Kneser graphs are Hamiltonian

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joint with Arturo Merino (TU Berlin) and Namrata (Warwick)

MSU Combinatorics and
Graph Theory Seminar

extended abstract in [STOC 2023]

Introduction

- **Kneser graph** $K(n,k)$

  vertices $= \binom{[n]}{k}$

  edges $=$ pairs of disjoint sets

  $$A \cap B = \emptyset$$

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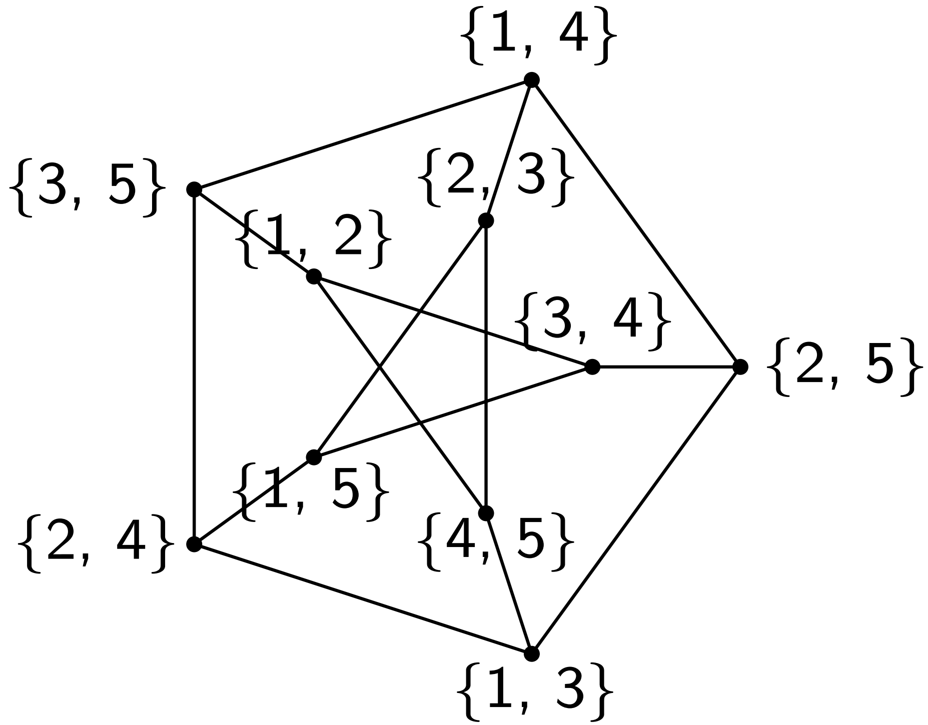

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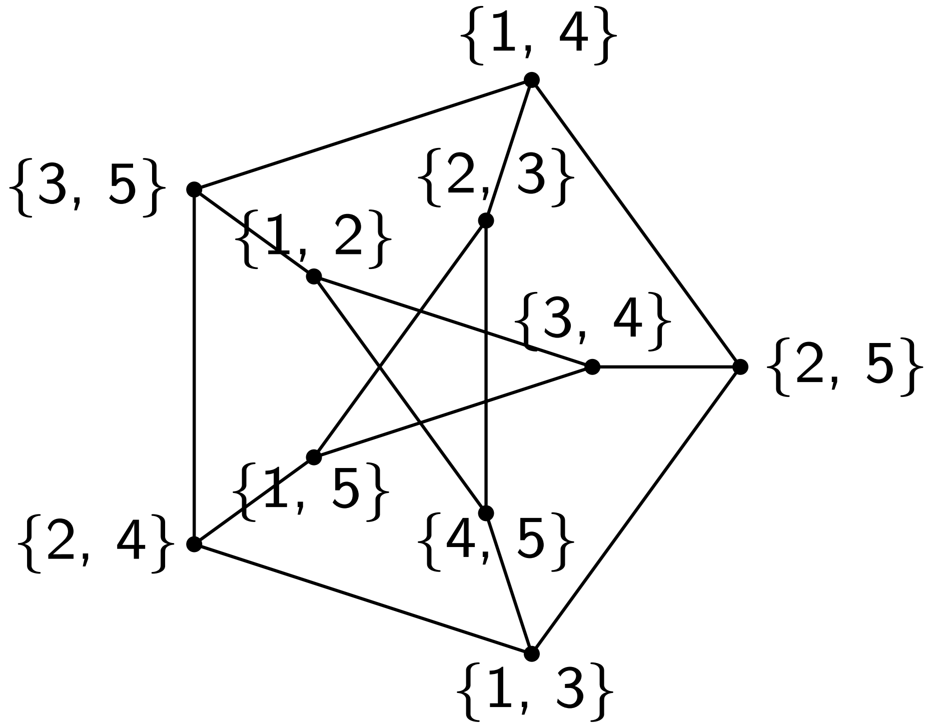

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- [Erdős, Ko, Rado 1961]:

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- Kneser graphs: should be easier for dense cases

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- [Y. Chen 2000]: $n \geq (1 + o(1))2.62 \cdot k$

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JOURNAL OF COMBINATORIAL THEORY (B) 20, 62–63 (1976)

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- sparsest open case: $n = 2k+3$

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- settles Hamiltonicity of $K(n,k)$ in full generality

Generalized Johnson graphs

- **generalized Johnson graphs** $J(n, k, s)$

  vertices $= \binom{[n]}{k}$

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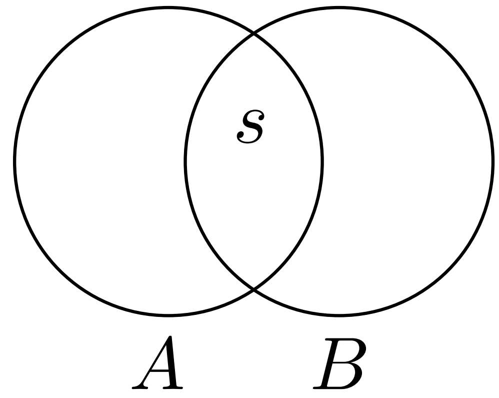
  $$|A \cap B| = s$$



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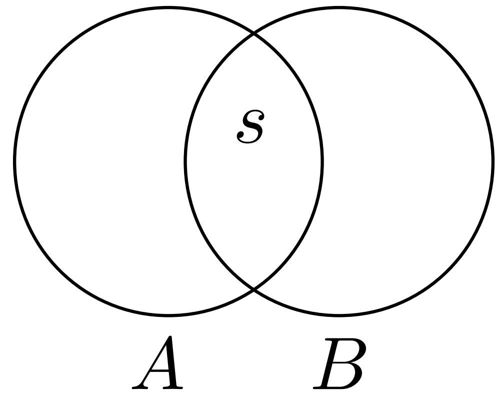

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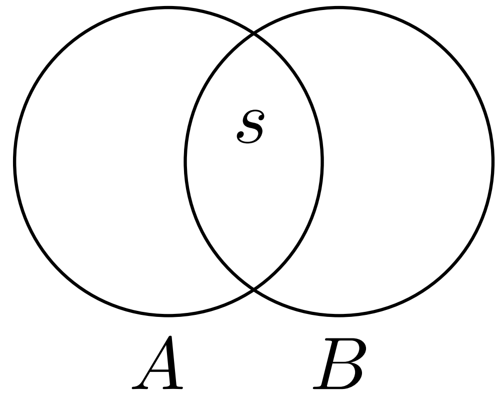

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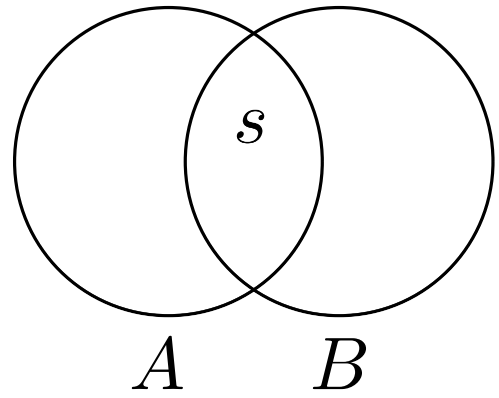

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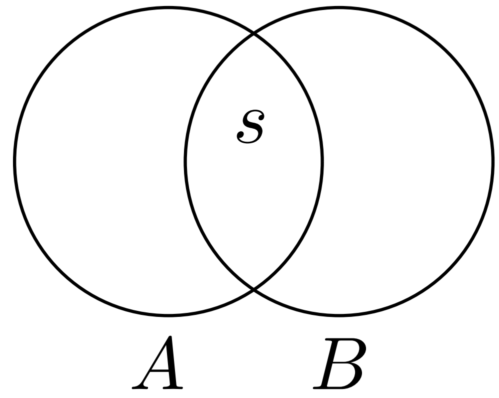

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- vertex-transitive

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- **Conjecture** [Chen, Lih 1987], [Gould 1991]:
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- results of [Tang, Liu 1973] settle the case $s = k - 1$

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- [Jiang, Ruskey 1994], [Knor 1994] proved that
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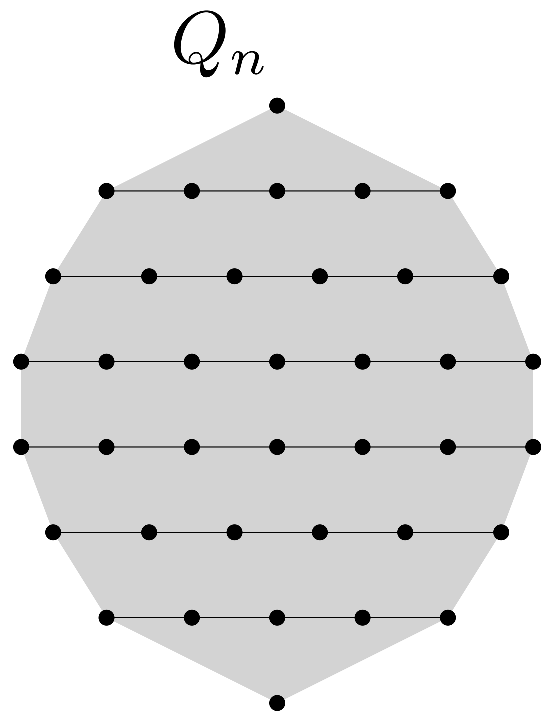

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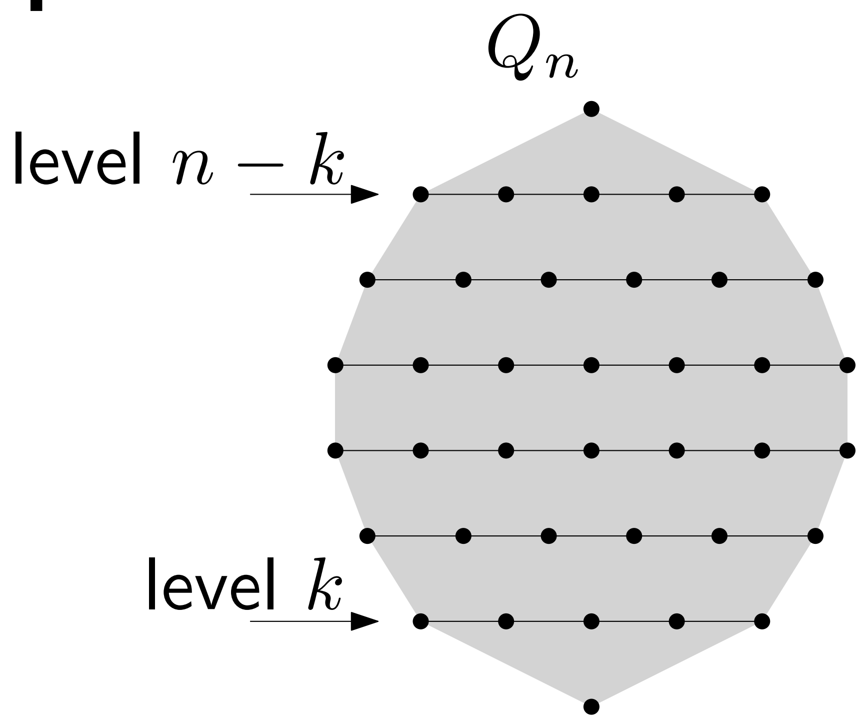

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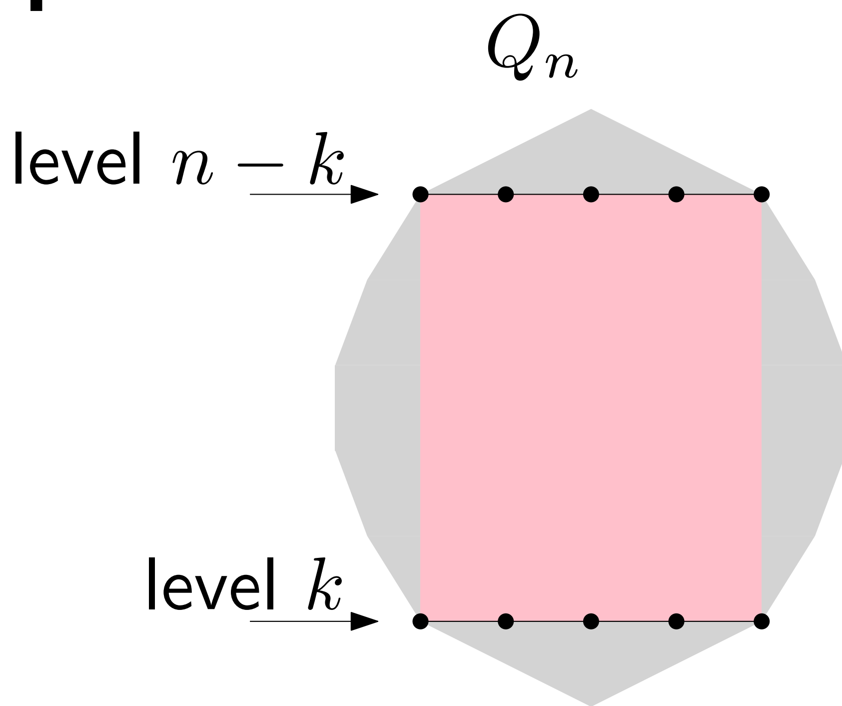

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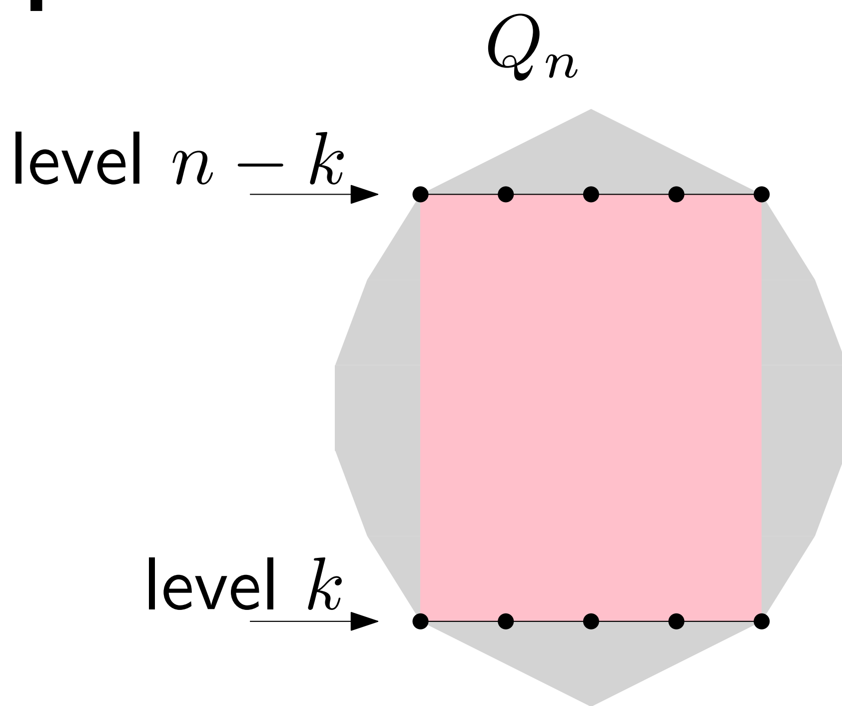

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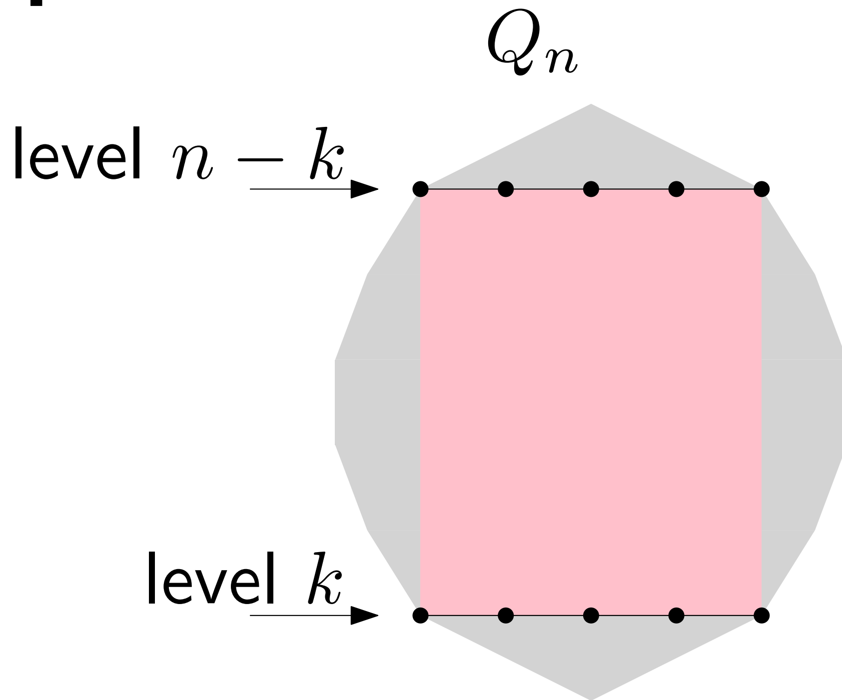

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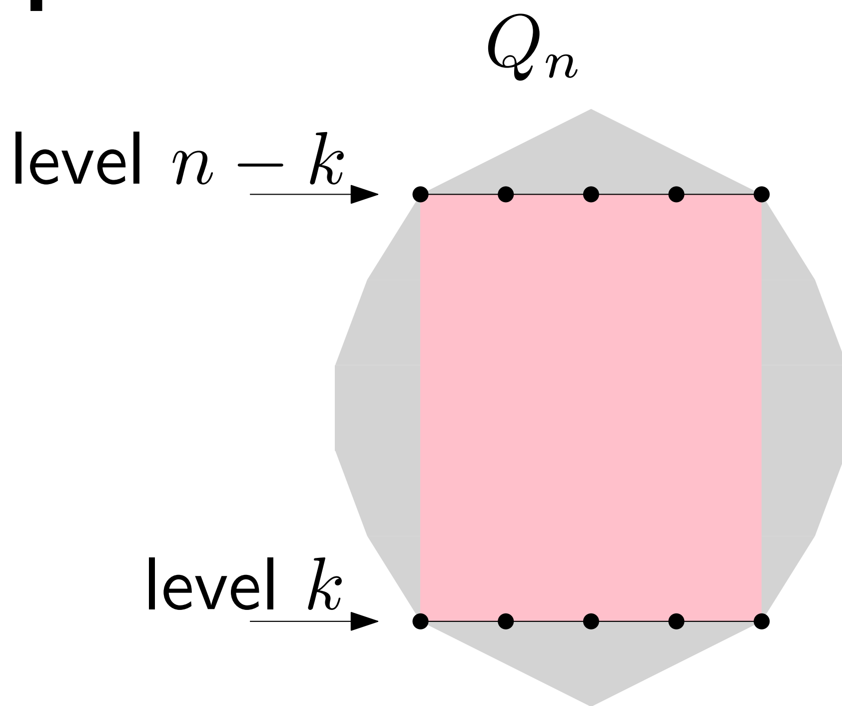

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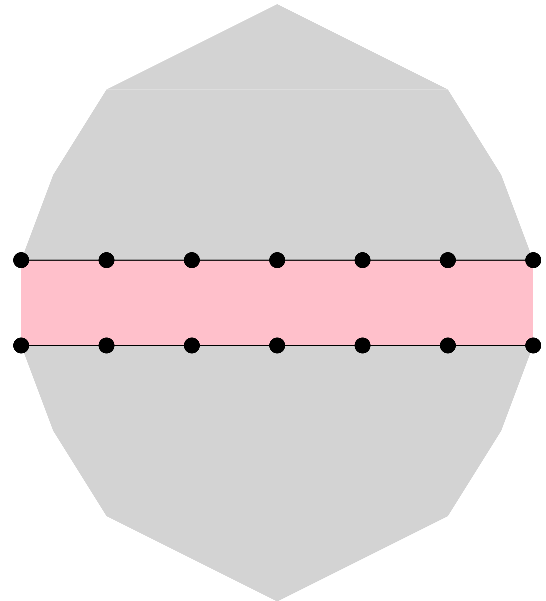

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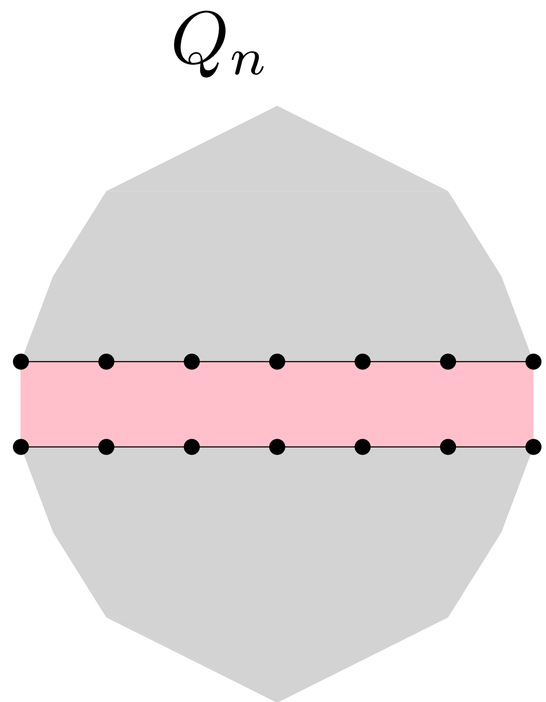

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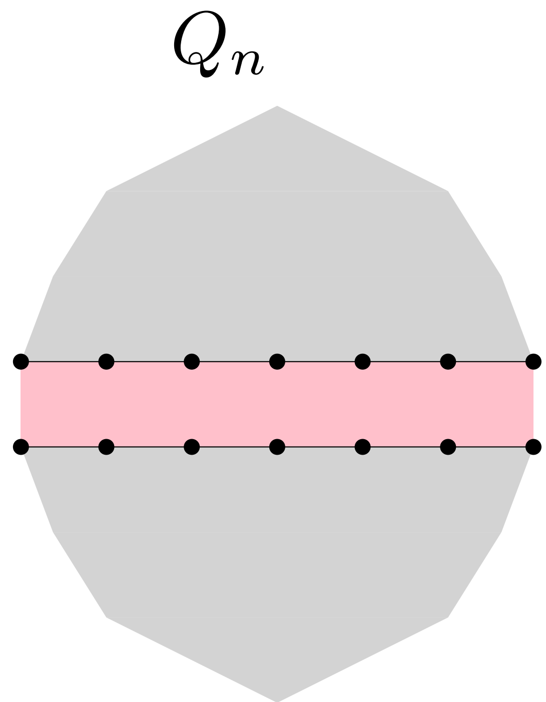

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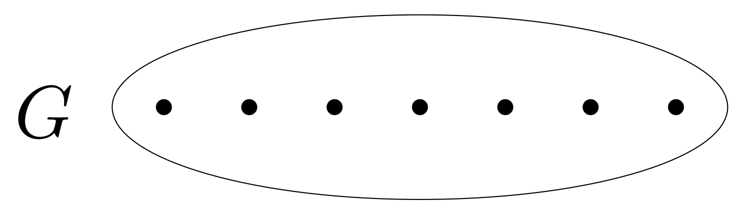

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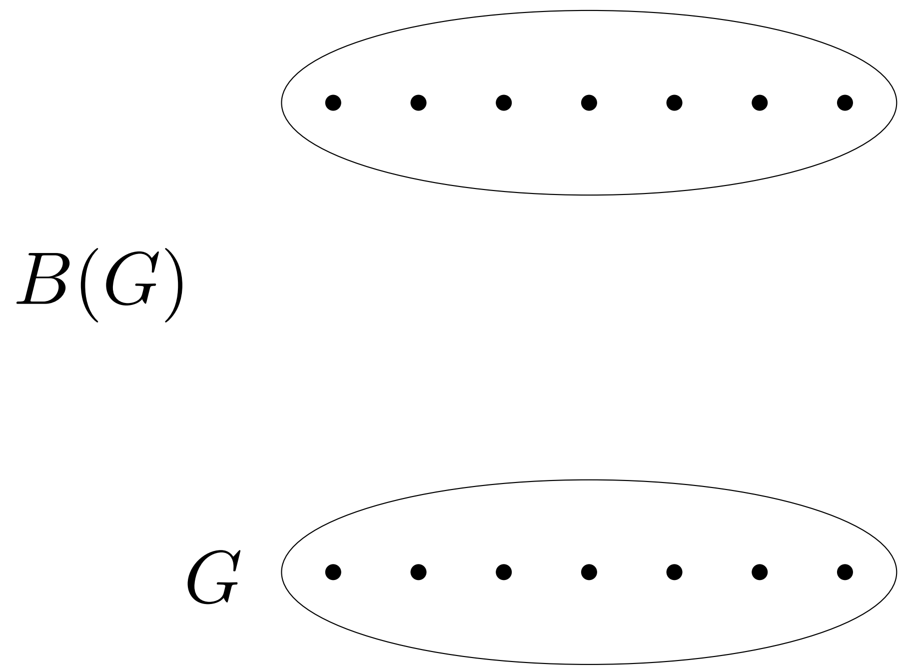

