theorem (Garsia-Remmel 86)

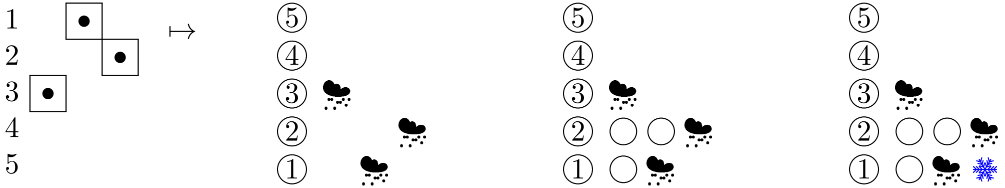
$$\sum_{R \in \mathsf{Rook}_n} q^{\mathsf{NW}(R)} = \sum_{R \in \mathsf{Rook}_n} q^{\binom{n}{2} - \mathsf{GR}(R)} = \mathsf{rev}(B_q(n)),$$

where rev *reverse the coefficients for the polynomial.*

Hilbert Series for $\widehat{V}_n$

Lemma (P.-Yu 23). There is a statistic-preserving bijection between Rook_n and snowy weak compositions in $\mathcal{C}_n$ which sends NW to raj.

Ex. The following rook diagram with $NW(R) = 7$ is sent to $\alpha = (2, 3, 1, 0, 0)$ with $\text{rajcode}(\alpha) = (3, 3, 1, 0, 0)$ and $\text{raj}(\alpha) = 7$.



$$\text{Hilb}(\widehat{V}_n; q) = \sum_{\alpha \in C_n,\ \text{snowy}} q^{\text{raj}(\alpha)} = \sum_{R \in \text{Rook}_n} q^{\text{NW}(R)} = \text{rev}(B_q(n))$$

Hilbert Series for $\widehat{V}$

Theorem (P.-Yu 23)

- $\widehat{V} = \cup_{n\geqslant 1}\widehat{V}_n$
- $\widehat{V}$ *has a basis* $\{\widehat{\mathfrak{L}}_\alpha \mid \alpha$ *is snowy*$\}$.
- $$\mathrm{Hilb}(\widehat{V};q) = \lim_{n\to\infty}\mathrm{Hilb}(\widehat{V}_n;q) = \prod_{m>0}(1+\frac{q^m}{1-q})$$

Thank you!

(I'm on the job market.)