Laurent Polynomials from the Super Ptolemy Relation

Nick Ovenhouse
(joint with Gregg Musiker and Sylvester Zhang)

University of Minnesota

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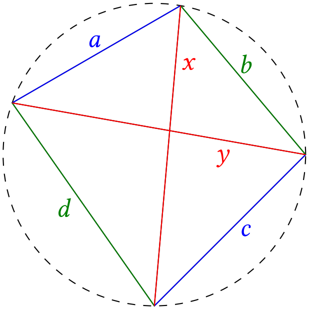
Ptolemy's Theorem

Take a quadrilateral inscribed in a circle, with lengths labelled as in the picture.

Then $xy = ac + bd$.



Triangulated Polygons

For a polygon (inscribed in a circle), let

$$x_{ij} = \text{ length of diagonal } (i, j)$$

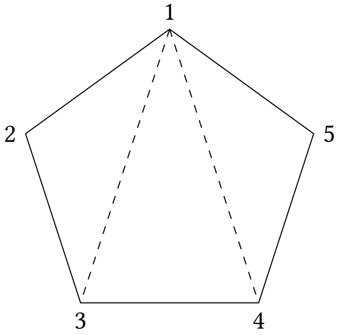
Fix a triangulation.
By repeatedly applying Ptolemy relations, we can express any x_{ij} in terms of the variables from the triangulation.

Example:

$$x_{25} = \frac{x_{15}x_{23} + x_{12}x_{35}}{x_{13}}$$

$$= \frac{x_{15}x_{23} + x_{12}\left(\frac{x_{15}x_{34}+x_{13}x_{45}}{x_{14}}\right)}{x_{13}}$$

$$= \frac{x_{15}x_{23}}{x_{13}} + \frac{x_{12}x_{15}x_{34}}{x_{13}x_{14}} + \frac{x_{12}x_{45}}{x_{14}}$$



T-Paths

Consider the graph Γ coming from a triangulated polygon.

A "*T-path*"[1] from i to j is a path in Γ starting at vertex i, ending at j, such that

(T1) the path does not use any edge twice

(T2) the path has an odd number of edges

(T3) the even-numbered edges cross the digaonal (i, j)

(T4) the intersections of the path with the diagonal (i, j) get closer to j and farther from i along the path

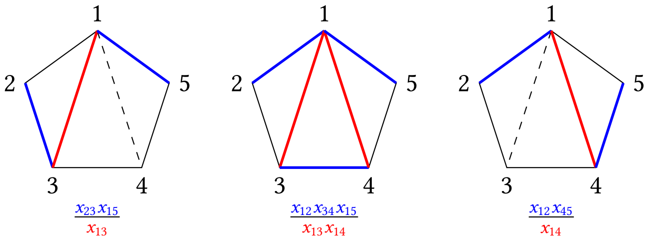
Let T_{ij} denote the set of T-paths from i to j.

[1] Schiffler. "A Cluster Expansion Formula (A_n case)". In: *Electronic Journal of Combinatorics* (2008)

Example

Here are all the T-paths in T_{25}

(odd steps are blue and even steps are red)



For a T-path α, using edges $(i_1, i_2), (i_2, i_3), \ldots$, define the Laurent monomial

$$x_\alpha := \prod_k x^{\varepsilon}_{i_k i_{k+1}} \qquad (\varepsilon = (-1)^{k+1})$$

The Laurent Formula

Theorem [Schiffler][1]

$$x_{ij} = \sum_{\alpha \in T_{ij}} x_\alpha$$

Corollary

Each x_{ij} is a Laurent polynomial in the lengths of the diagonals from any fixed triangulation.

The "*Laurent phenomenon*" also follows from the cluster algebra structure[2].