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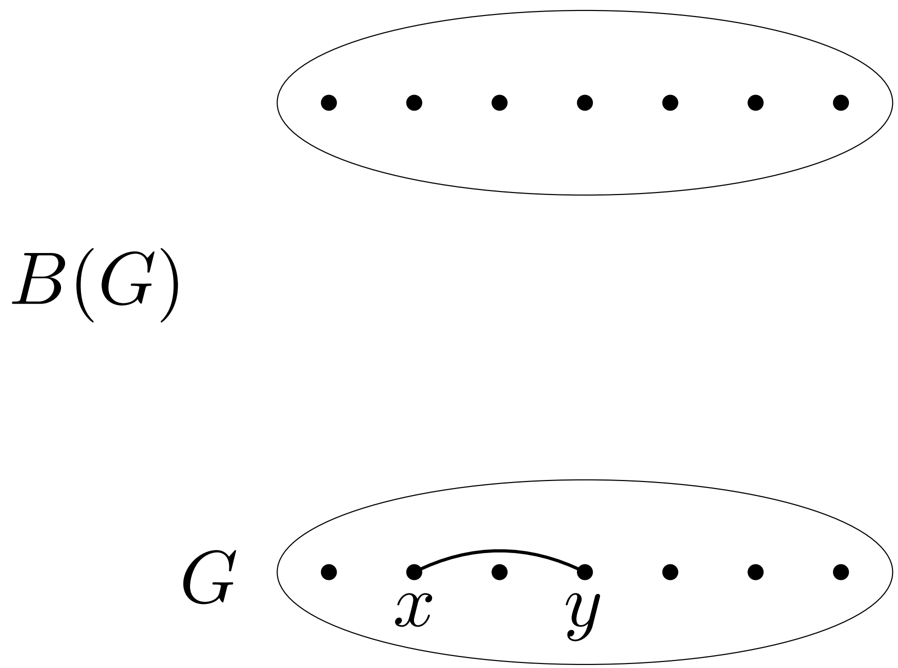

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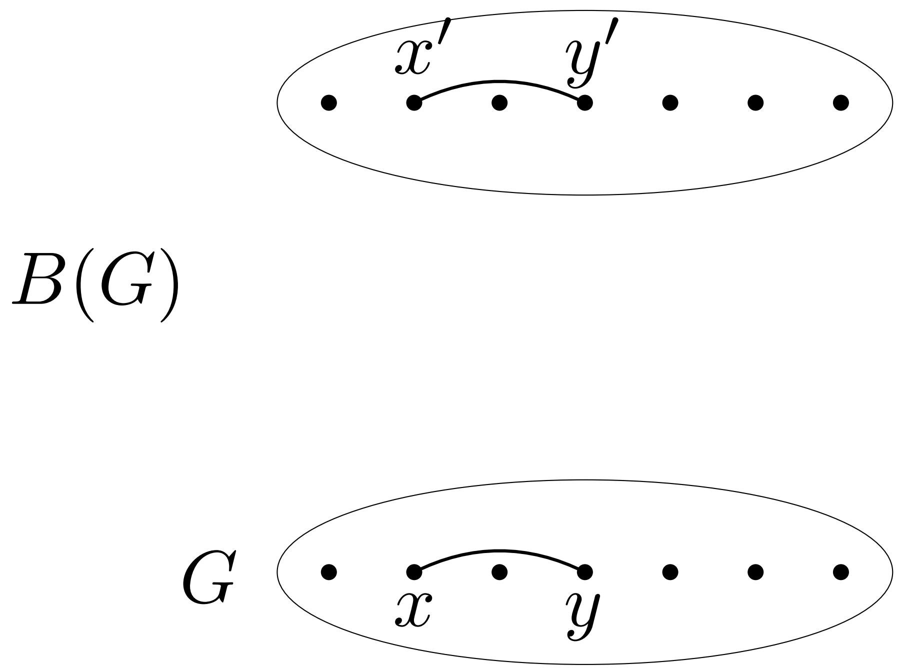

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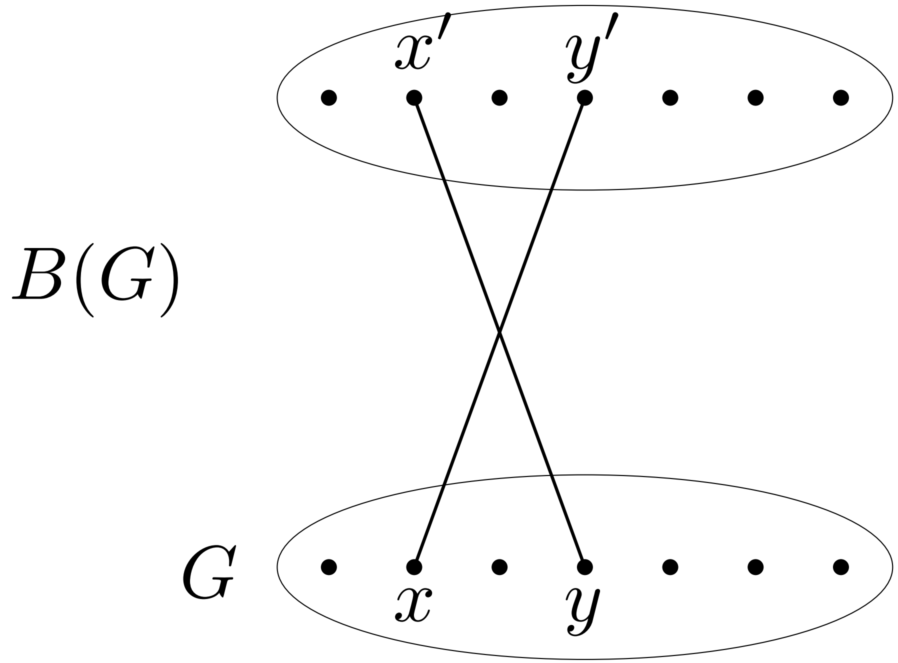

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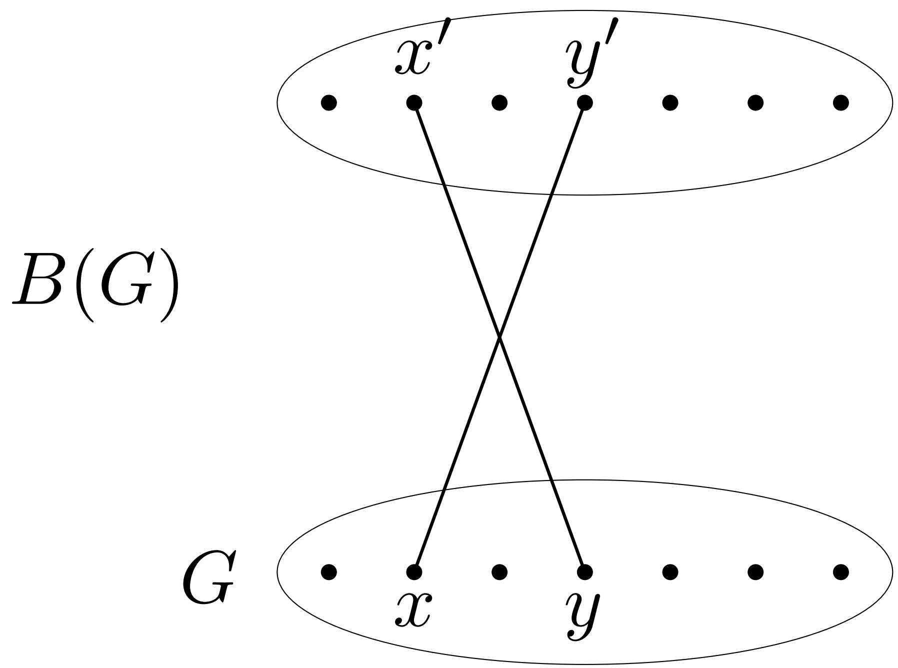

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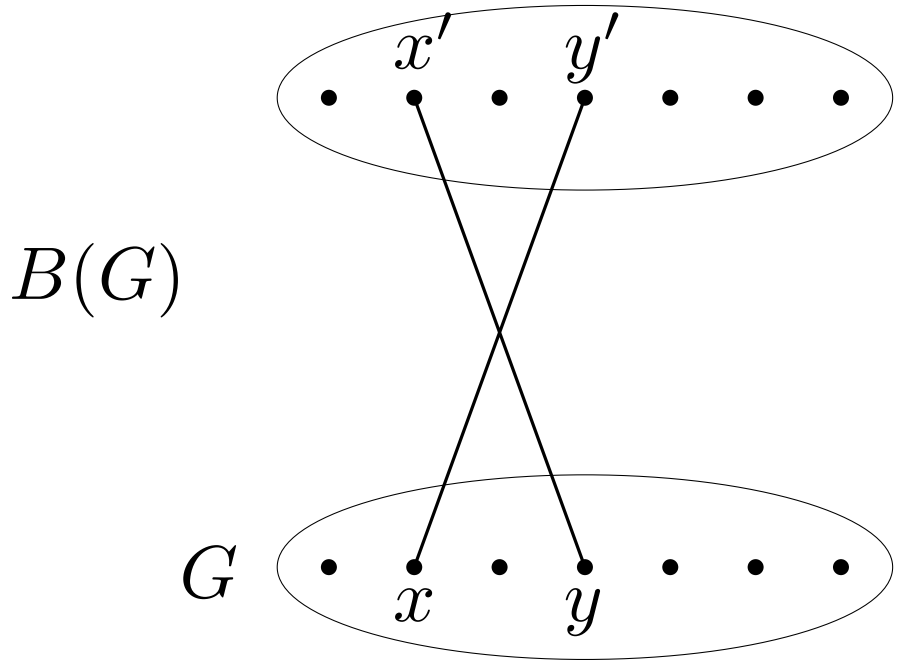

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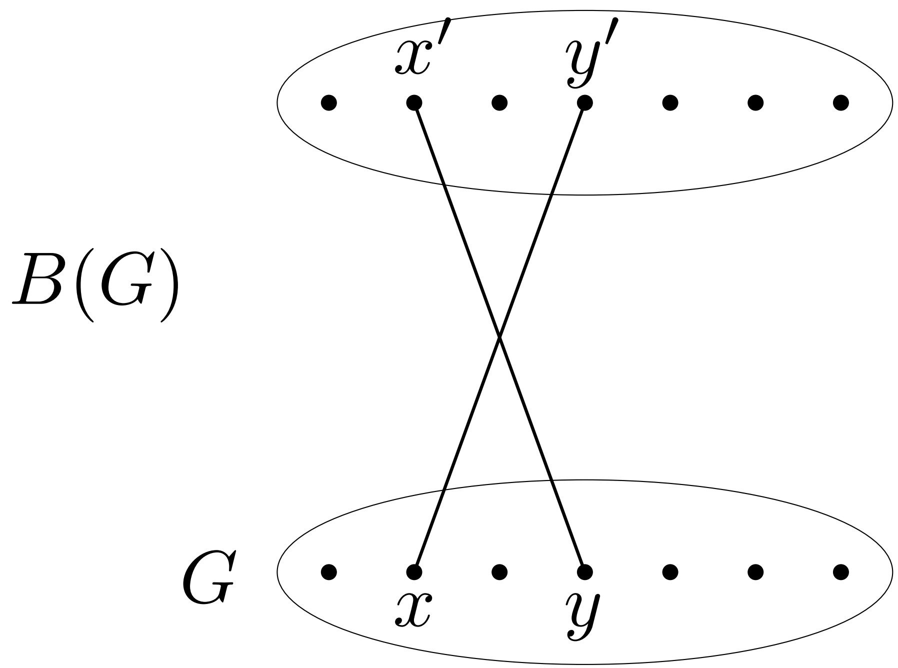
- we thus obtain a new proof for Hamiltonicity of $H(n,k)$



Summary of old and new results

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$n = 2k+1$ ↓

odd graphs
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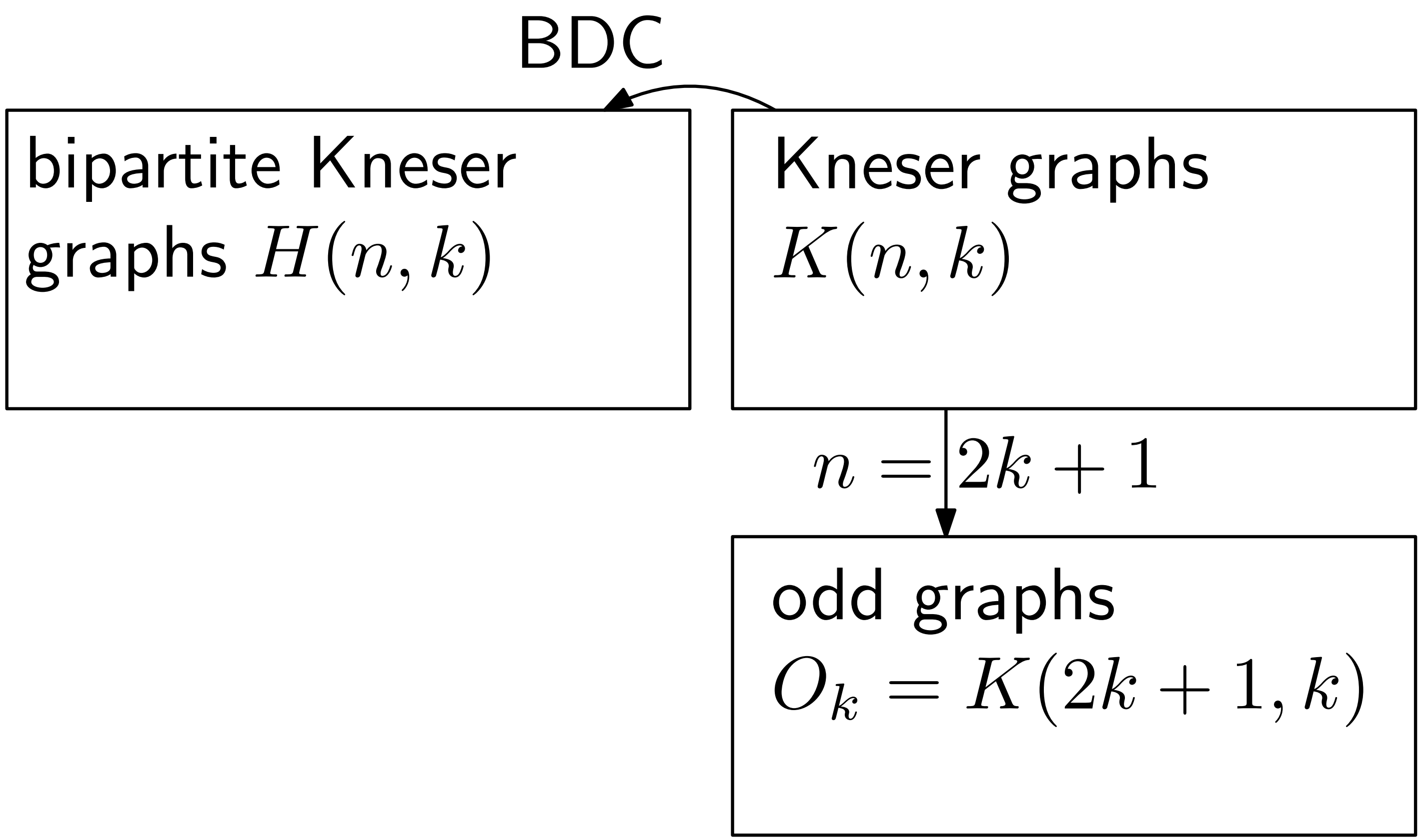

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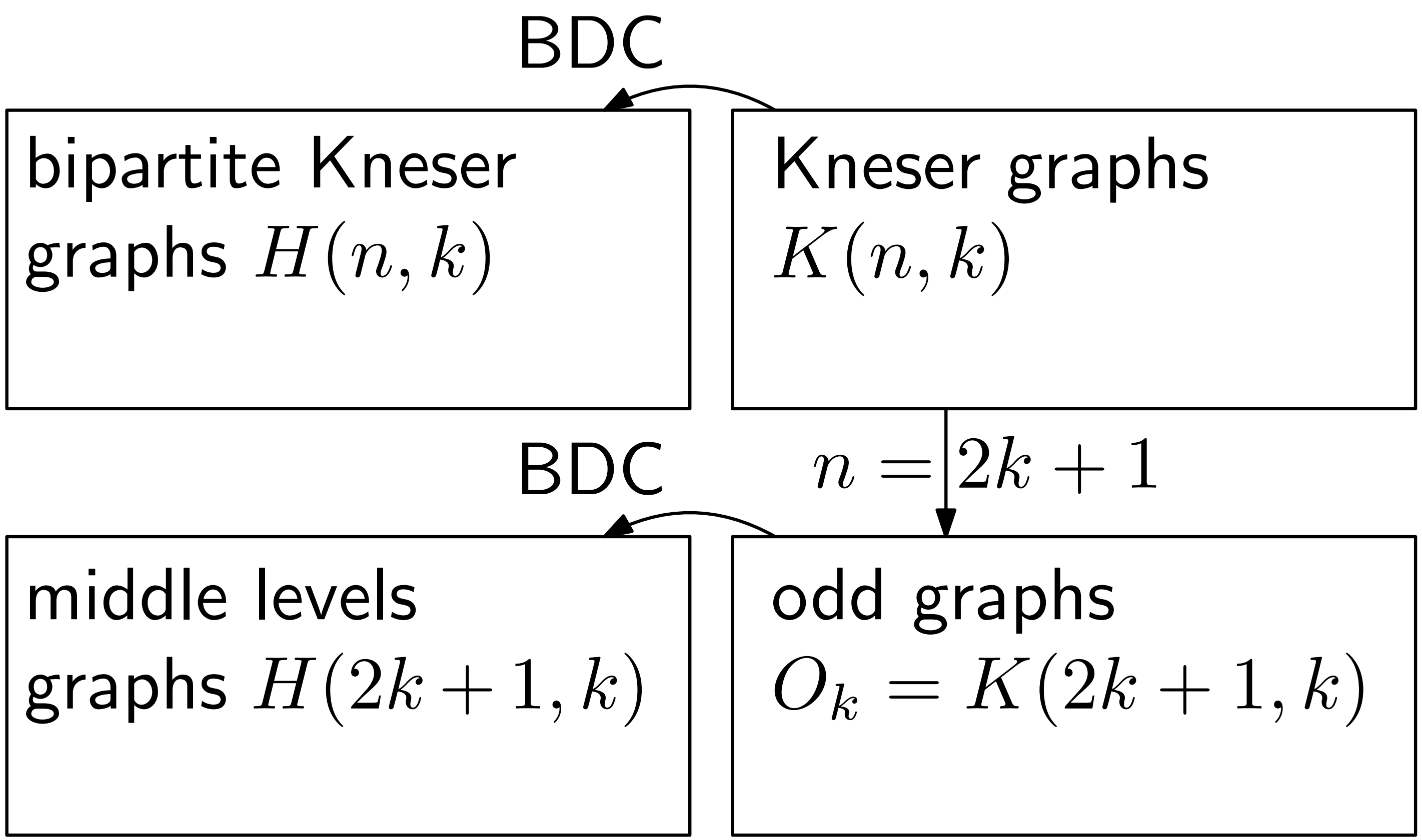

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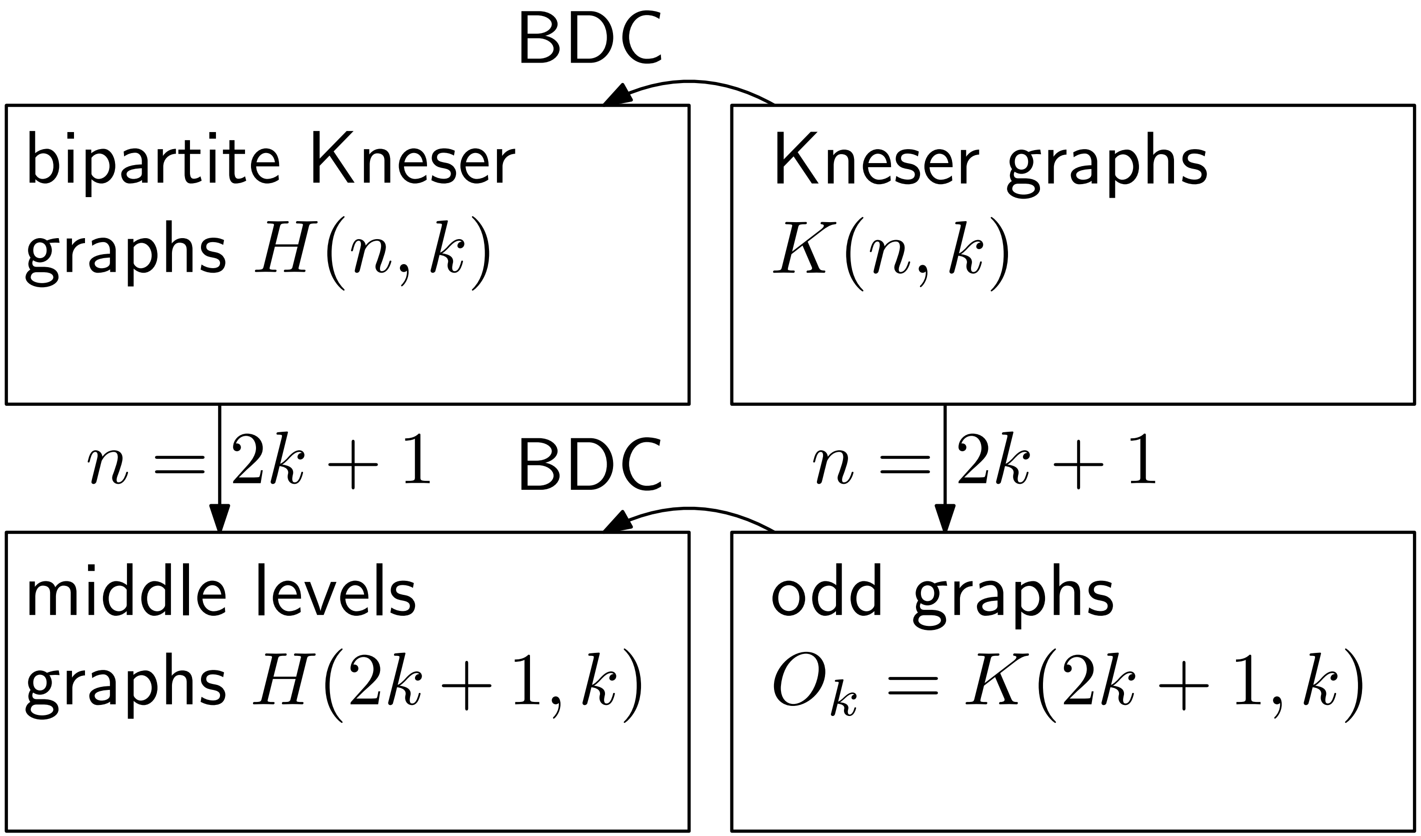

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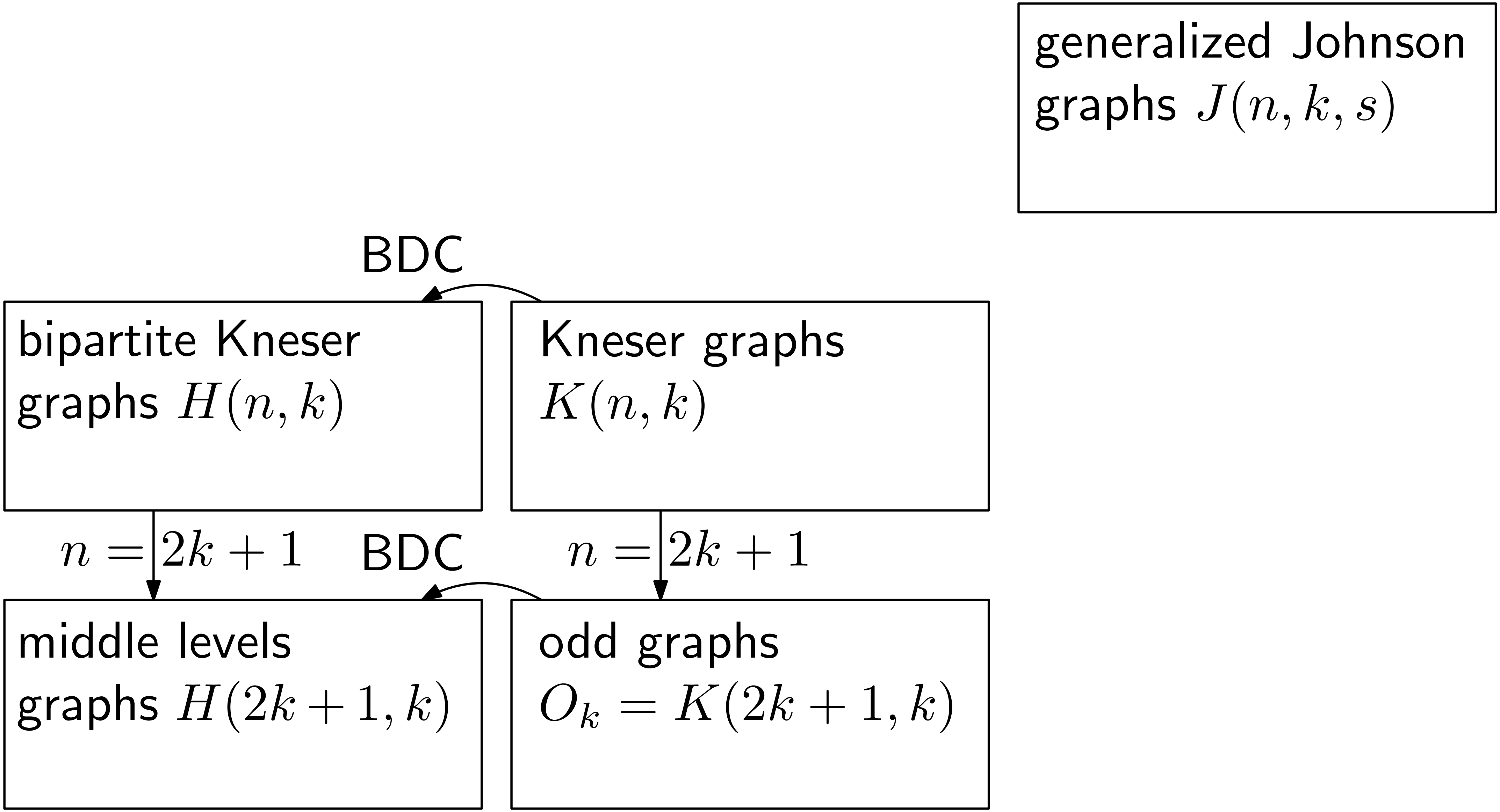

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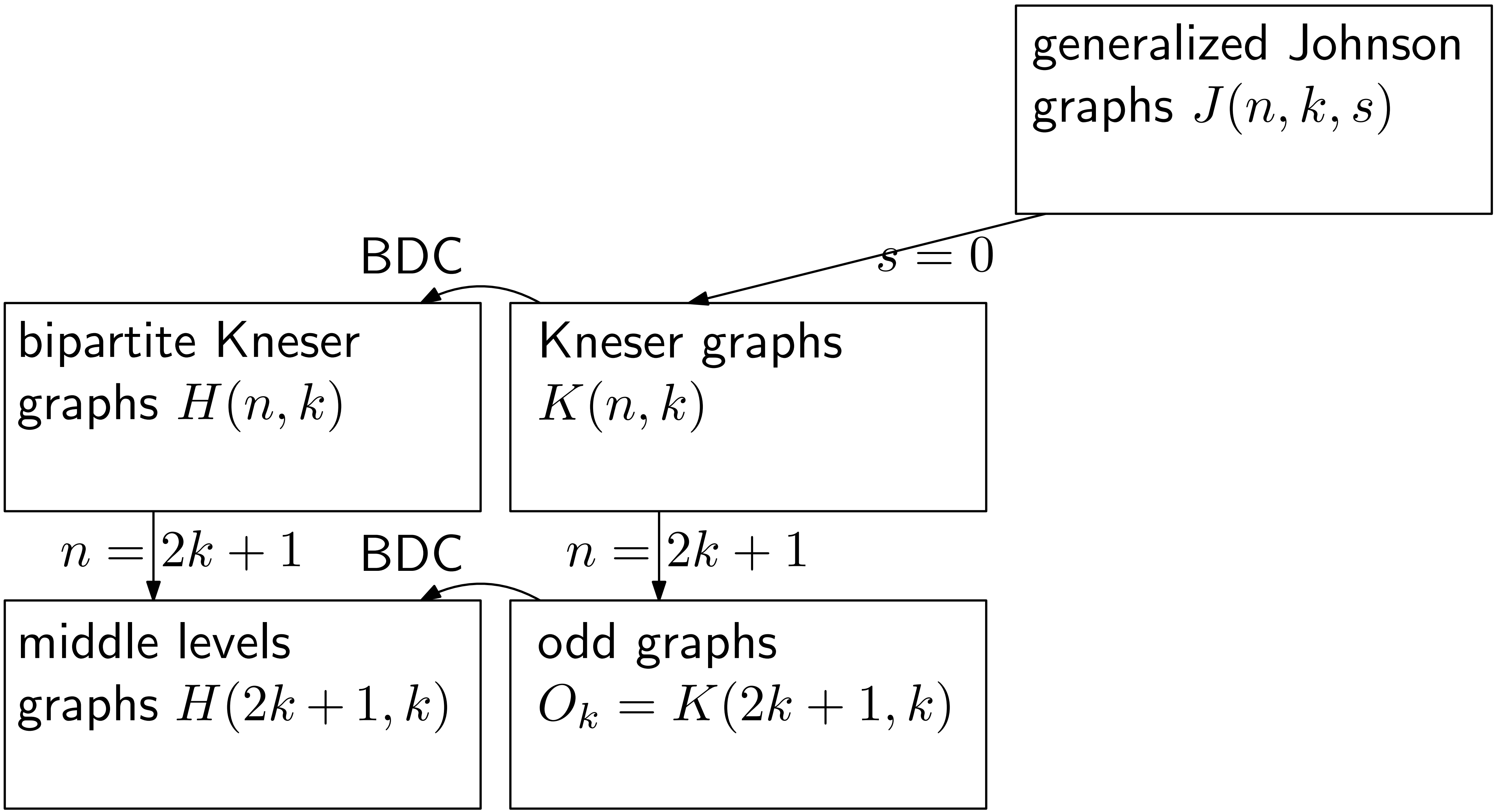

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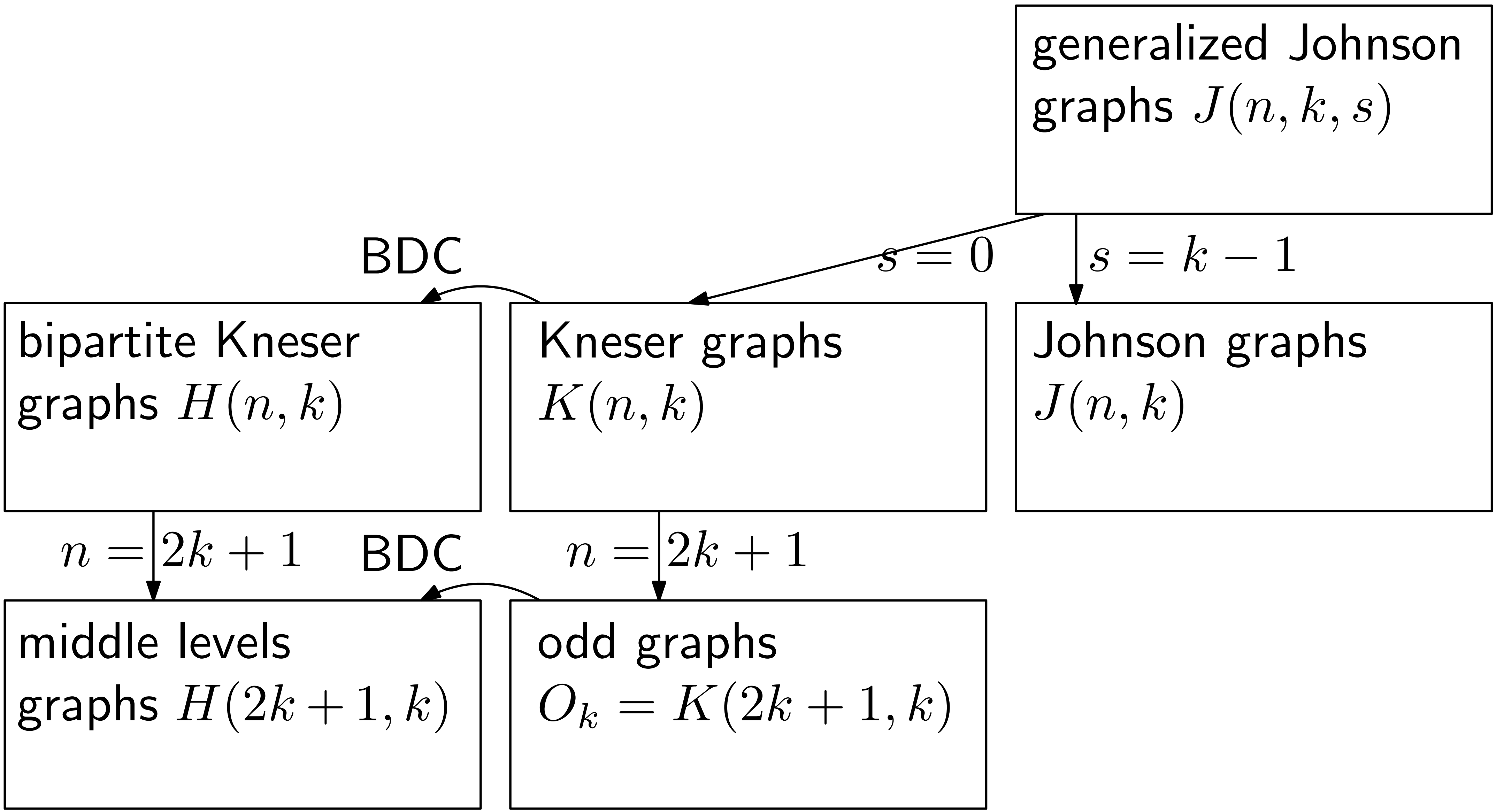

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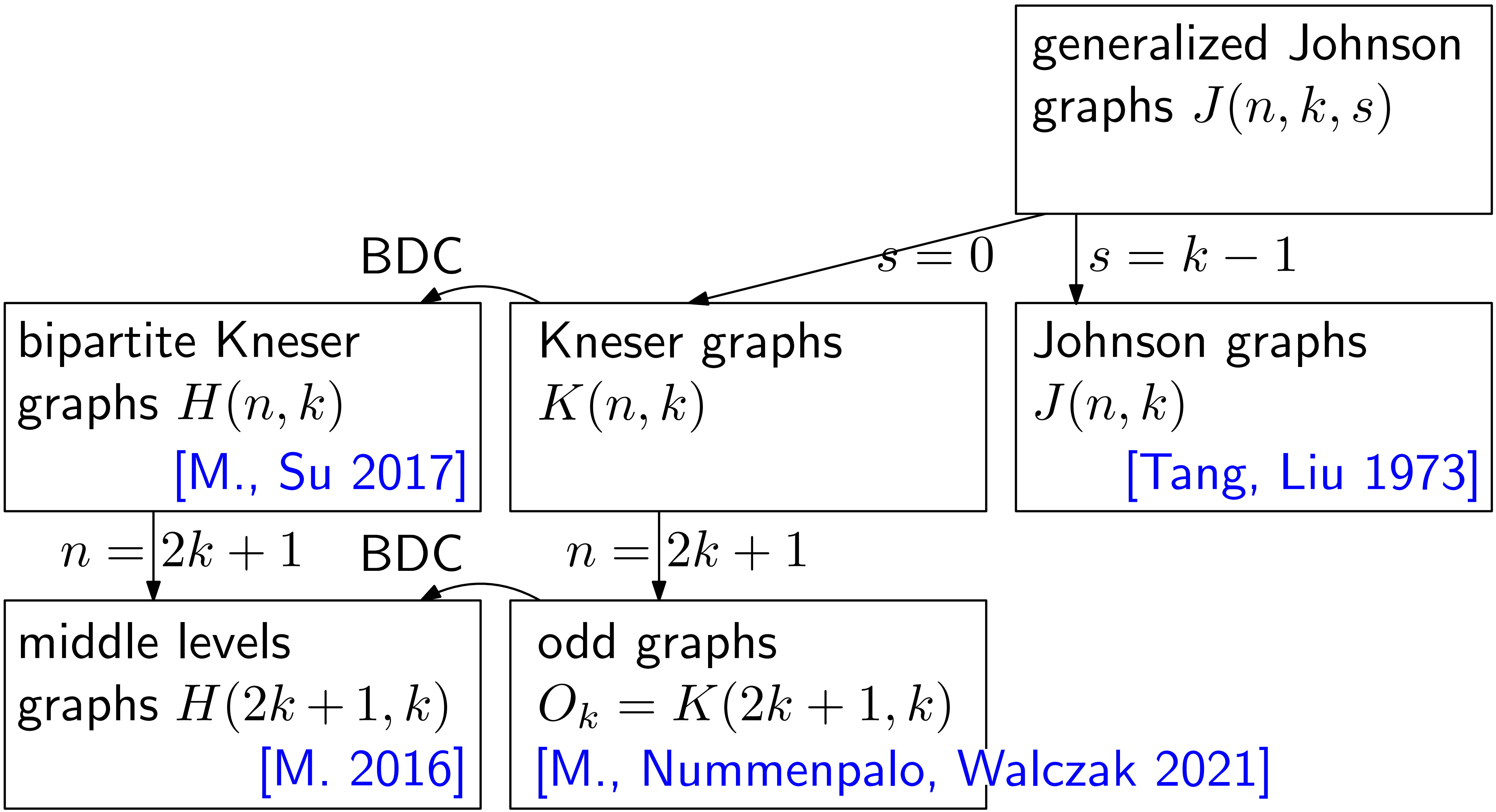

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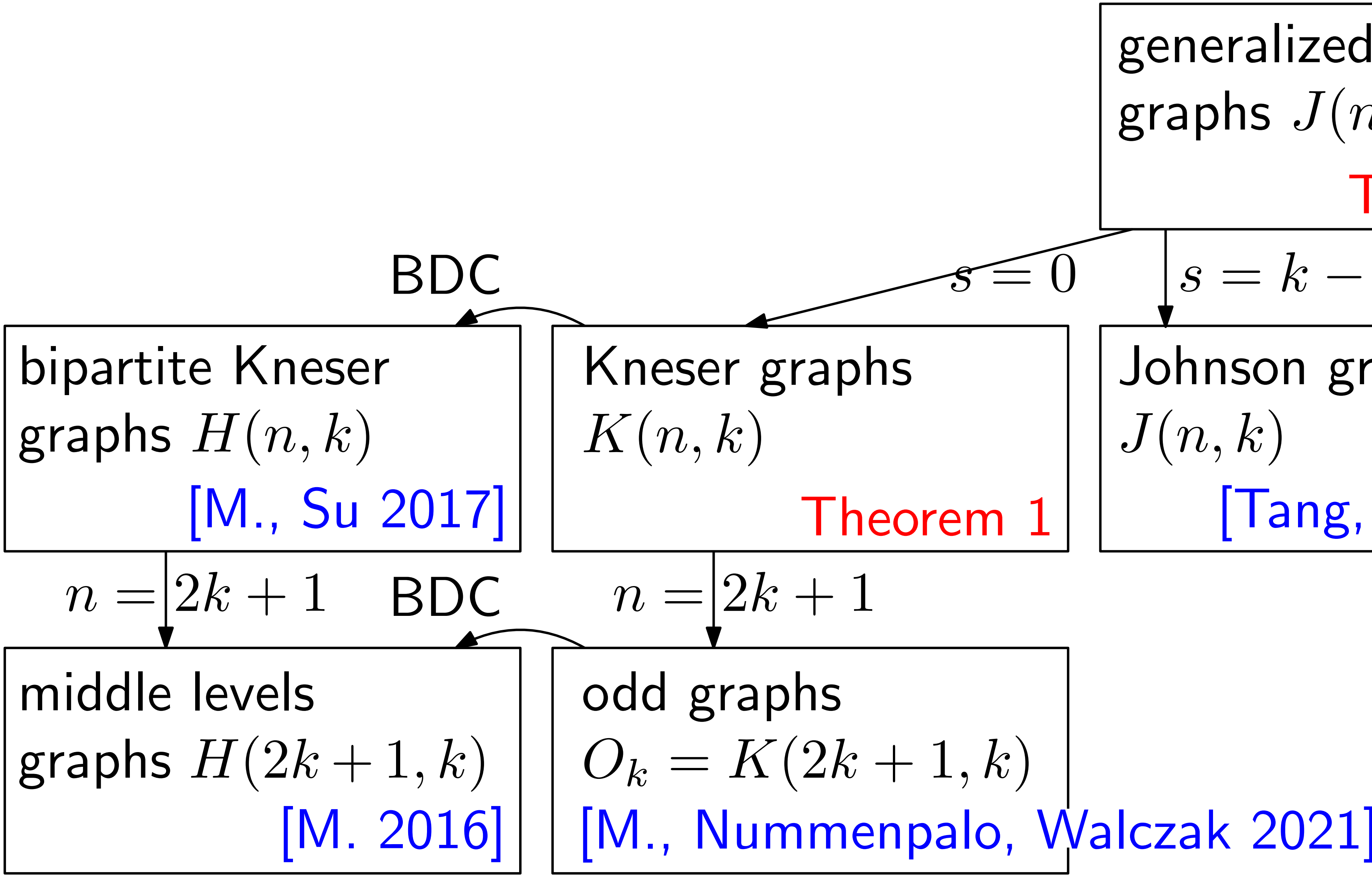

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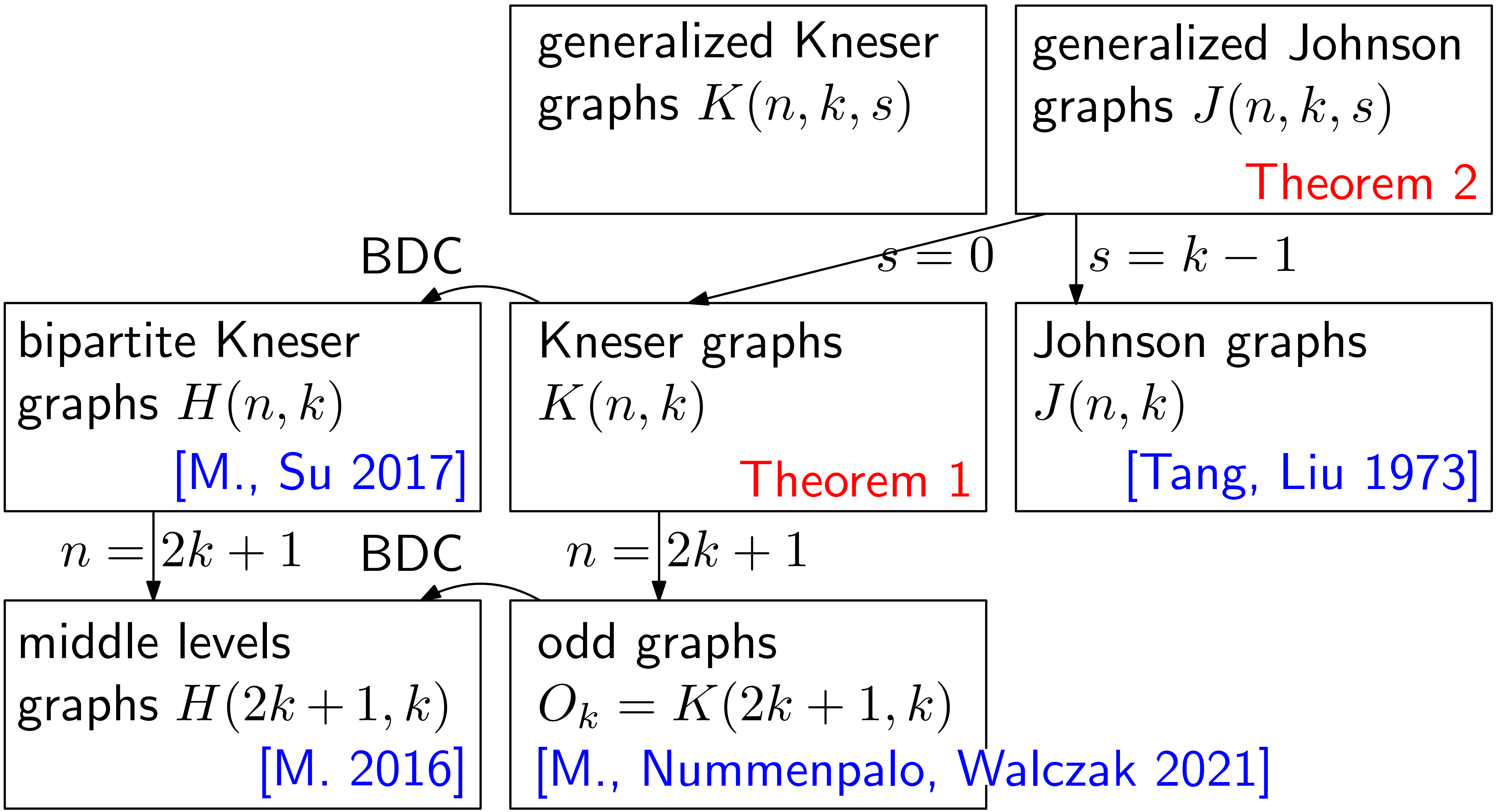

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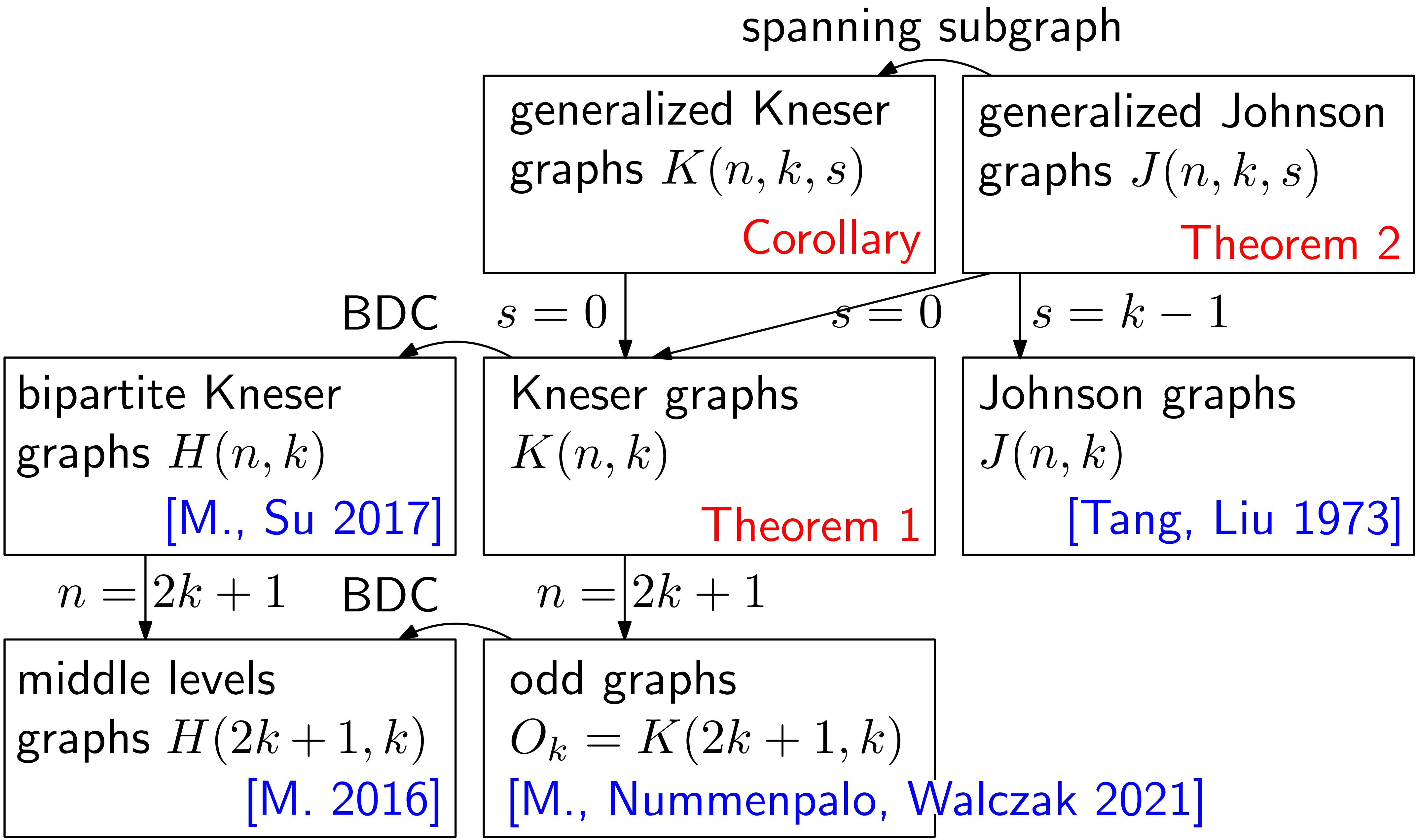

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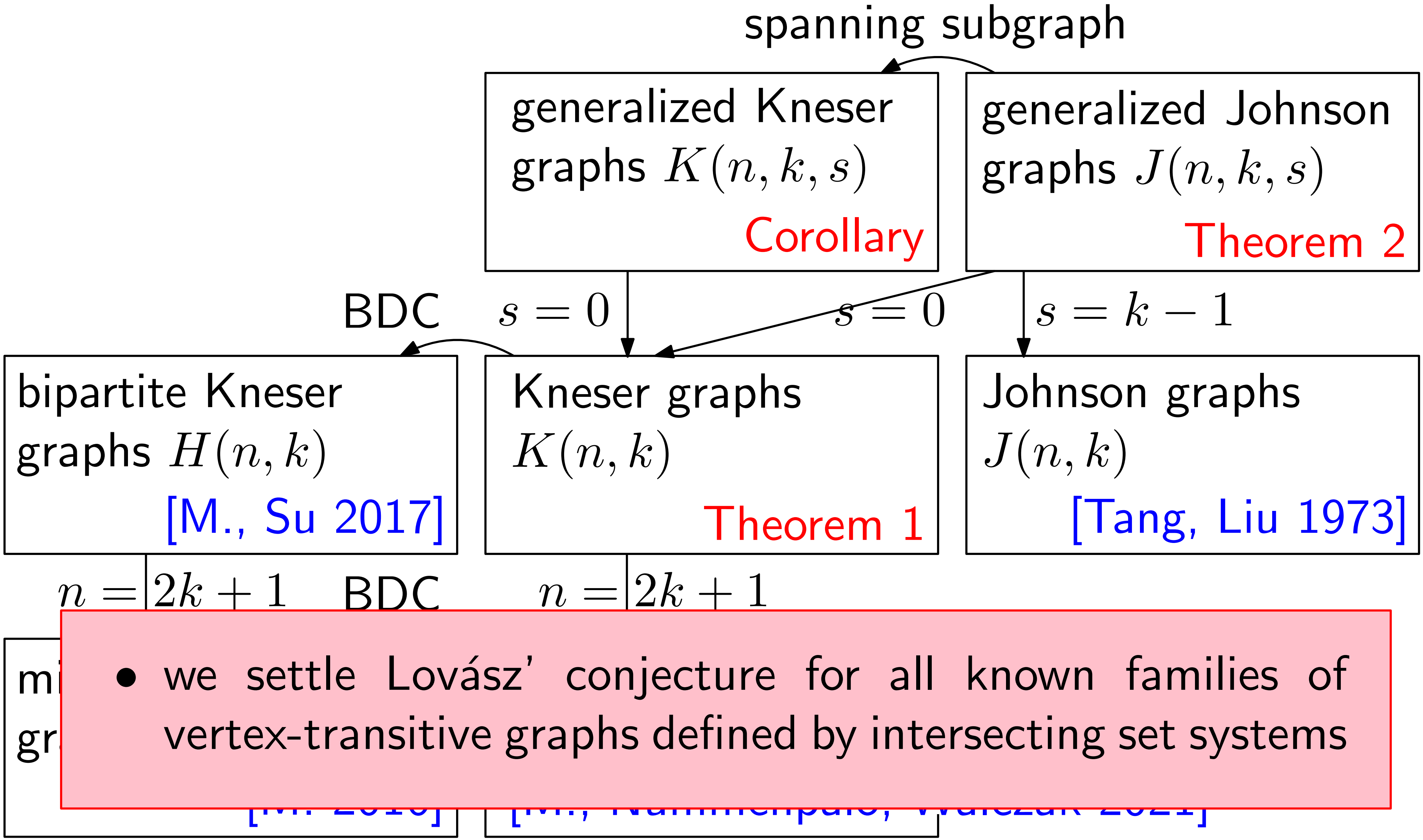

- we settle Lovász' conjecture for all known families of vertex-transitive graphs defined by intersecting set systems

Proof outline

- two sparsest cases $n = 2k + 1$ and $n = 2k + 2$ settled by [M., Nummenpalo, Walczak 2021]+[Johnson 2011]

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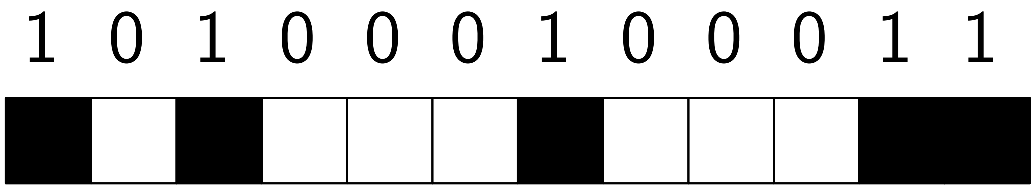