[1] Schiffler. "A Cluster Expansion Formula (A_n case)". In: *Electronic Journal of Combinatorics* (2008)
[2] Fomin and Zelevinsky. "Cluster Algebras I: Foundations". In: *Journal of the AMS* (2002)

Super Algebras

A "*super algebra*" is a $\mathbb{Z}_2$-graded algebra.

i.e. $A = A_0 \oplus A_1$, (the "*even*" and "*odd*" parts) and

$$A_i A_j \subseteq A_{i+j}$$

A basic example is the algebra generated by $x_1, \dots, x_n, \theta_1, \dots, \theta_m$, subject to the relations

$$x_i x_j = x_j x_i, \quad x_i \theta_j = \theta_j x_i, \quad \theta_i \theta_j = -\theta_j \theta_i$$

in particular, $\theta_i^2 = 0$

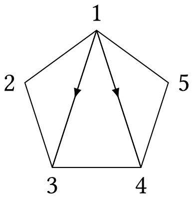
The x's are the "*even generators*" and the θ's are the "*odd generators*".

In this example, A_0 is spanned by monomials with either no θ's, or an even number of θ's, and A_1 is spanned by monomials containing an odd number of θ's.

Super Algebra from a Triangulation

Given an n-gon, choose:

- a triangulation T
- an orientation of each edge in T (We usually do not draw the boundary orientations)

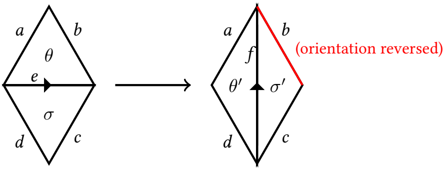


Consider the super algebra with one even generator x_{ij} for each diagonal in T, and one odd generator θ_{ijk} for each triangle in T.

The example pictured above would have 7 even generators x_{ij}, and 3 odd generators θ_{ijk}.

The Super Ptolemy Relation

Given a quadrilateral, which is part of some triangulated polygon, we get a new triangulation by "*flipping*" the diagonal:



We define the new variables via the relations[1]:

$$ef = ac + bd + \sqrt{abcd}\,\sigma\theta$$

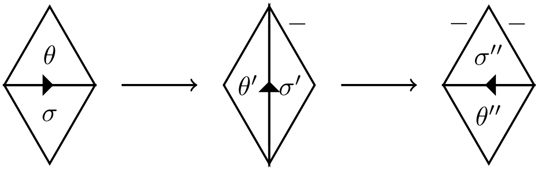
$$\theta' = \frac{\sqrt{bd}\,\theta + \sqrt{ac}\,\sigma}{\sqrt{ac+bd}}$$

$$\sigma' = \frac{\sqrt{bd}\,\sigma - \sqrt{ac}\,\theta}{\sqrt{ac+bd}}$$

[1] Penner and Zeitlin. "Decorated Super-Teichmüller Space". In: *Journal of Differential Geometry* (2019)

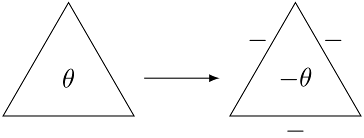
The Odd Variables

Unlike the ordinary Ptolemy relation, this one is *not* an involution.



Using the super Ptolemy relation twice, one gets that $\theta'' = \sigma$ and $\sigma'' = -\theta$.

Thus reversing the orientations around a triangle corresponds to negating the odd variable.



The Main Question

Starting with a fixed triangulation, we can reach any diagonal by a sequence of flips. Using the super Ptolemy relation, we will get some algebraic expression attached to this diagonal.

Question: Can we explicitly describe this algebraic expression?

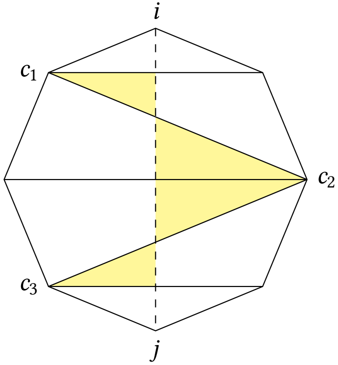
Question: Does it have a nice combinatorial description (analogous to T-paths)?