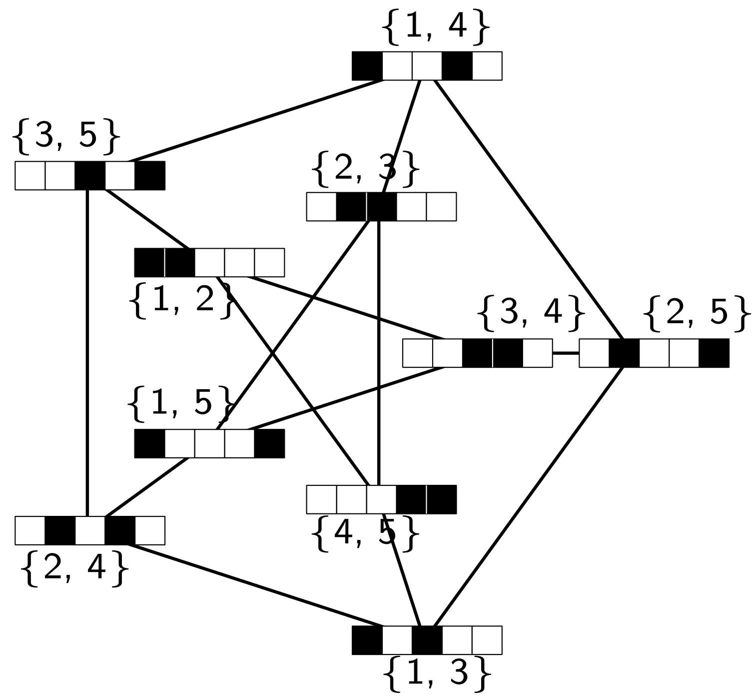

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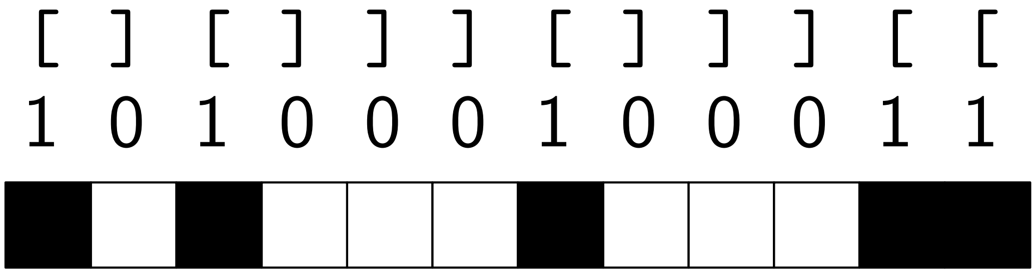

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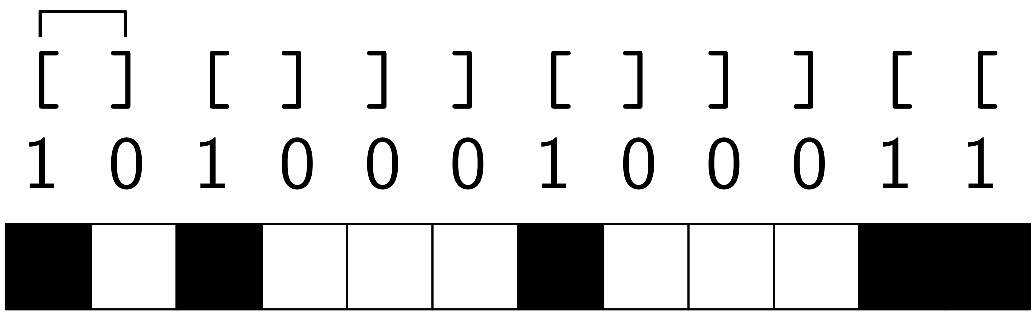

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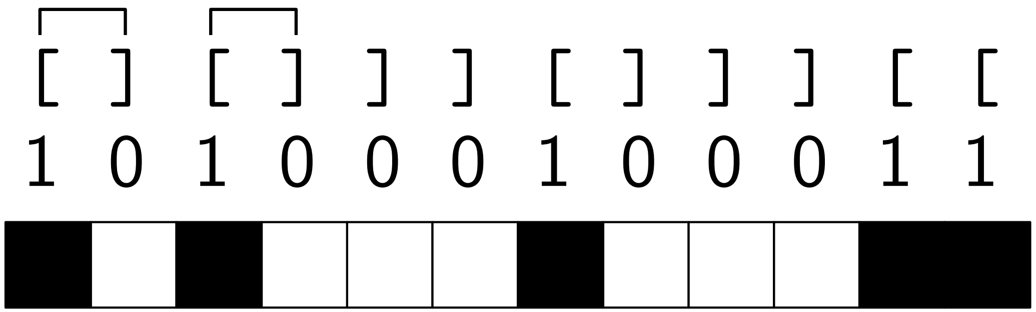

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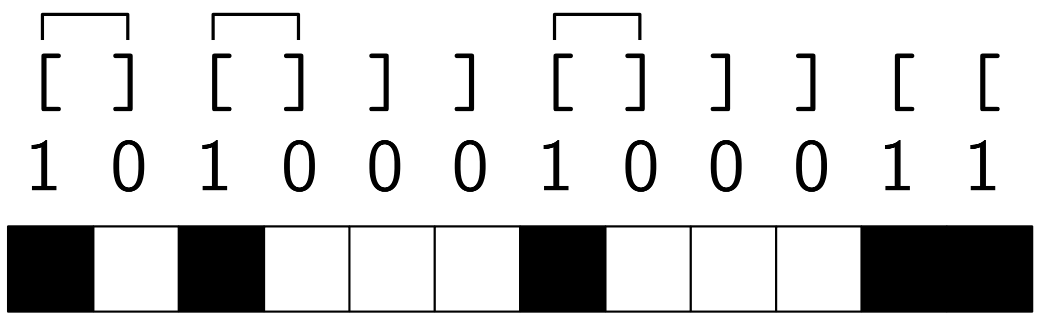

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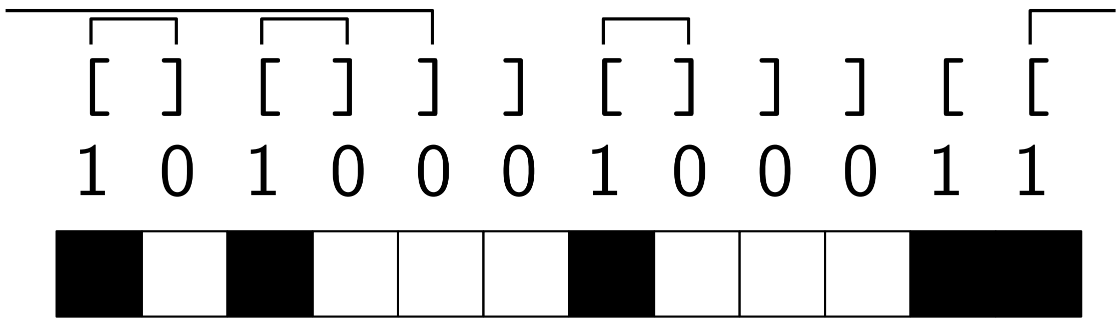

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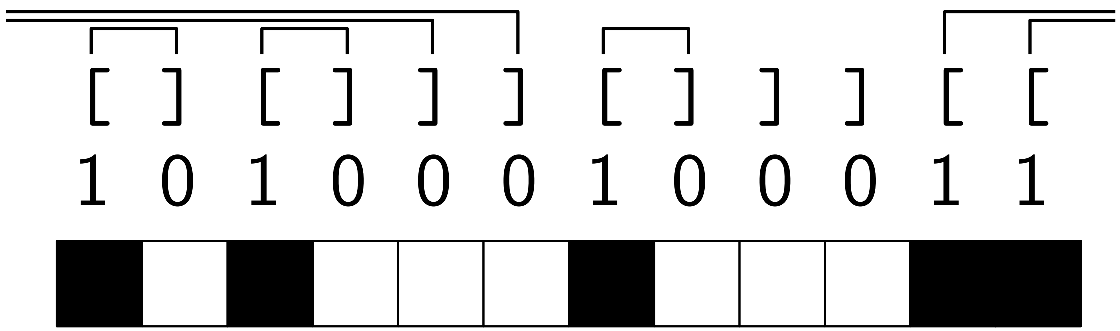

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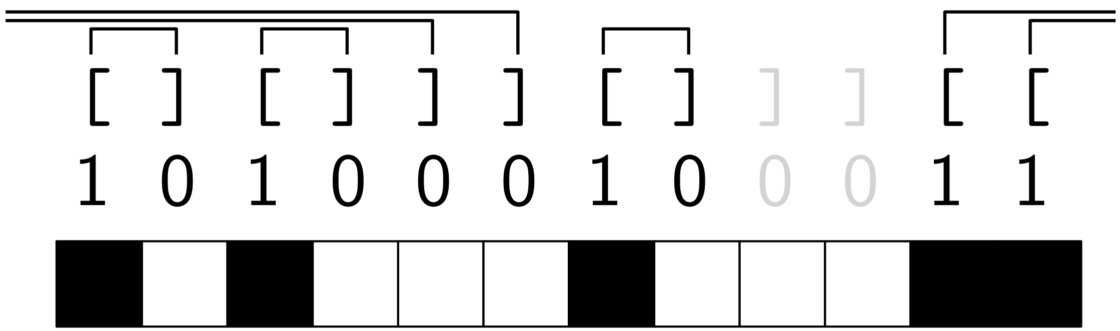

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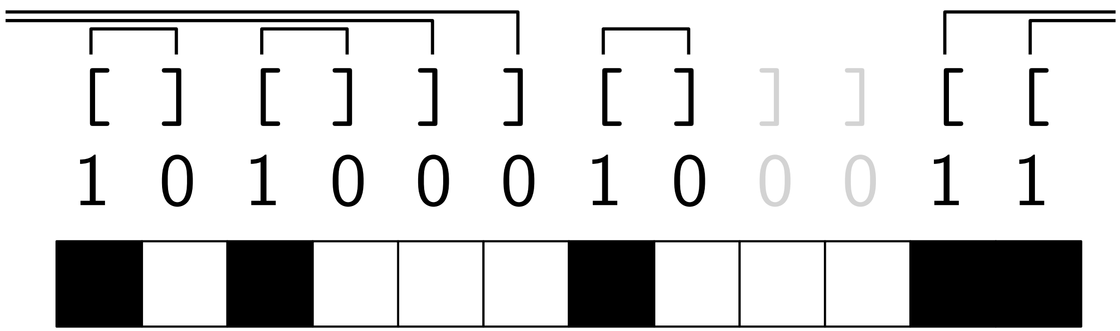

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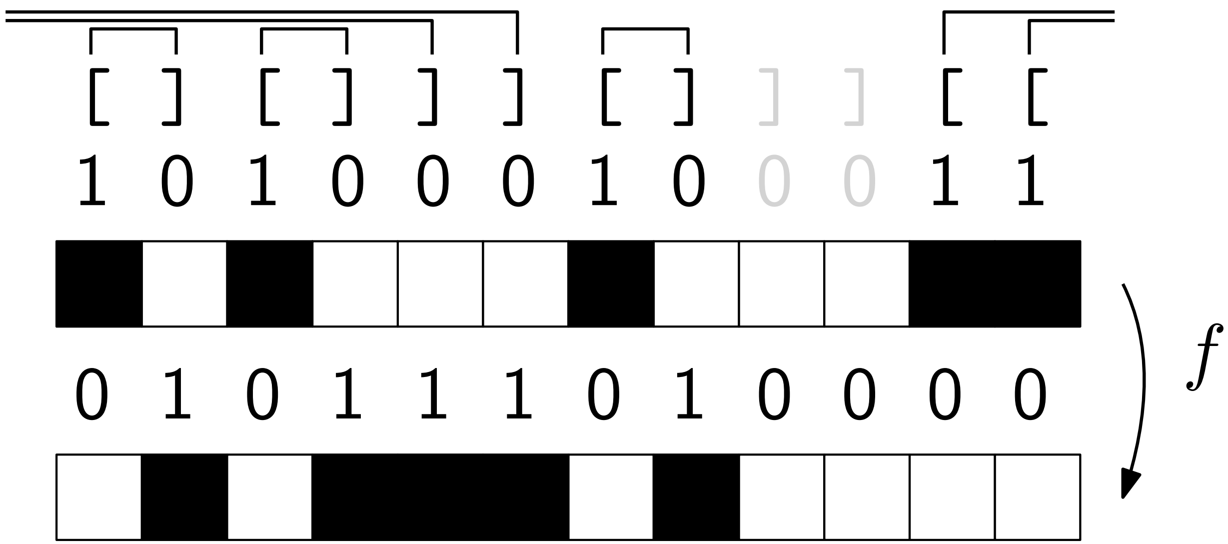

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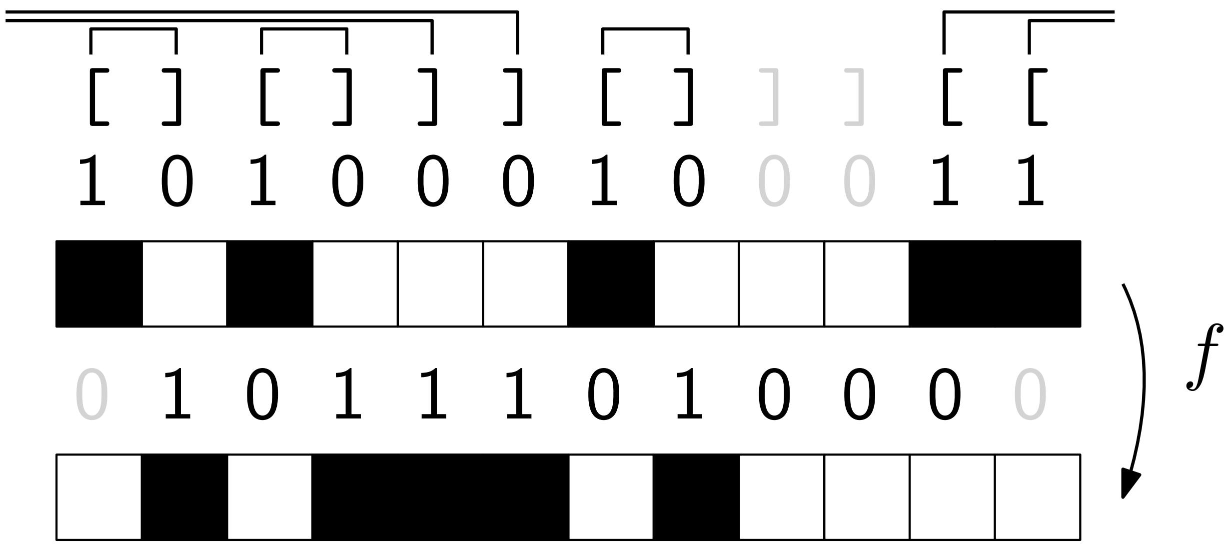

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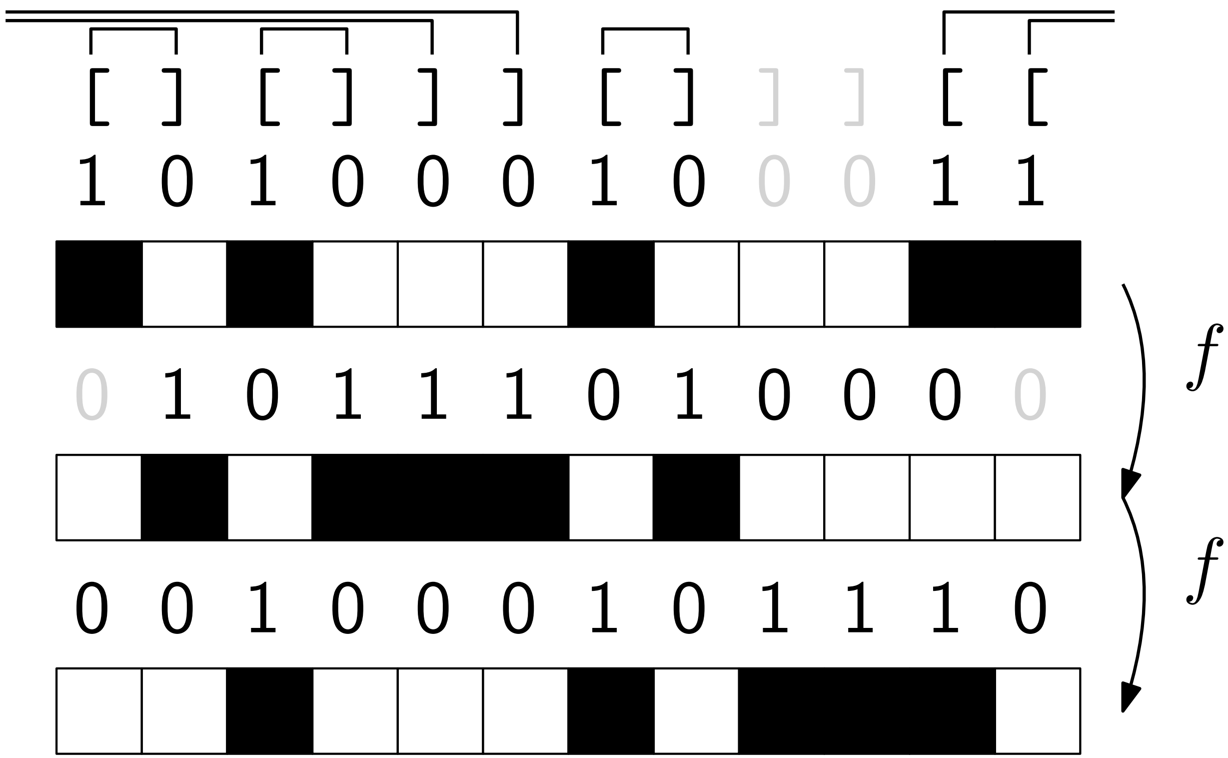

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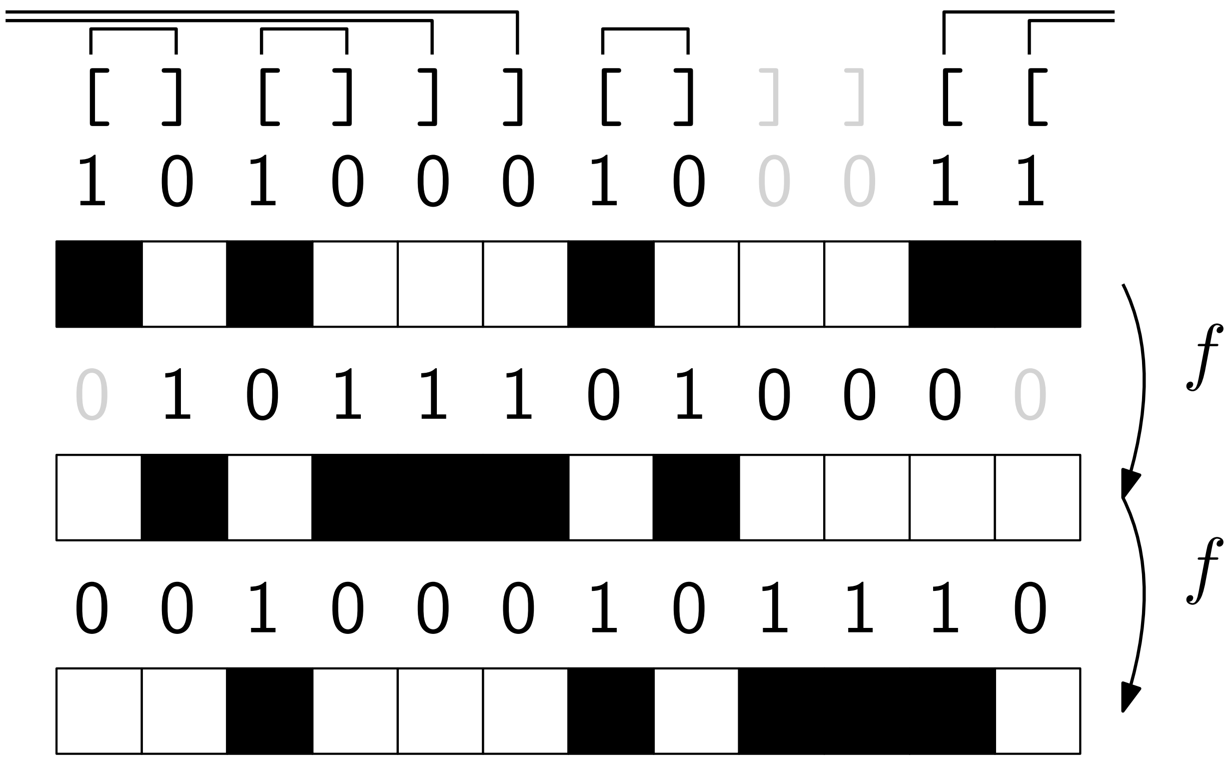
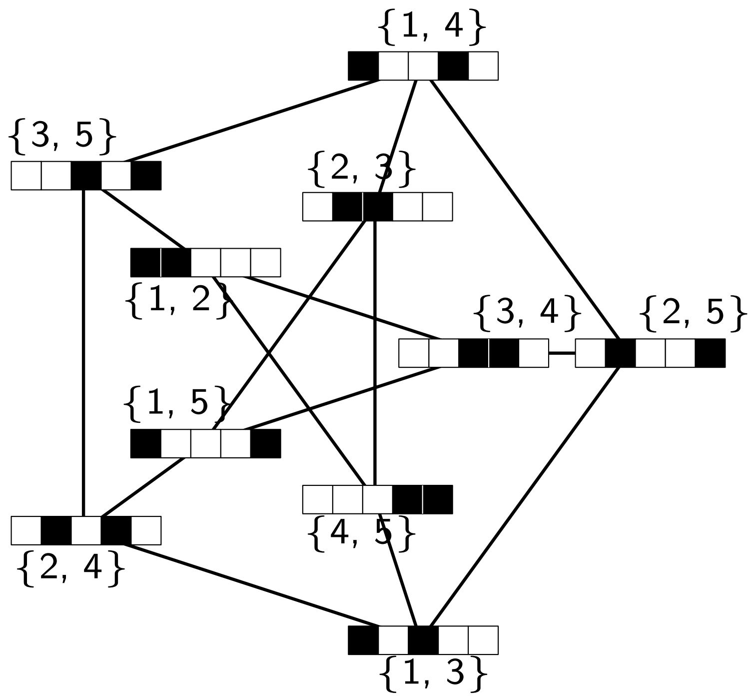


- f is invertible $\rightarrow$ partition of $K(n,k)$ into disjoint cycles

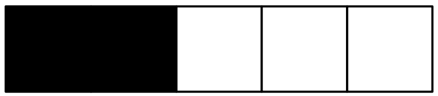
Cycle factor

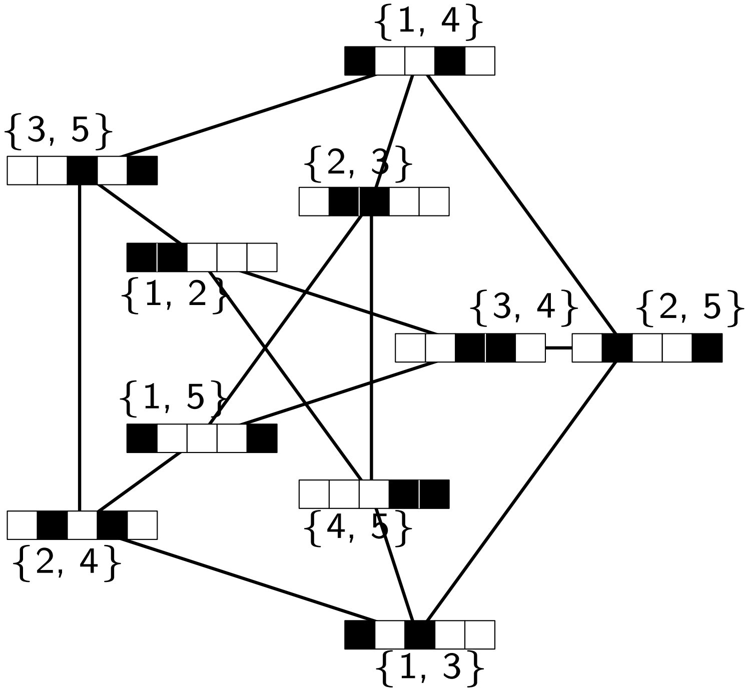
- **Example:** $K(5,2)$



Cycle factor

- **Example:** $K(5, 2)$





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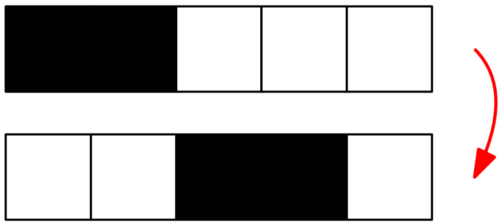

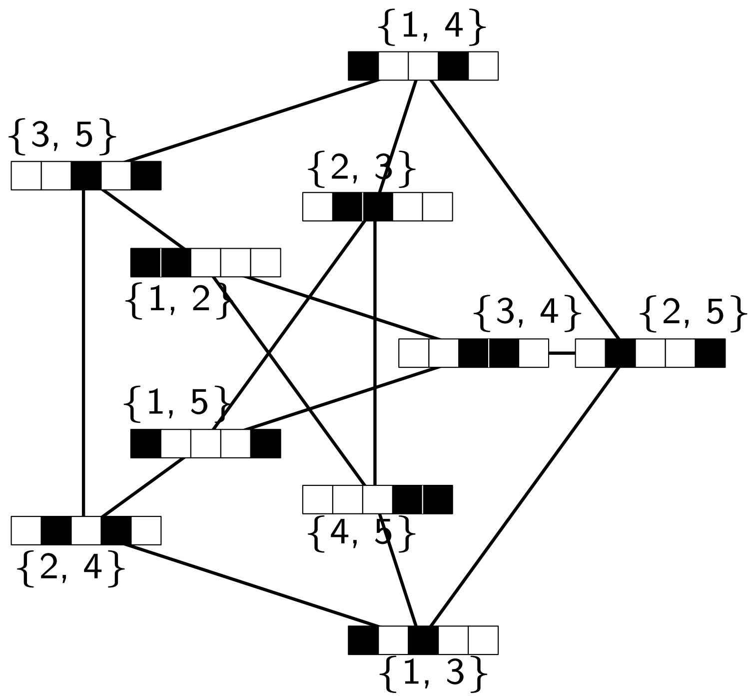

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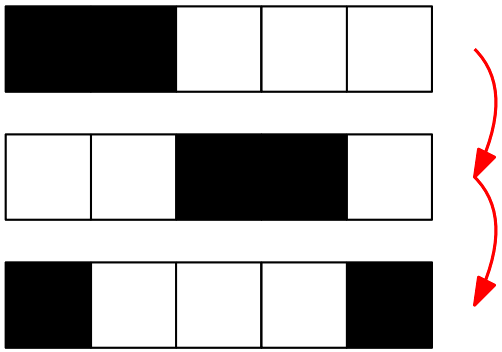

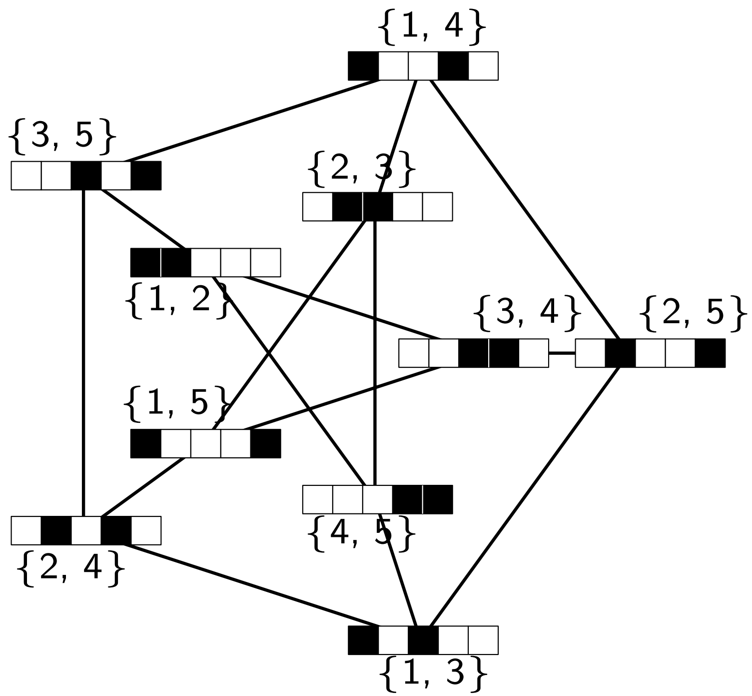

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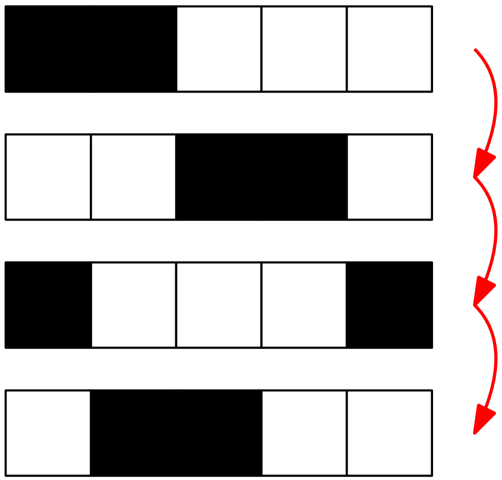

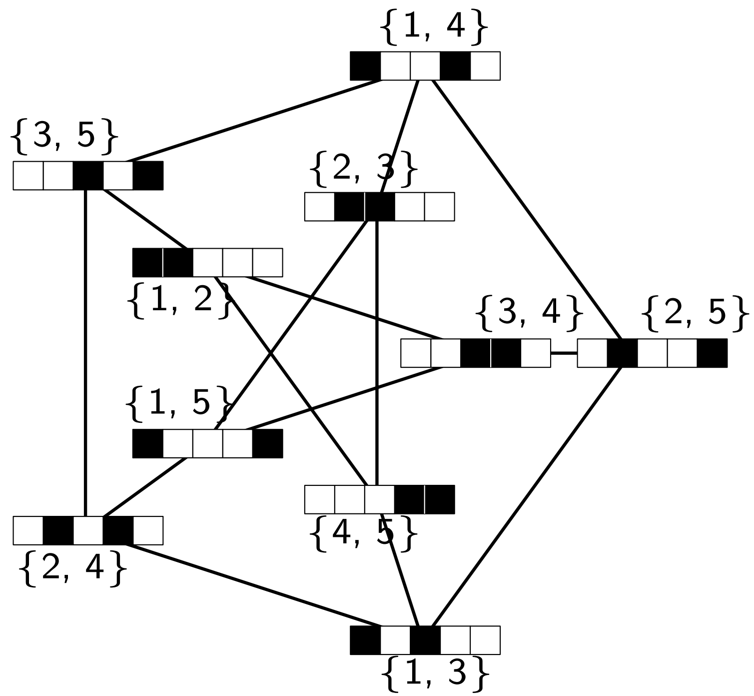

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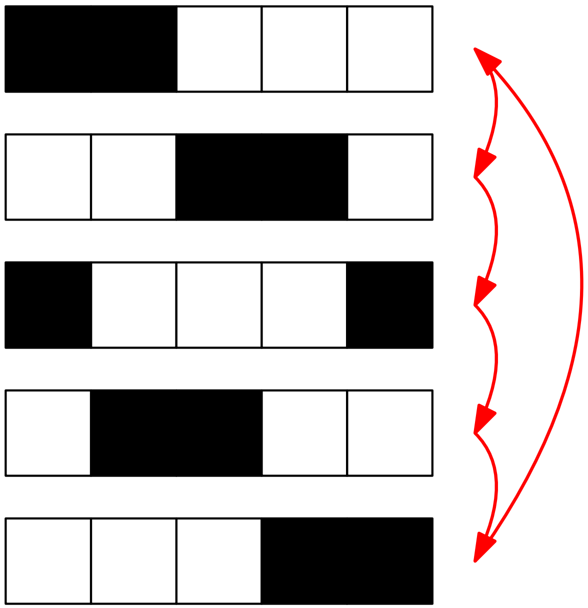

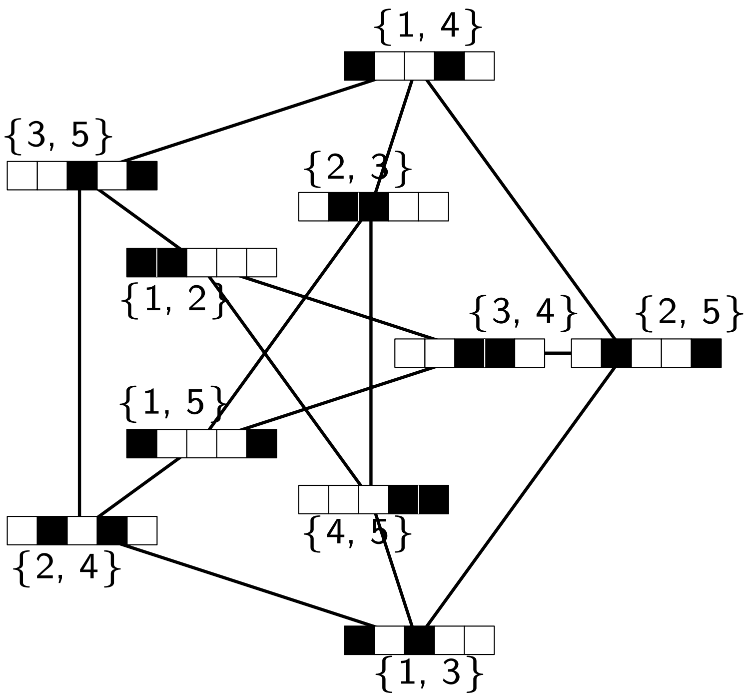

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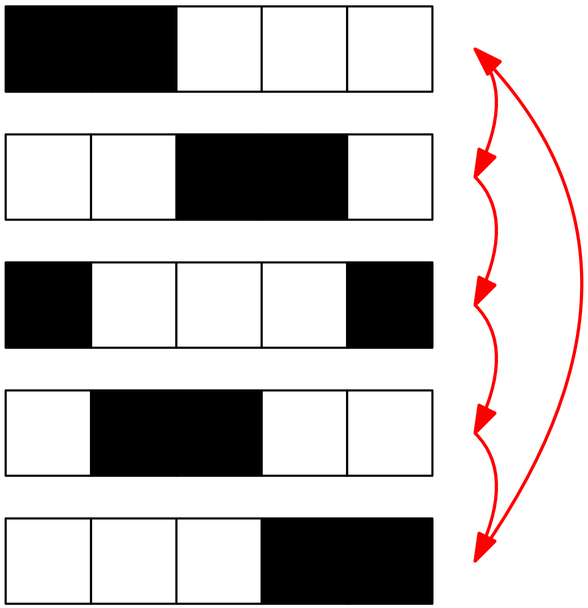

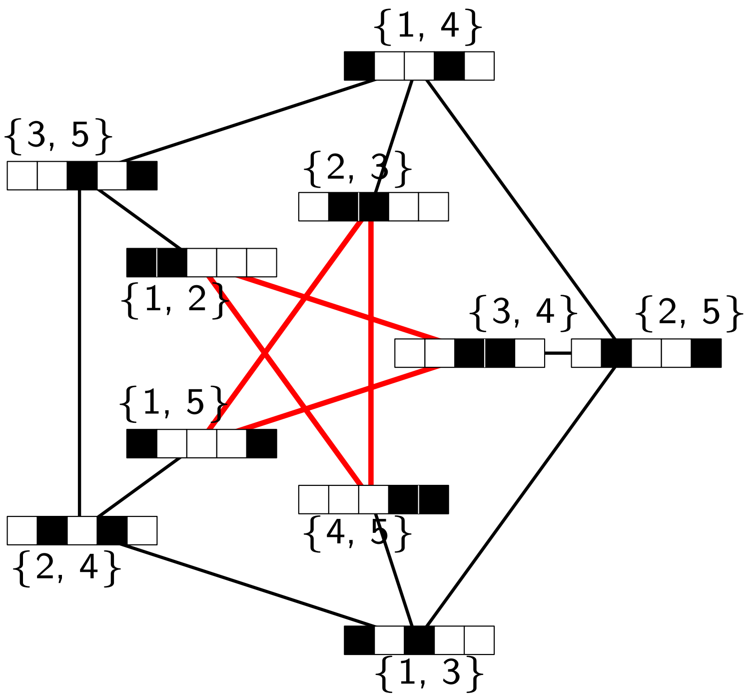

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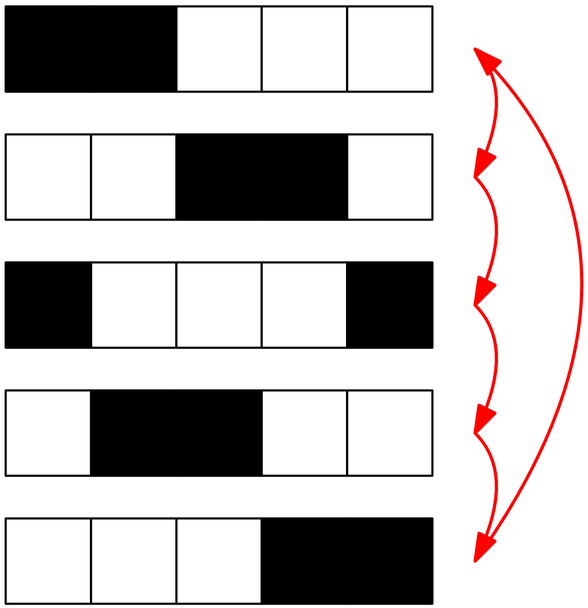

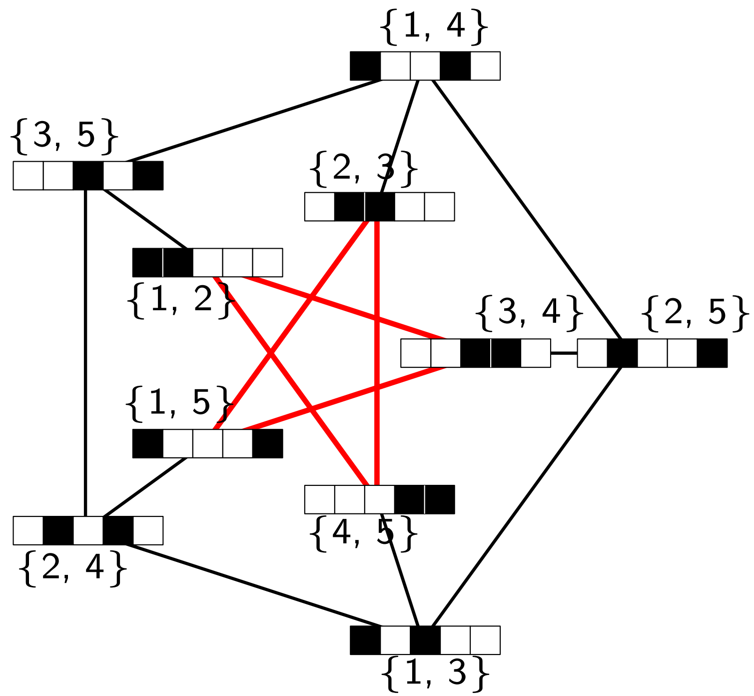

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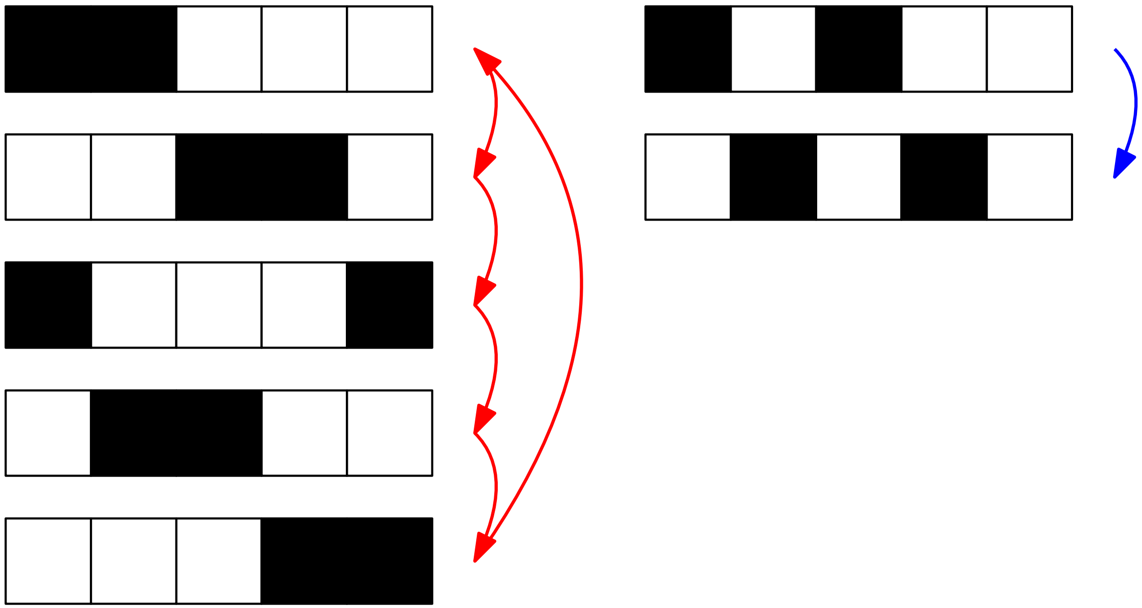

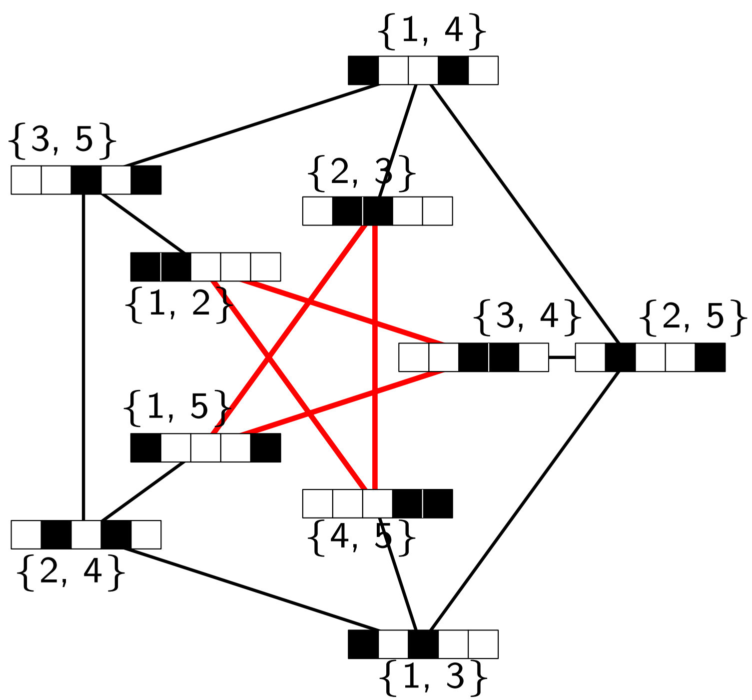

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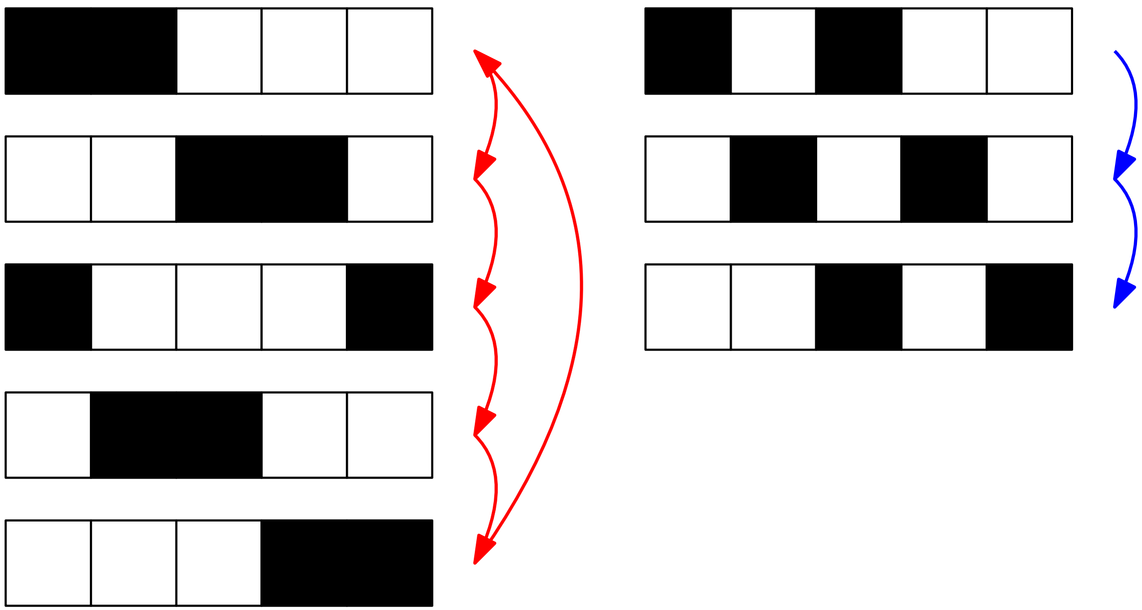

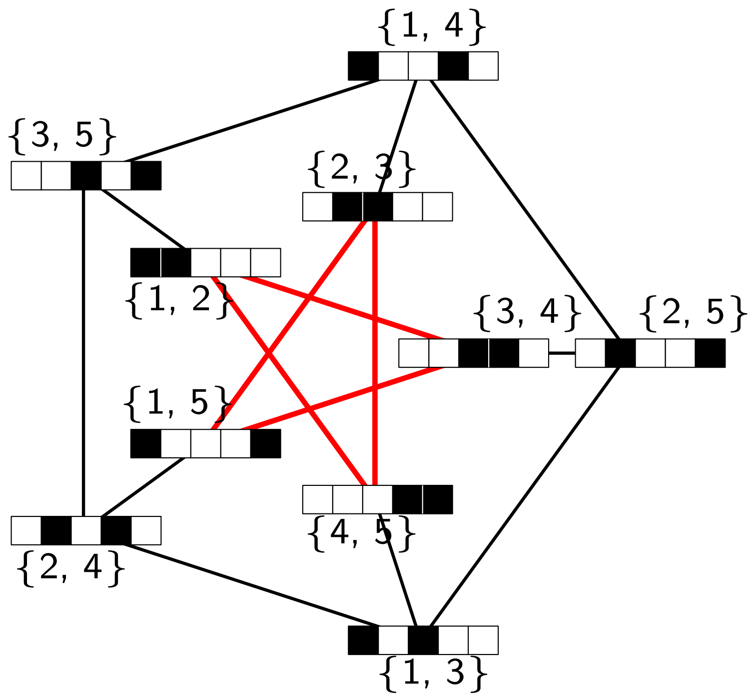

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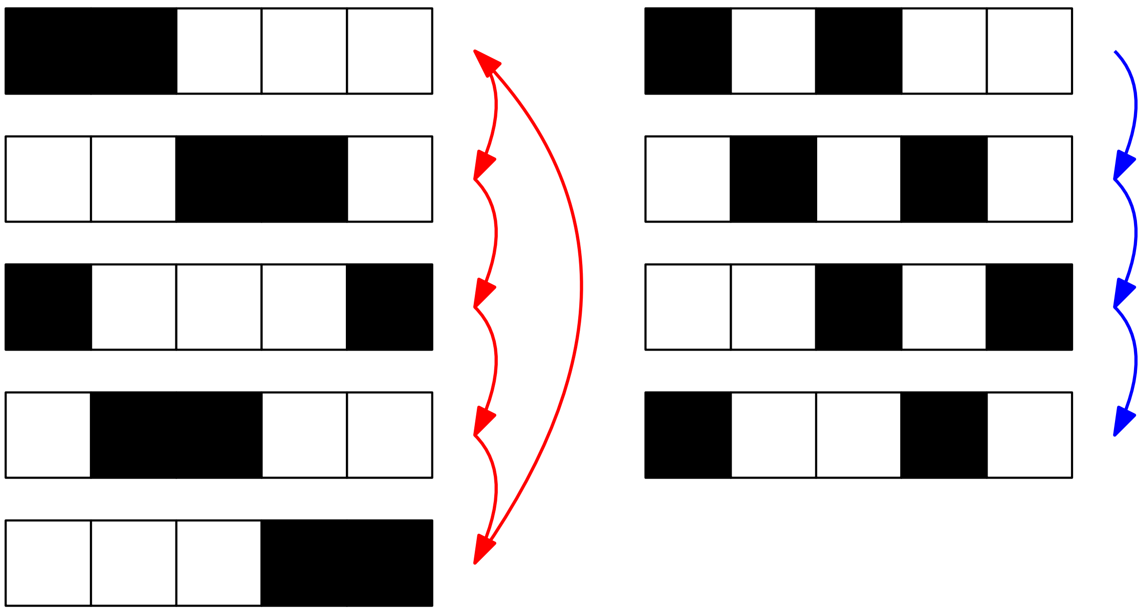

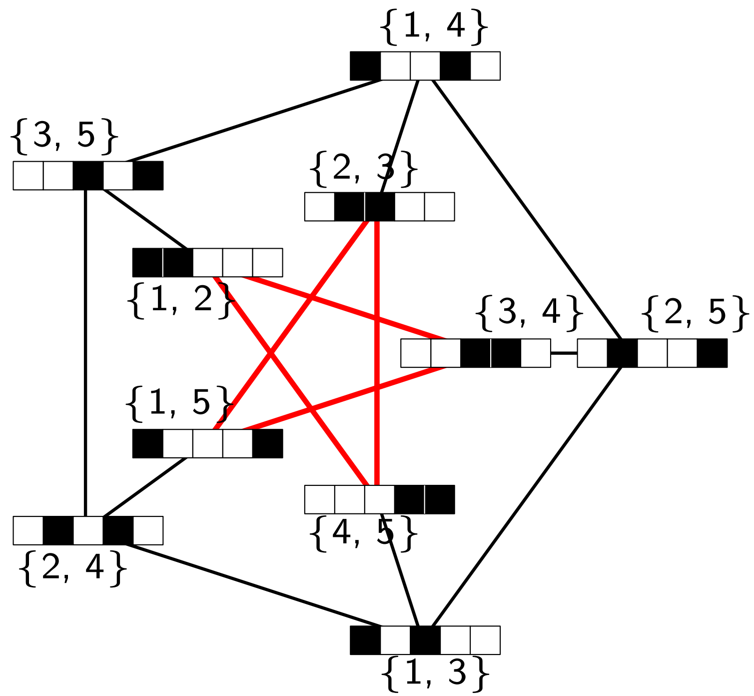

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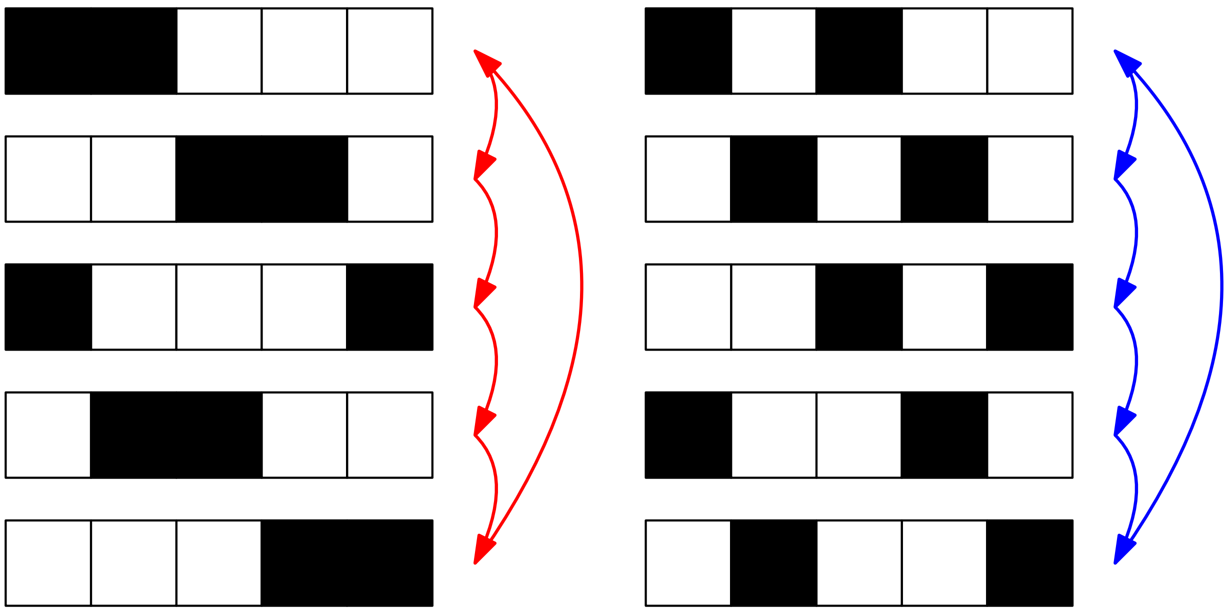