Fans

Fix a triangulation T of a polygon. We only consider triangulations that have a "*longest edge*" (an edge which crosses all internal diagonals of T). Call the endpoints of the longest edge i and j.

The longest edge splits the triangles in T into triangular and quadrilateral regions:

The vertices incident to the triangular regions will be called "*fan centers*", and we will label them $c_1, c_2, \ldots,$



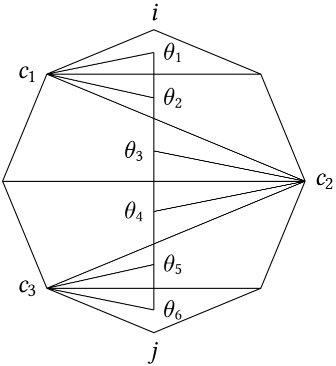
The Auxiliary Graph

Given a triangulated polygon, we construct the "*auxiliary graph*" Γ:

The vertices and edges of the triangulation are in Γ.

There is a vertex within each triangle (labelled by odd variables θ_i), connected by an edge to the adjacent fan center. These edges are labelled σ_i.

For each pair of triangles, there is an edge (labelled τ_{ij}) connecting θ_i and θ_j.



A "*super T-path*" from i to j is a path in Γ which satisfies:

(T1) the path does not use any edge twice

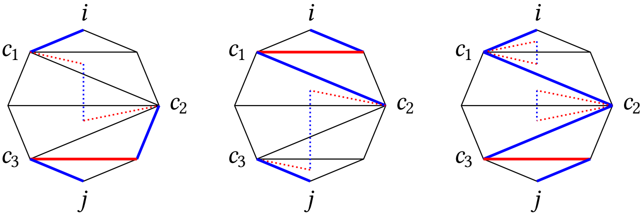
(T2) the path has an odd number of edges

(T3) the even-numbered edges cross (i, j)

(T4) σ-edges can only be even steps (σ-edges are considered to be crossing (i, j)), and τ-edges can only be odd steps

(T5) The points where the path crosses (i, j) progressively move from i to j

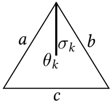
Examples:



Weights

If a super T-path uses edges $t_1, t_2, \ldots$, we define weights, with values in our super algebra:

- If $t = (k, \ell)$ is a diagonal in the triangulation, then:
  $\mathrm{wt}(t) = x_{k\ell}$ if t is an odd step, and
  $\mathrm{wt}(t) = x_{k\ell}^{-1}$ if t is an even step
- If $t = \tau_{a,b}$, then $\mathrm{wt}(t) = 1$
- If $t = \sigma_k$, then $\mathrm{wt}(t) = \mu_k := \sqrt{\frac{c}{ab}}\,\theta_k$



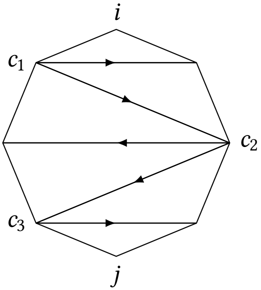
If α is a super T-path with edges $t_1, t_2, \ldots$, define $\mathrm{wt}(\alpha) = \prod_i \mathrm{wt}(t_i)$.

Default Orientation

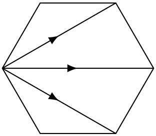
First, we define a “*default orientation*” on the diagonals of the triangulation:

Edges connecting fan centers are oriented $c_i \to c_{i+1}$.

Within each fan segment, orient edges *away* from the fan center.



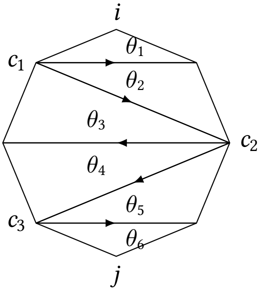
Another Example:



Positive Ordering

Label the triangles $\theta_1, \theta_2, \ldots$ from i to j.