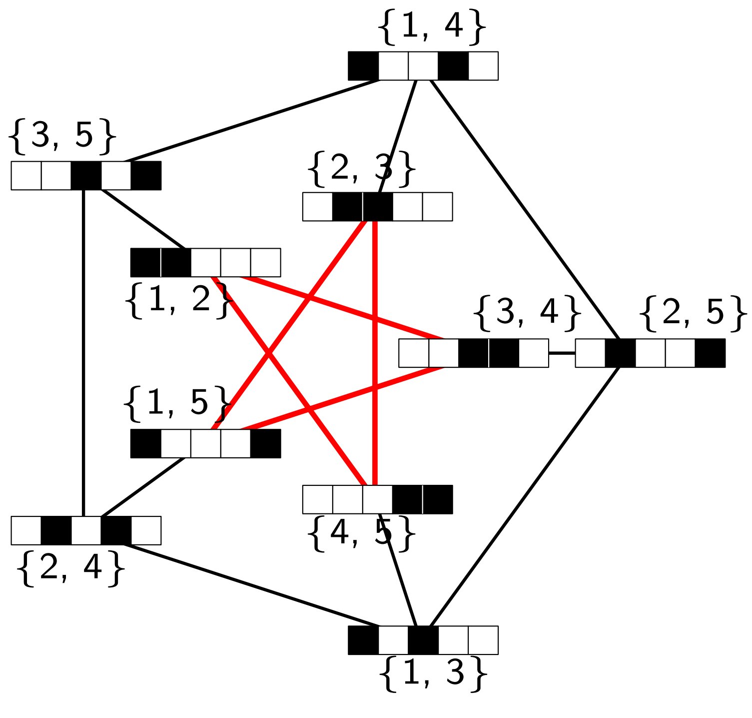

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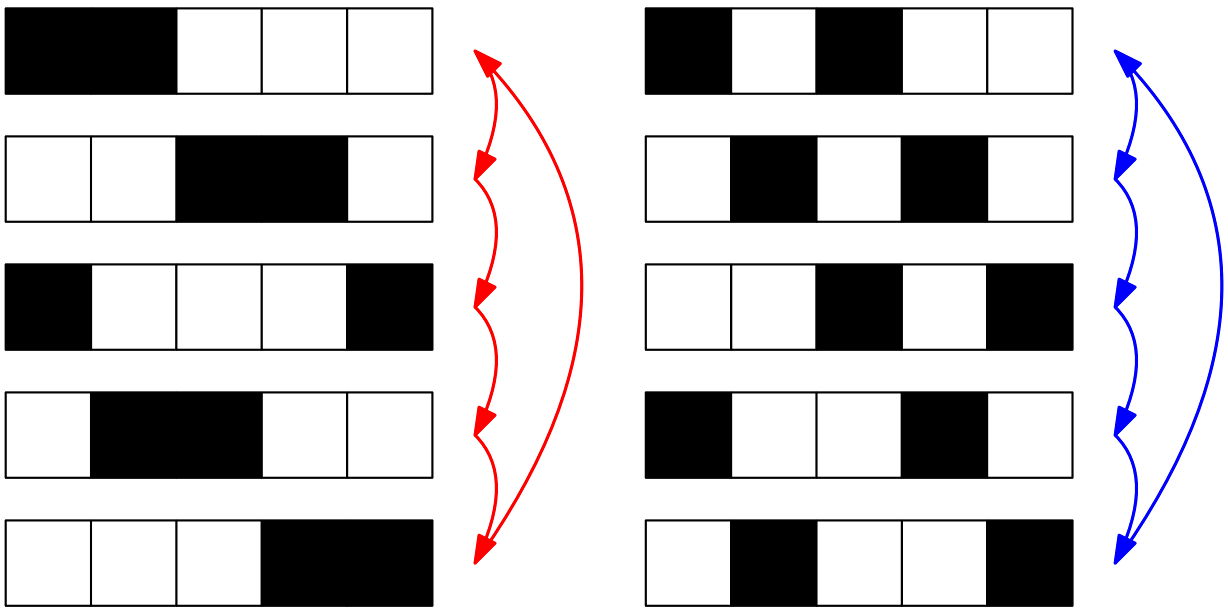
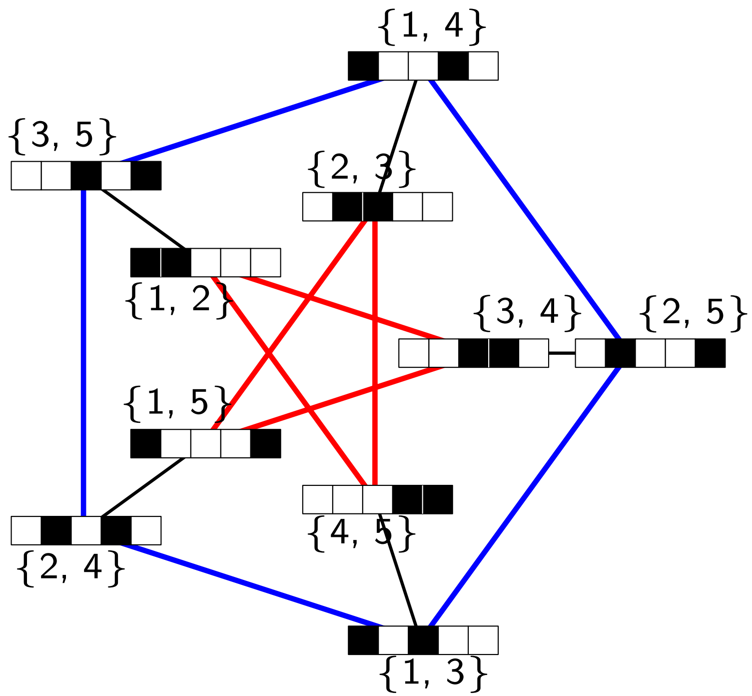
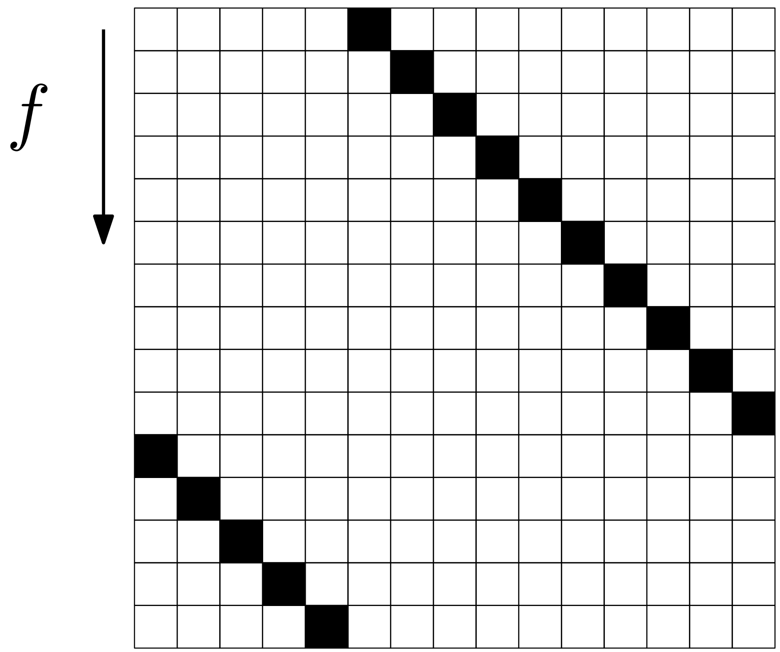




Analyzing the cycles

$$(n, k) = (15, 1)$$



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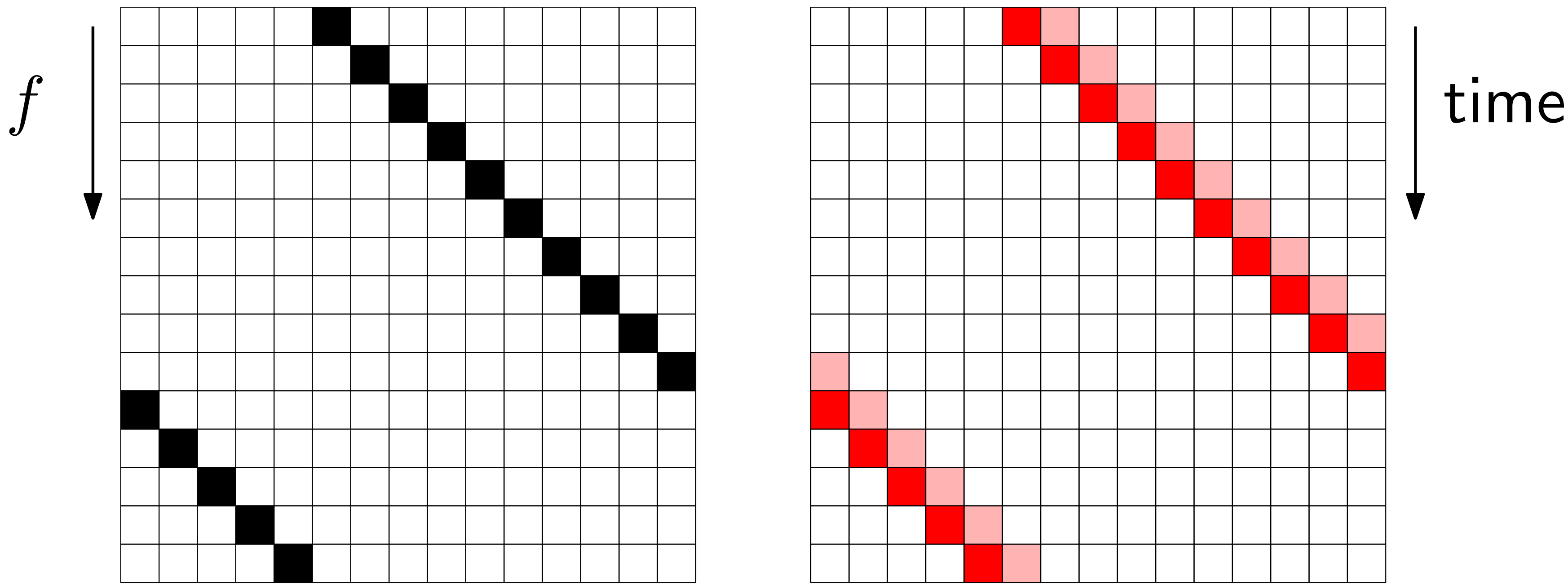

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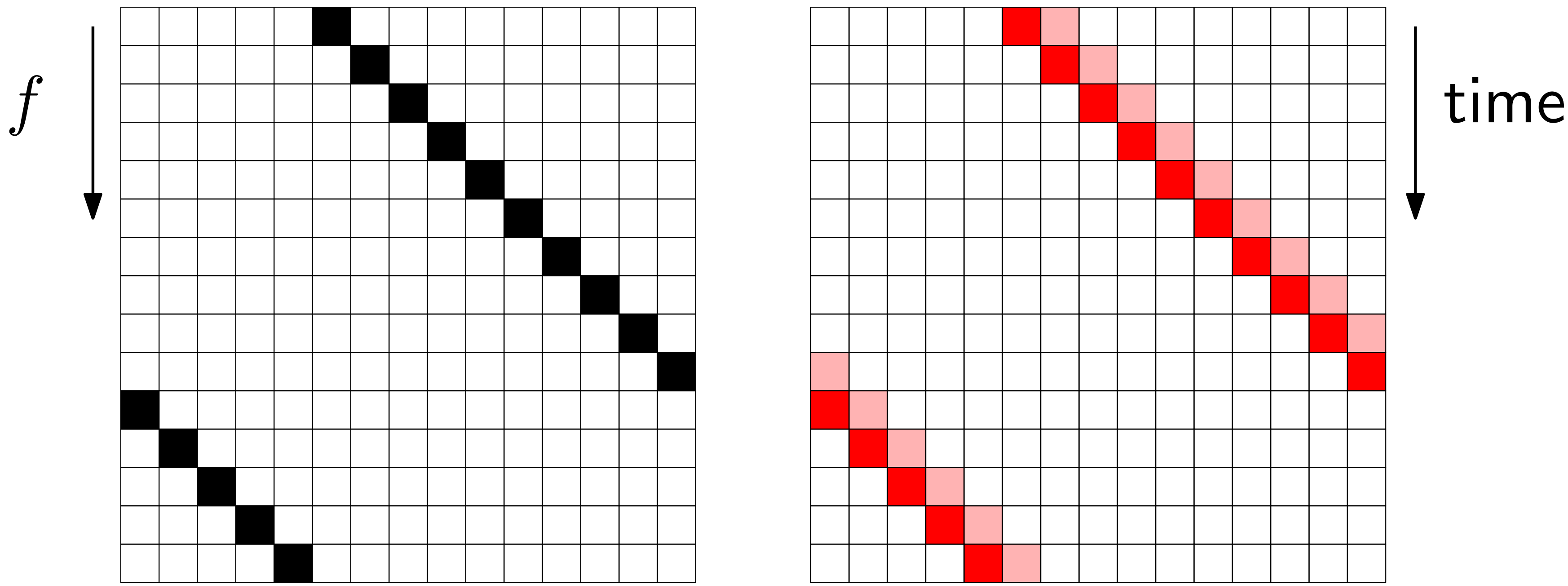

- Two matched bits form a glider
- Glider moves forward by 1 unit per step

Analyzing the cycles

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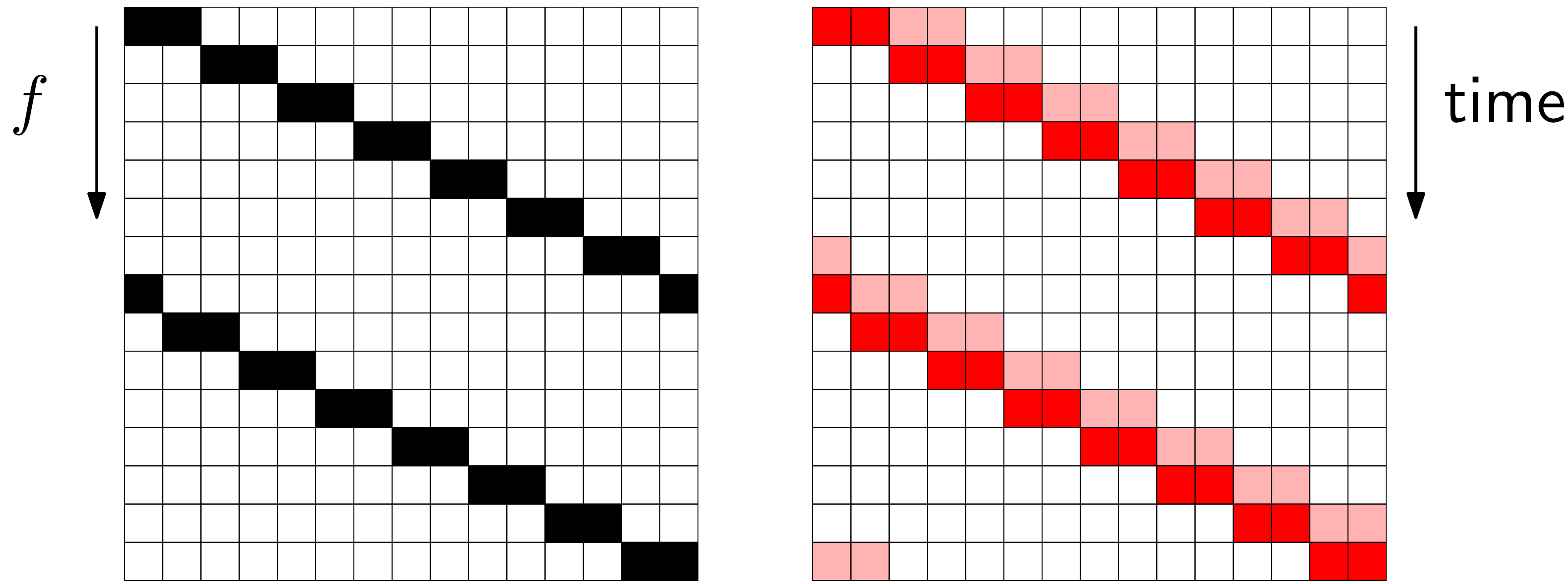
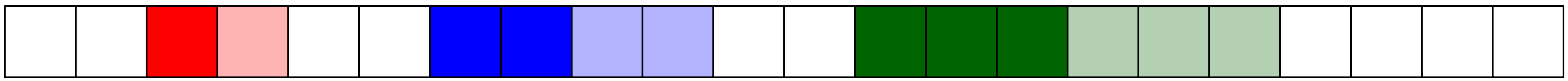


- Four matched bits form one glider
- Glider moves forward by 2 units per step

Gliders

- **glider** := set of matched 1s and 0s (same number of each)



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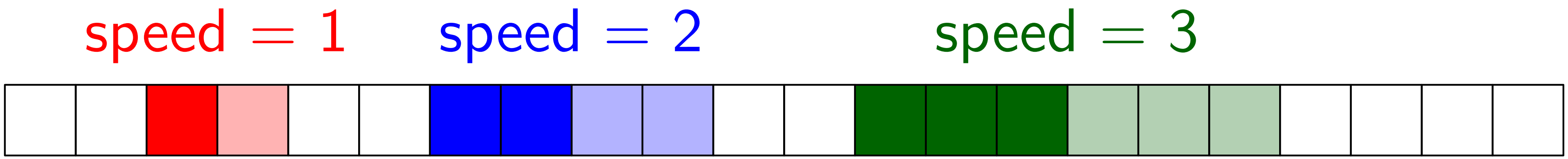

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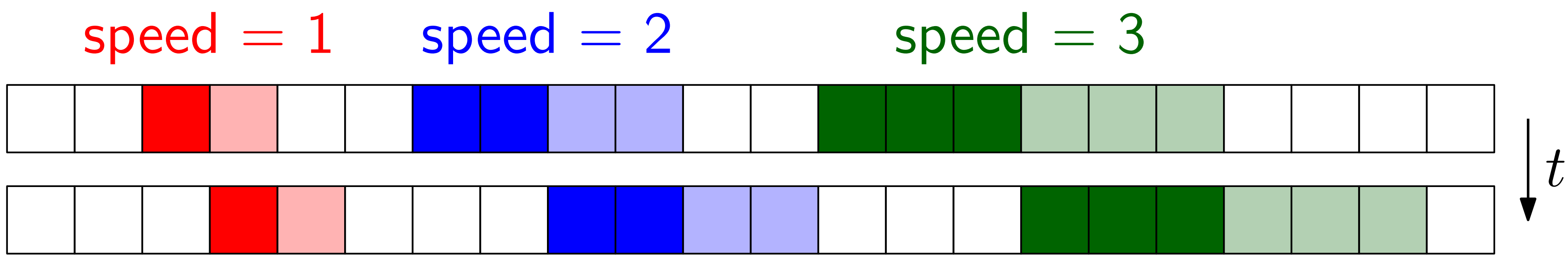

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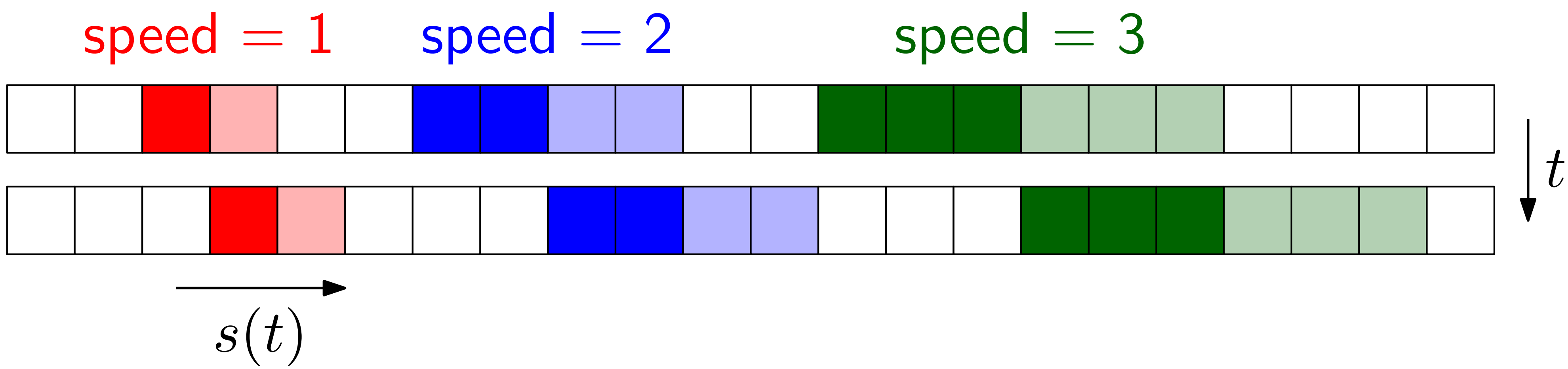

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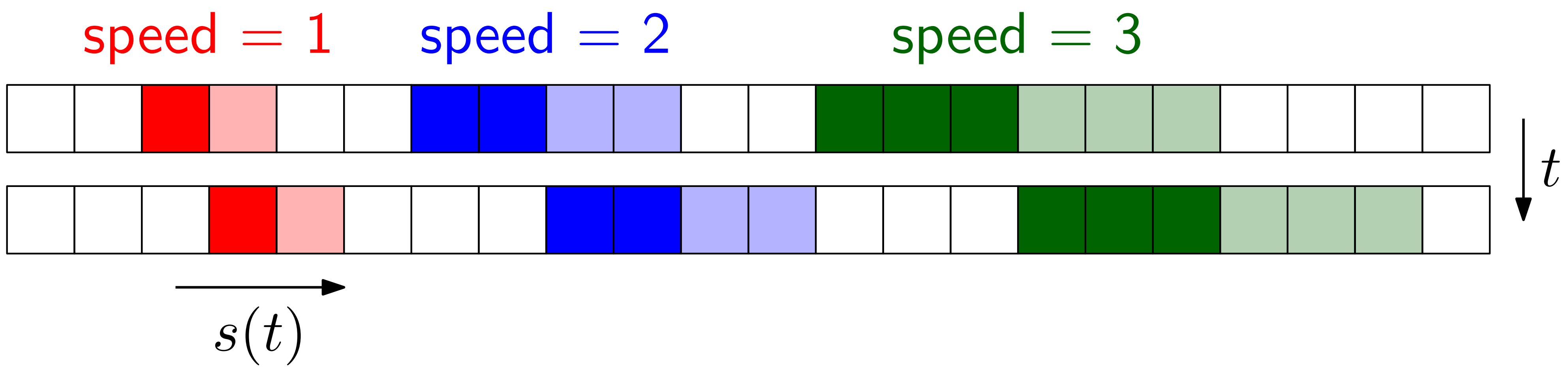

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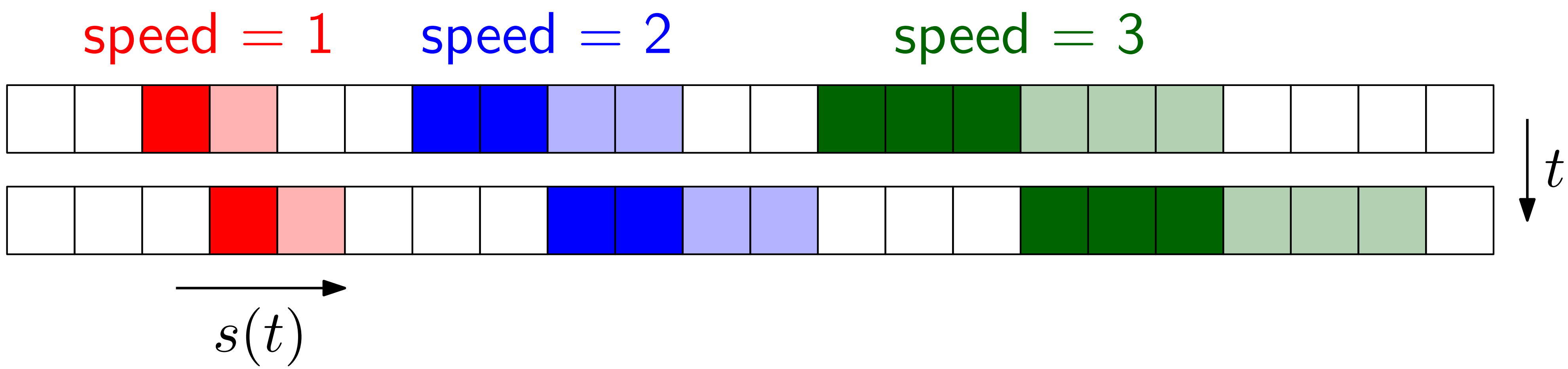


- Uniform equation of motion: $s(t) = v \cdot t + s(0)$

$s(t)$: position (modulo n)

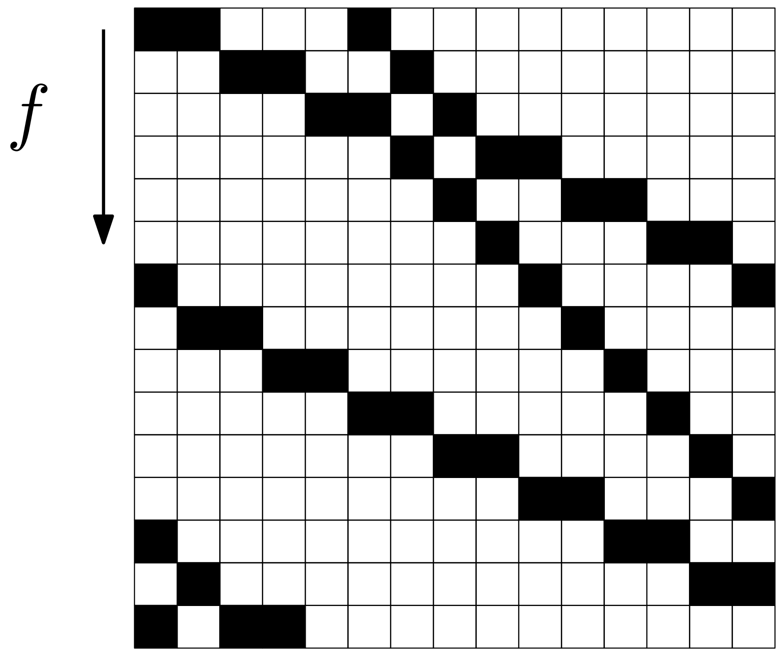
v: speed

t: time t = number of applications of f

$s(0)$: starting position

Overtaking of gliders

$(n, k) = (15, 4)$



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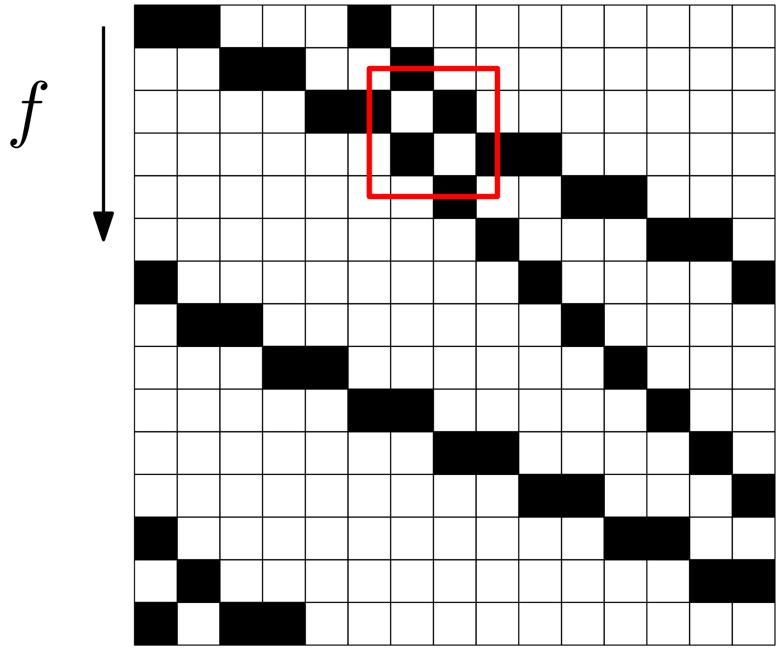

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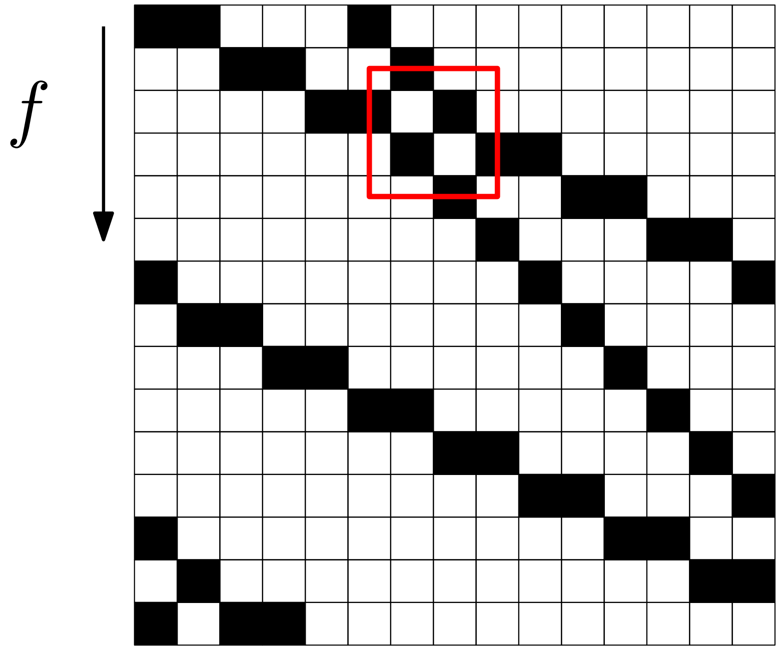

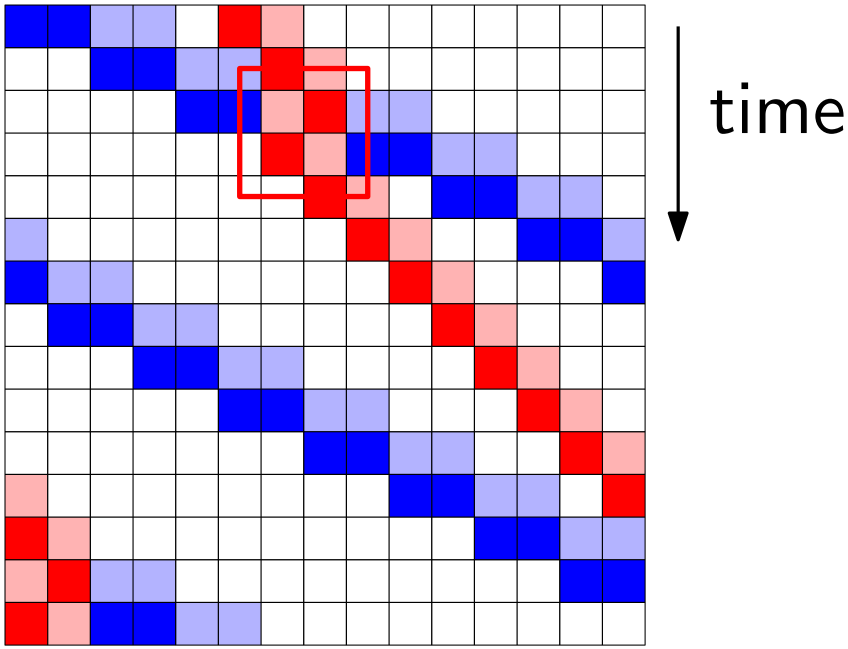

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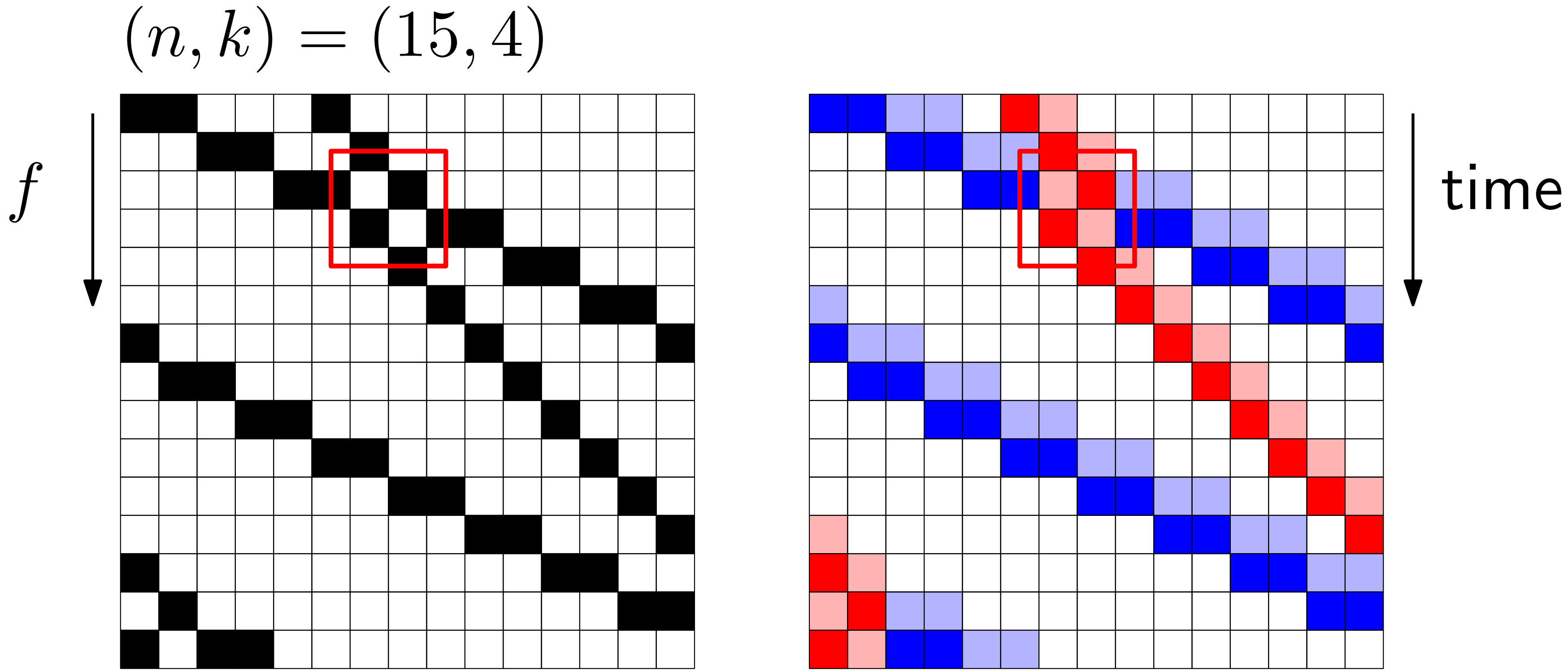

- during overtaking, slower glider stands still for two time steps

Overtaking of gliders

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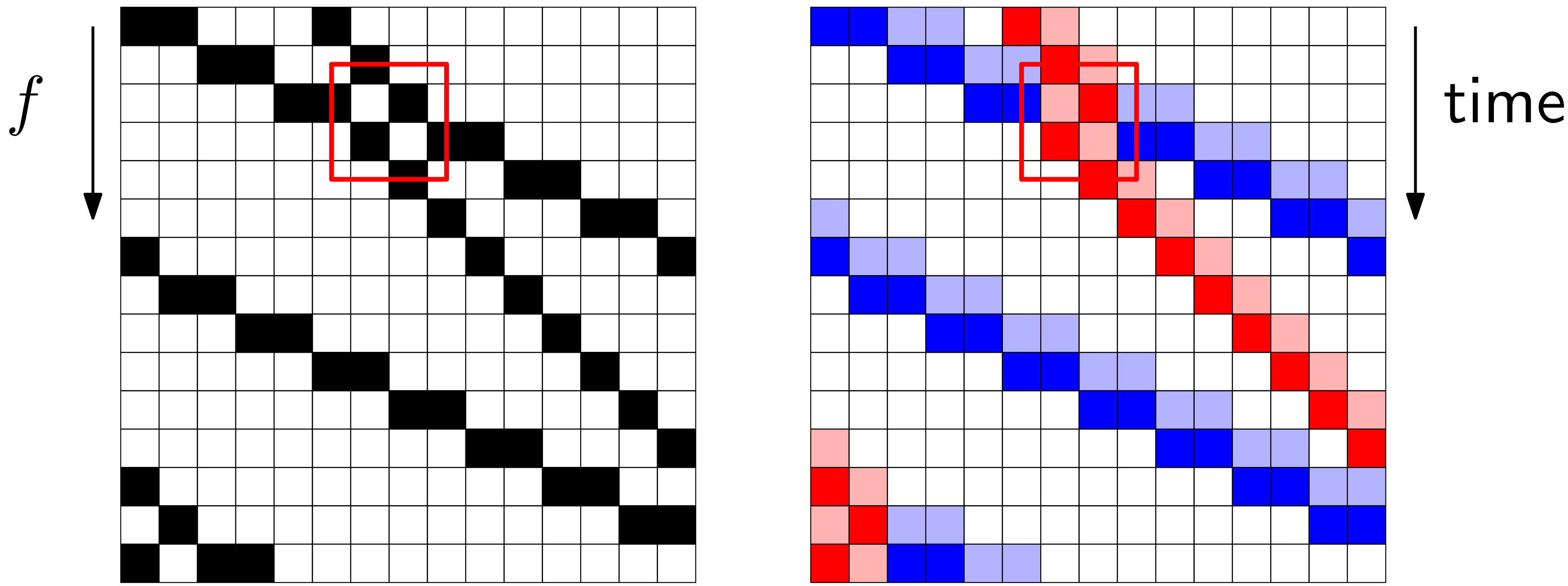

- during overtaking, slower glider stands still for two time steps
- faster glider is boosted by twice the speed of slower glider

Overtaking of gliders

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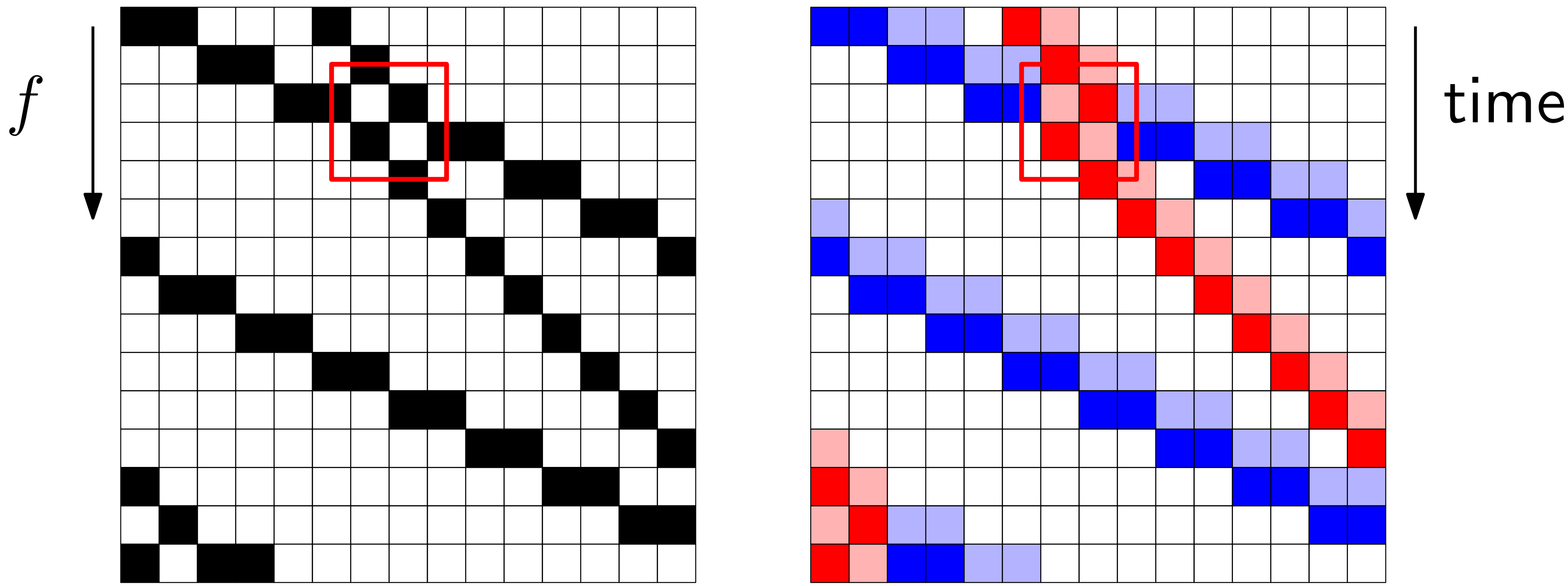

- non-uniform equations of motion:

$$s_1(t) = v_1 \cdot t + s_1(0)$$
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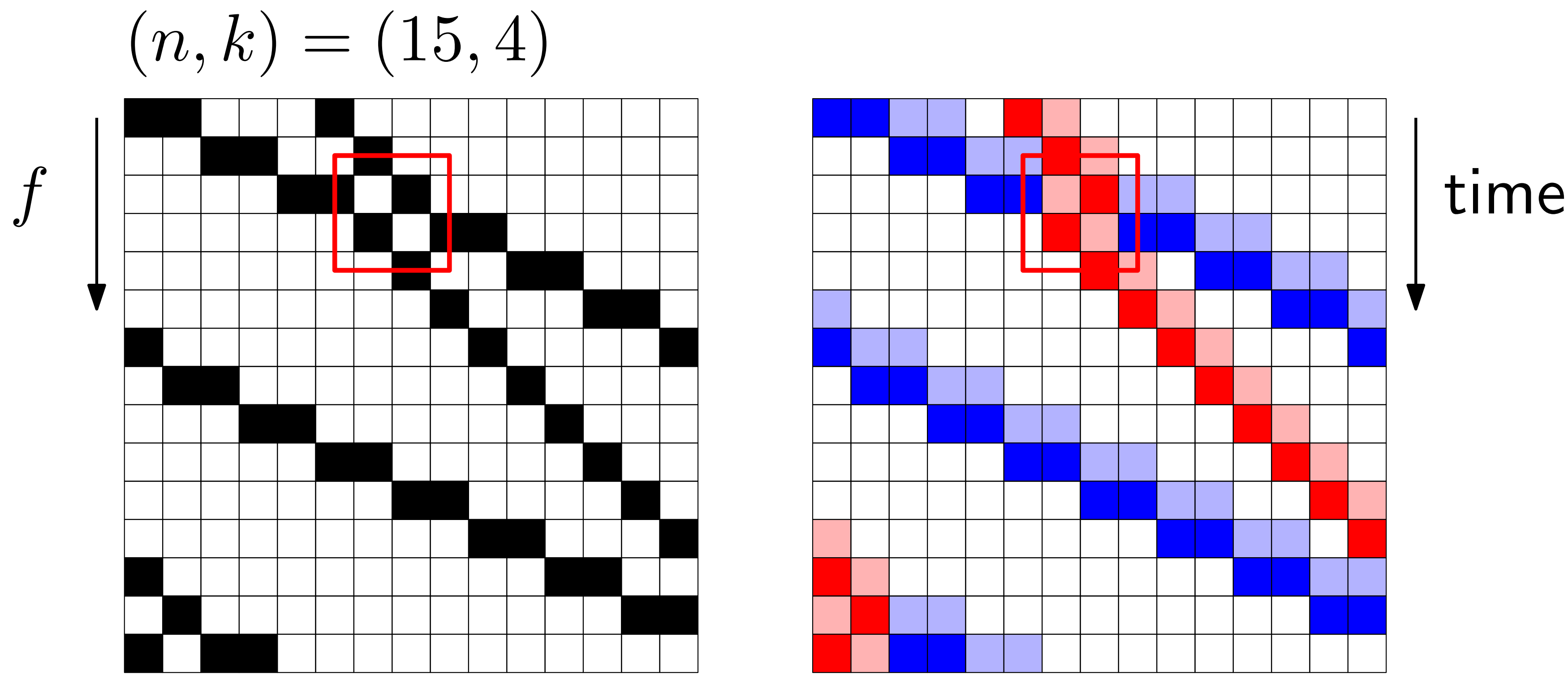

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Overtaking of gliders

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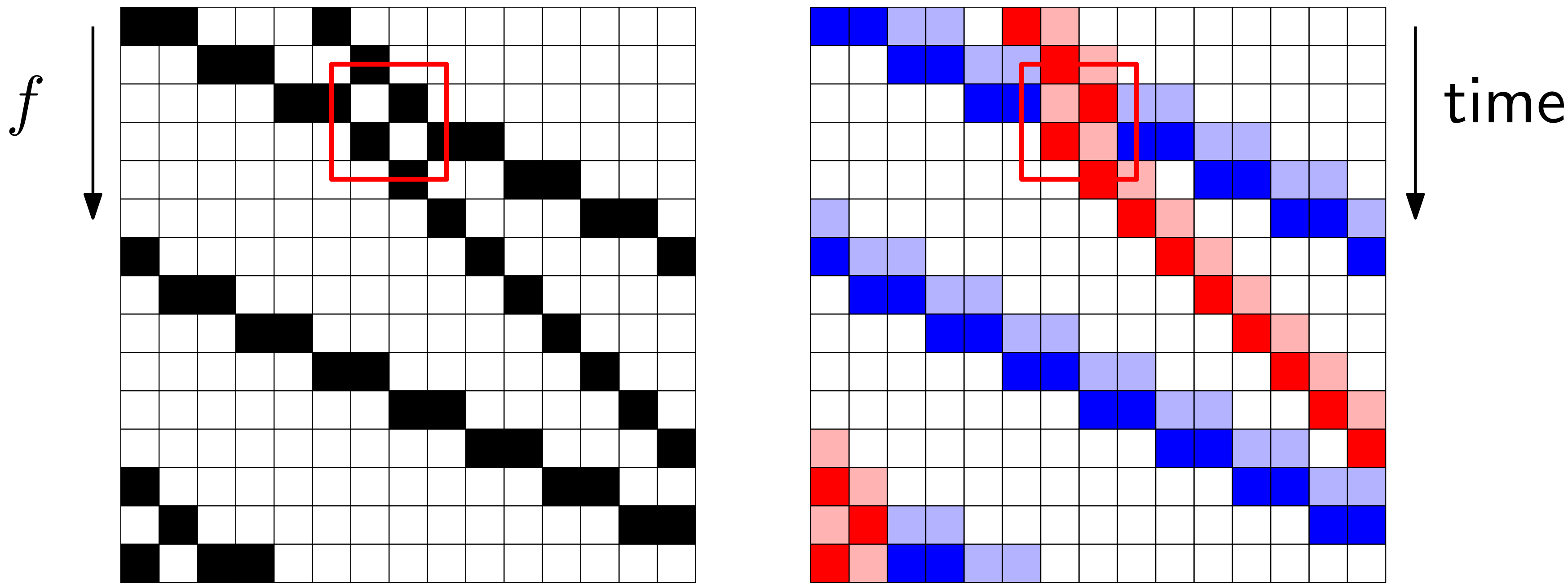

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Overtaking of gliders

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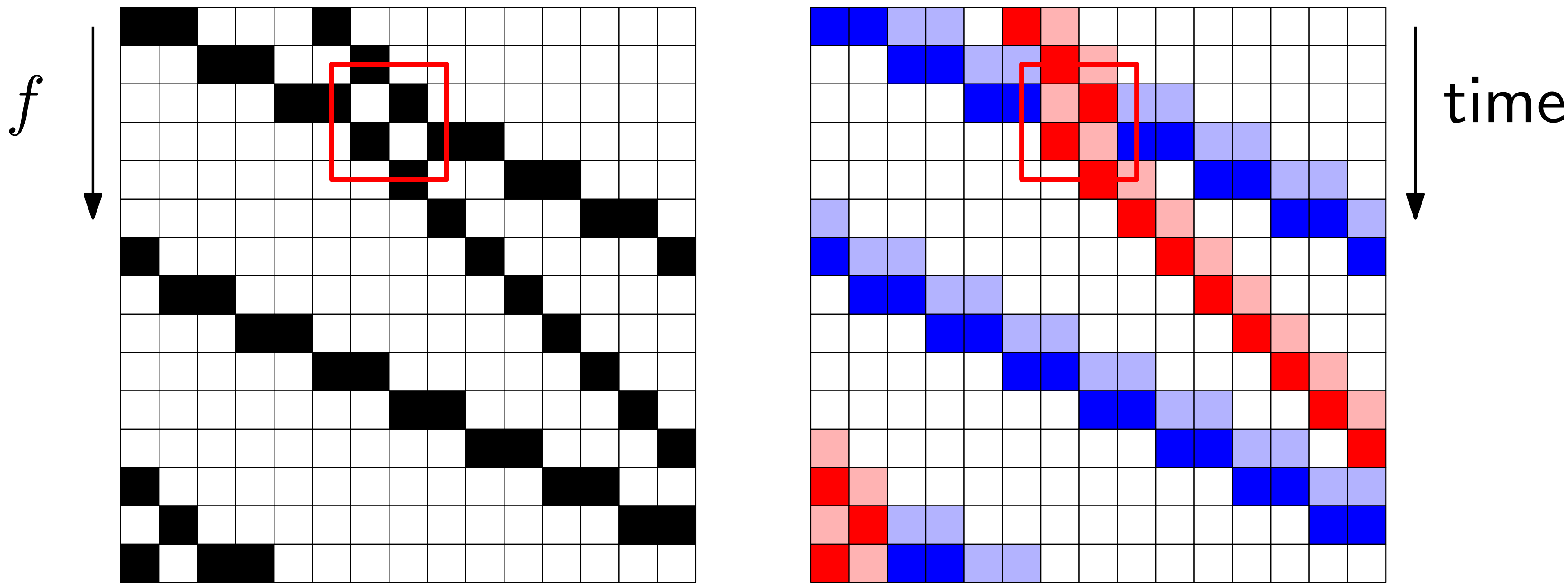

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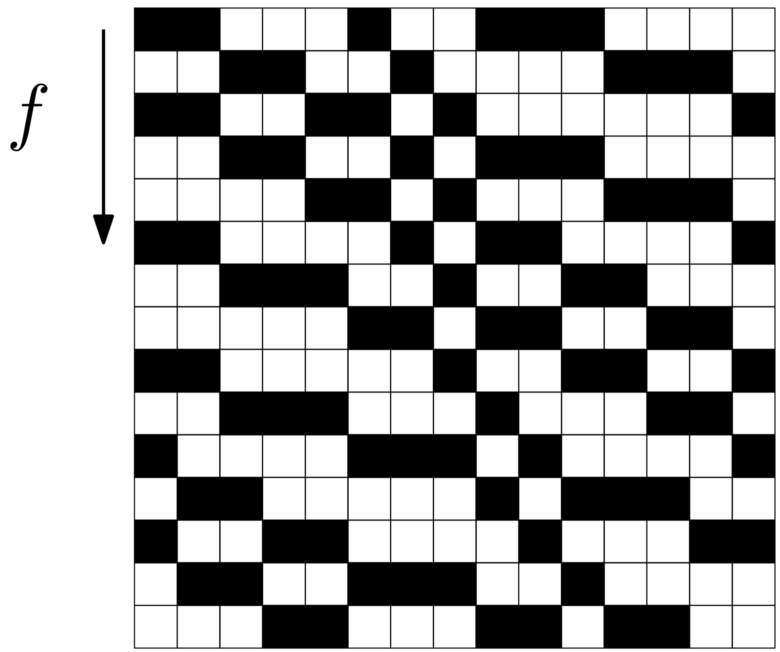
energy conservation!