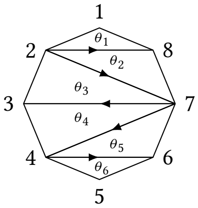
Look at the oriented edge separating θ_k and θ_{k+1}.
If θ_k is on the right, then define $\theta_k > \theta_i$ for all $i > k$.
If θ_k is on the left, define $\theta_k < \theta_i$ for all $i > k$.



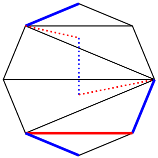
$$\theta_3 > \theta_4 > \theta_6 > \theta_5 > \theta_2 > \theta_1$$

When we write $\text{wt}(\alpha) = \prod_i \text{wt}(t_i)$, this product is taken with respect to this ordering.

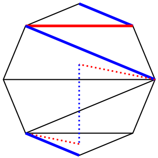
Examples



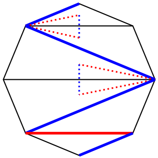
$$\theta_3 > \theta_4 > \theta_6 > \theta_5 > \theta_2 > \theta_1$$



$\frac{x_{12}x_{67}x_{45}}{x_{46}}\ \mu_4\mu_2$



$\frac{x_{18}x_{27}x_{45}}{x_{28}}\ \mu_3\mu_6$



$\frac{x_{12}x_{27}x_{47}x_{56}}{x_{46}}\ \mu_3\mu_4\mu_2\mu_1$

Laurent Formula

Theorem[Musiker, O., Zhang][1]

Given a fixed triangulation (with the default orientation),

$$x_{ij} = \sum_{\alpha \in \mathcal{T}_{ij}} \mathrm{wt}(\alpha)$$

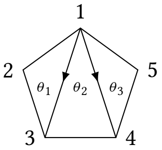
Corollary

Each term of x_{ij}:

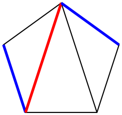
- is a Luarent monomial in the x's times a monomial in the μ's.
- has a positive coefficient when the μ's are written in the positive order

[1] Musiker, Ovenhouse, and Zhang. "An Expansion Formula for Decorated Super-Teichmüller Spaces". In: *to appear on ArXiv soon!* (2021)

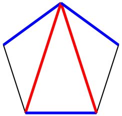
A Complete Example



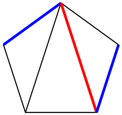
$\theta_1 > \theta_2 > \theta_3$



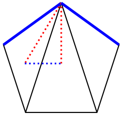
$\frac{x_{23}x_{15}}{x_{13}}$



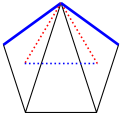
$\frac{x_{12}x_{34}x_{15}}{x_{13}x_{14}}$



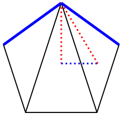
$\frac{x_{12}x_{45}}{x_{14}}$



$x_{12}x_{15}\ \mu_1\mu_2$



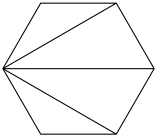
$x_{12}x_{15}\ \mu_1\mu_3$



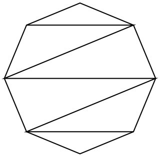
$x_{12}x_{15}\ \mu_2\mu_3$

Sketch of Proof

Step 1: Prove for "*fan triangulations*".



Step 2: Prove for "*zig-zag triangulations:*



Sketch of Proof

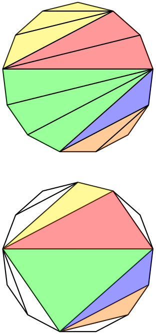
Step 3: Perform the following flip sequence.

Flip the diagonals within each fan segment (but *not* the edges $c_i \to c_{i+1}$).

We get a zig-zag triangulation containing the longest edge.

By **Part 2**, we can express the longest edge in terms of this triangulation.

By **Part 1**, we can express everything in this triangulation in terms of the original triangulation.



Thank You!