$c_{1,2} :=$ number of overtakings

Glider partition

$(n, k) = (15, 6)$



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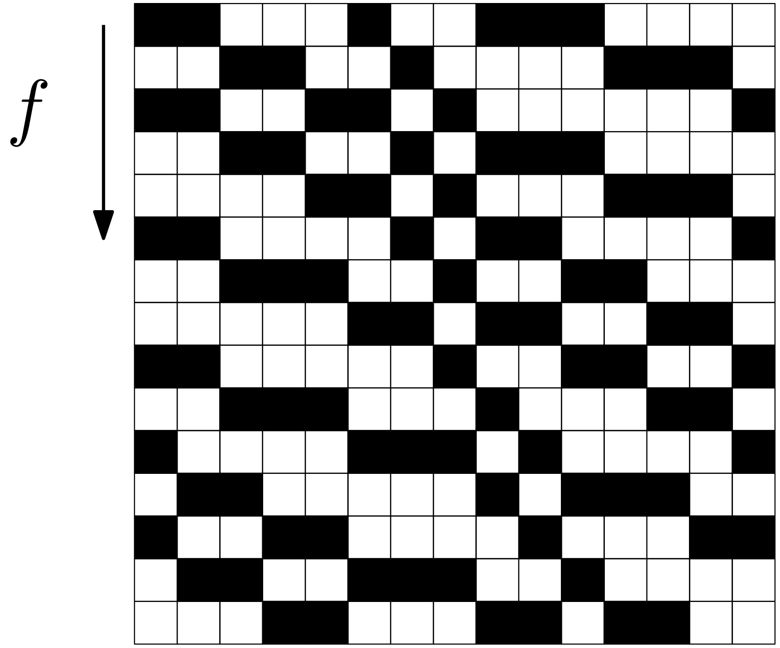

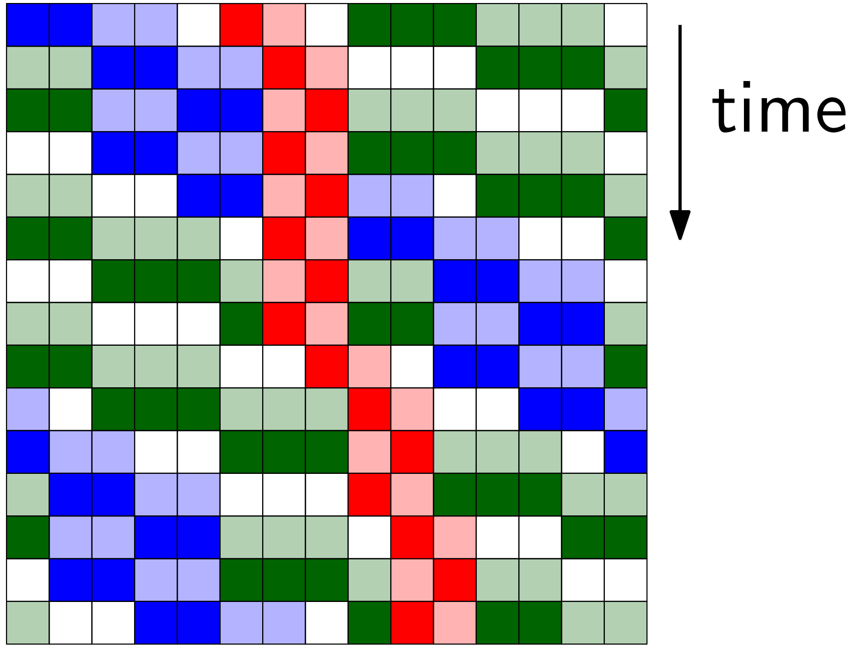

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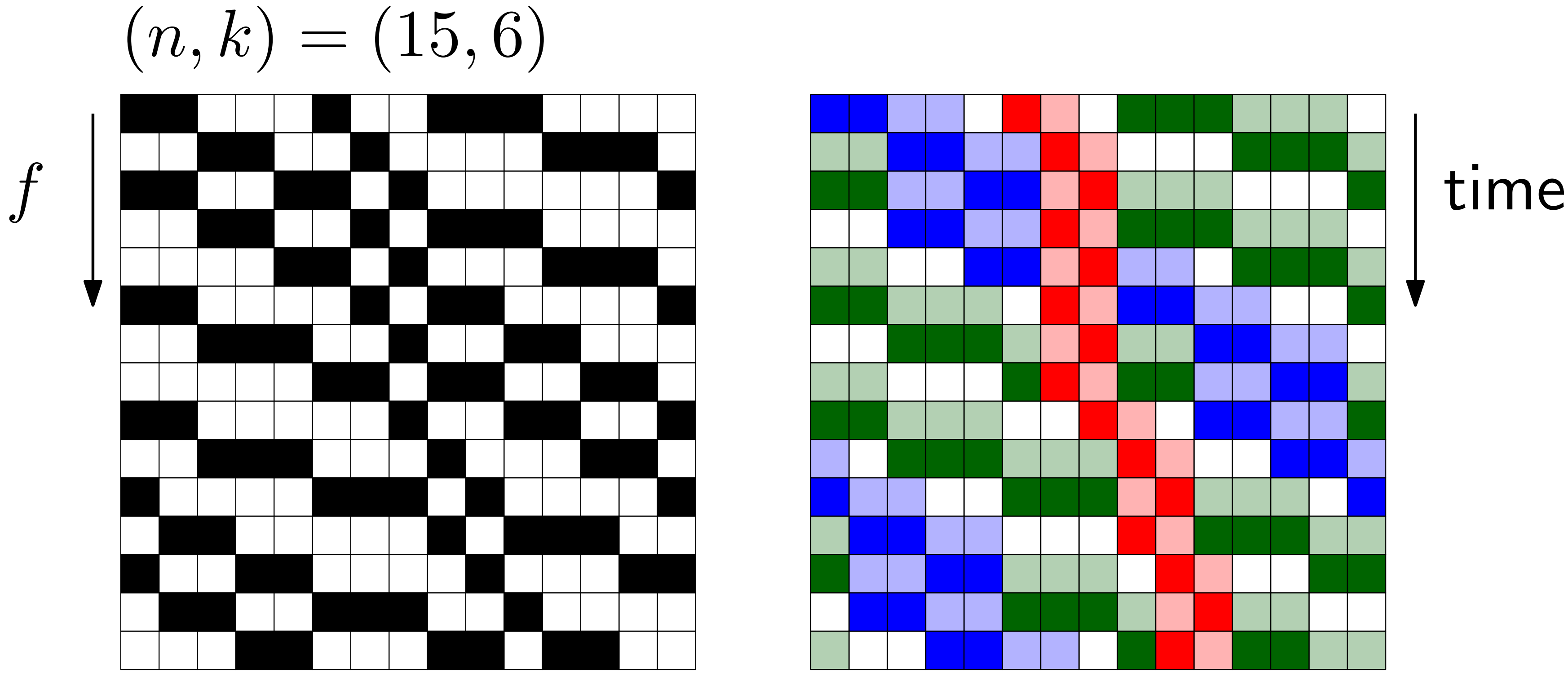

- gliders can be interleaved in complicated ways

Glider partition

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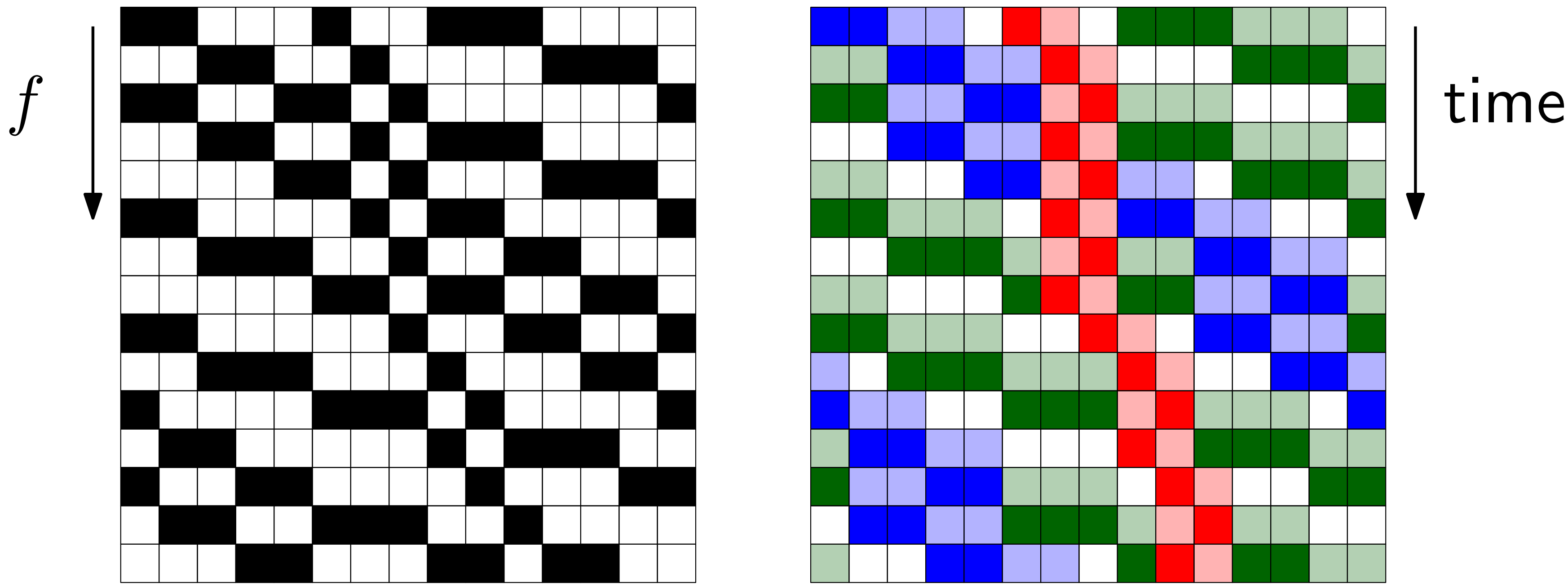

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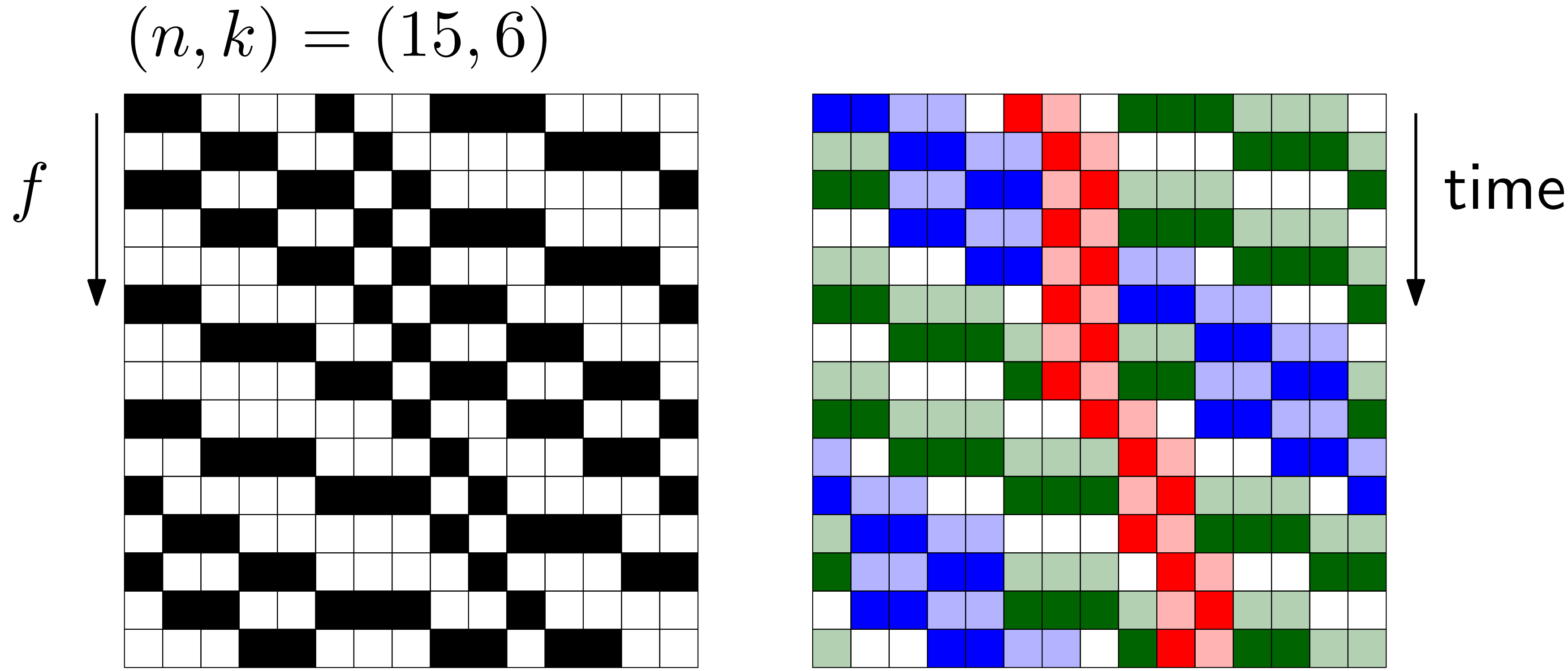
Glider partition



- gliders can be interleaved in complicated ways
- general glider partition rule works recursively on Motzkin path
- general equations of motion have overtaking counters $c_{i,j}$ for all pairs of gliders i, j

Cycle invariant

- **Lemma:** For any cycle in $K(n,k)$ defined by f, the set of gliders is invariant.

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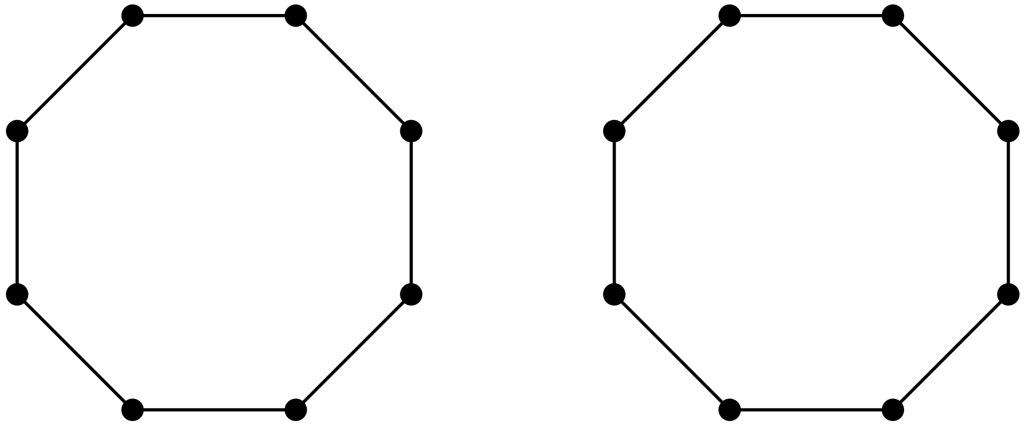
- don't know number of cycles

Gluing cycles



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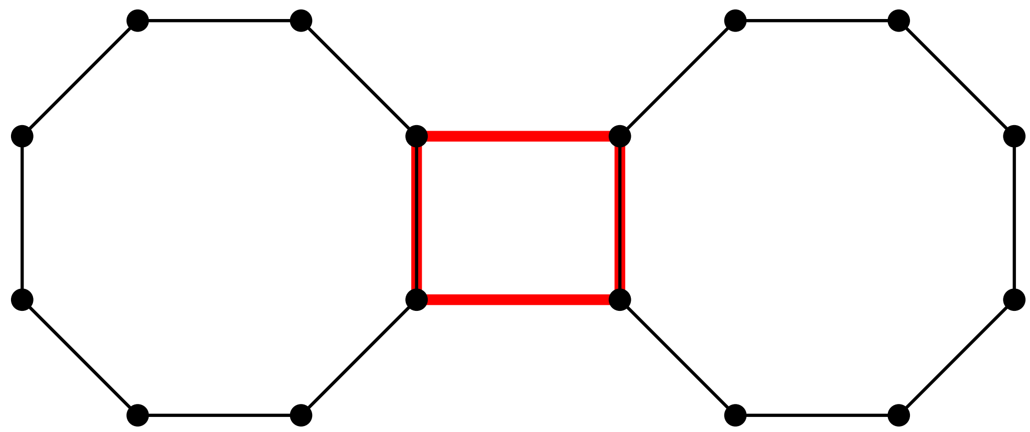

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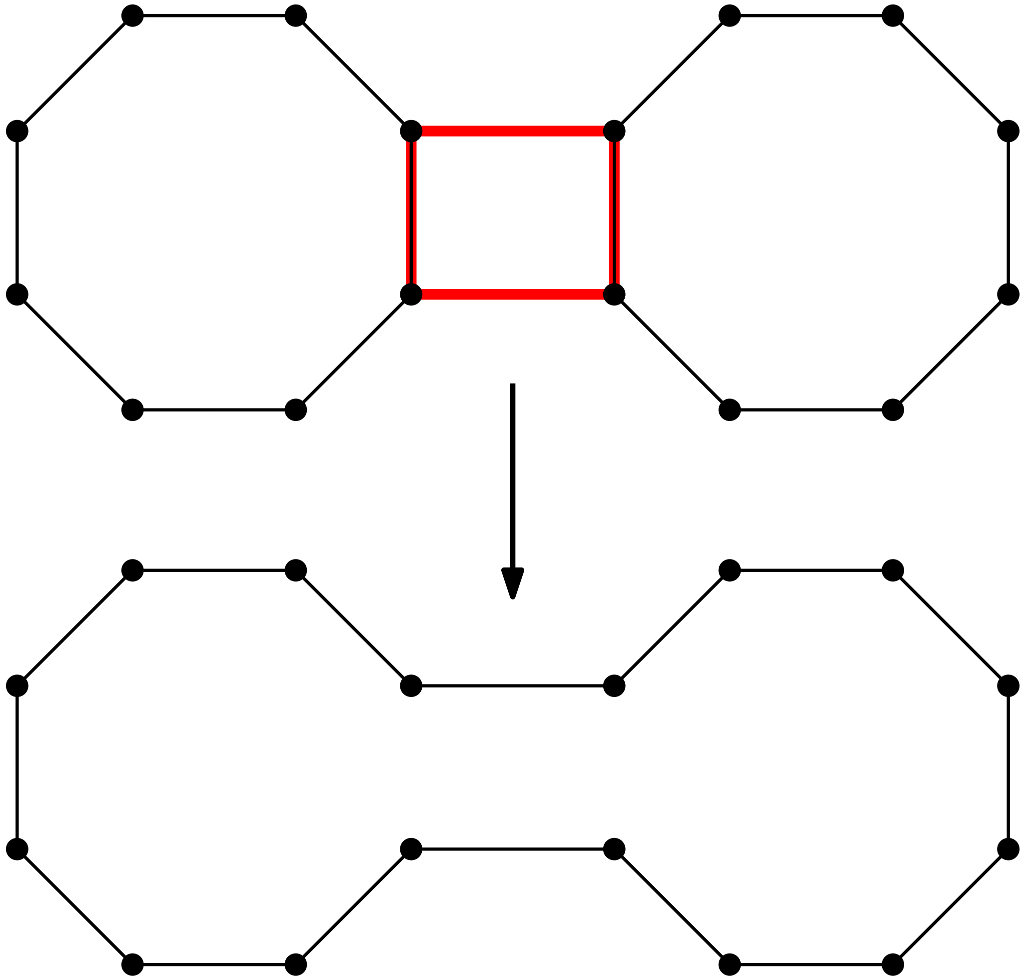

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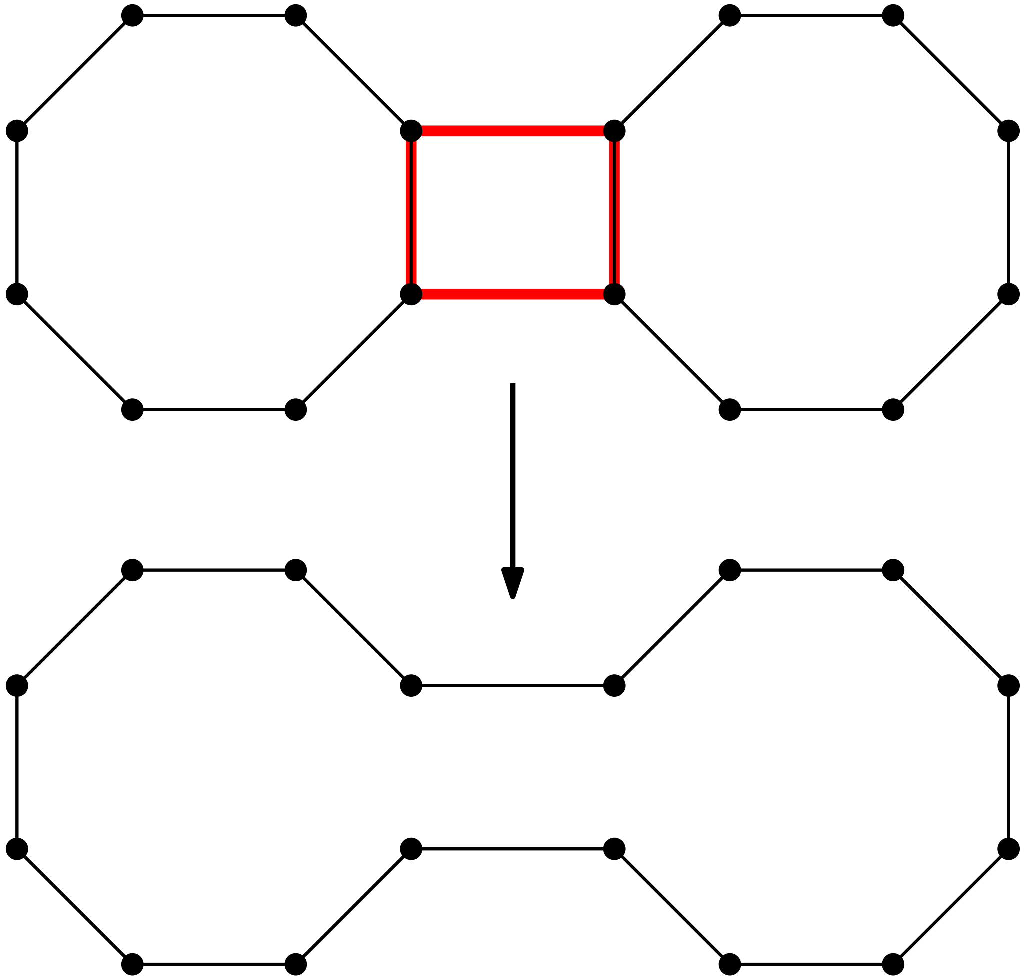
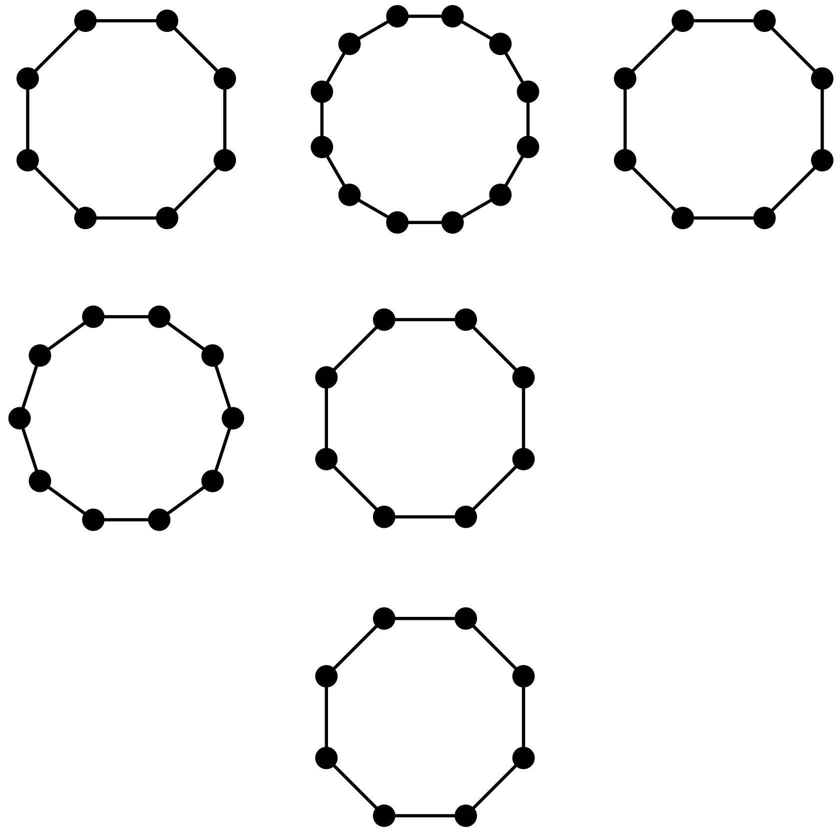


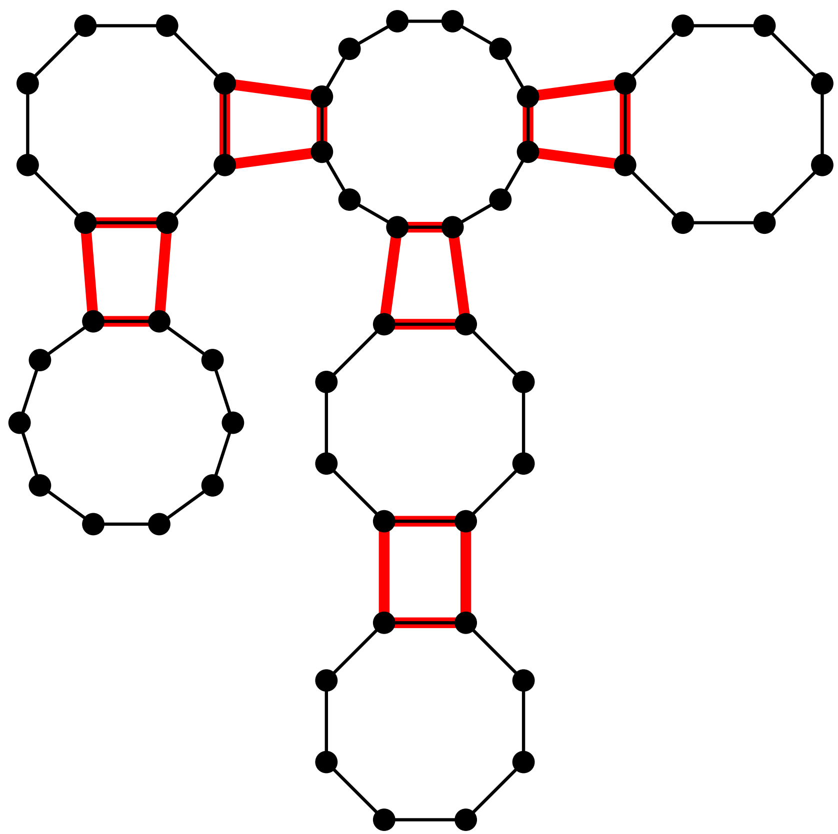
4-cycles exist as $n \geq 2k + 3$

Gluing cycles



Gluing cycles

- connect cycles of factor to a single Hamilton cycle (tree-like)



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- connect cycles of factor to a single Hamilton cycle (tree-like)
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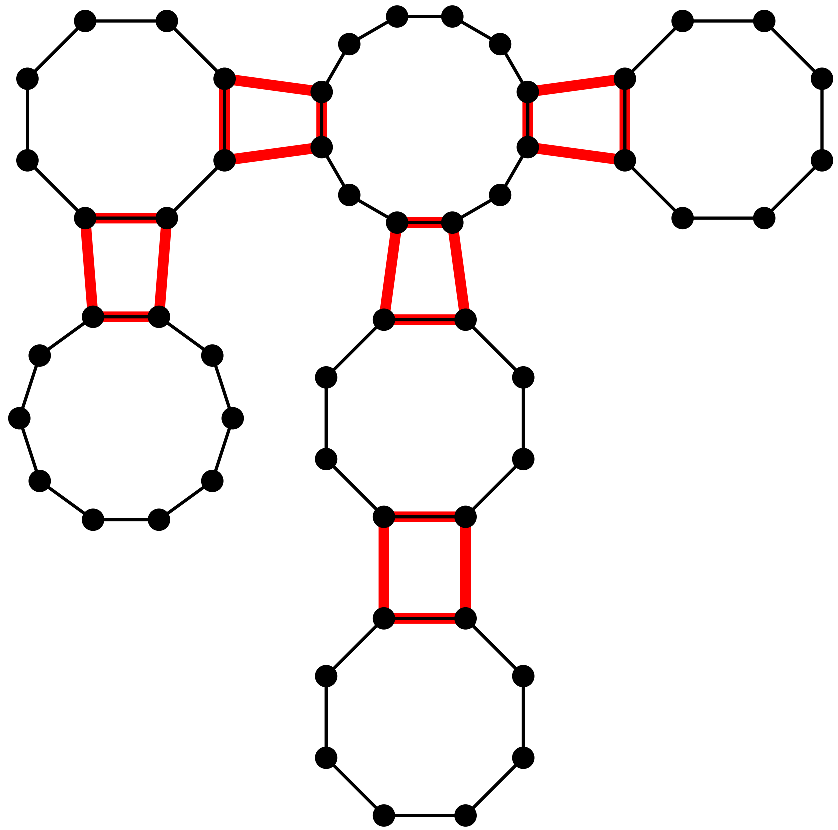

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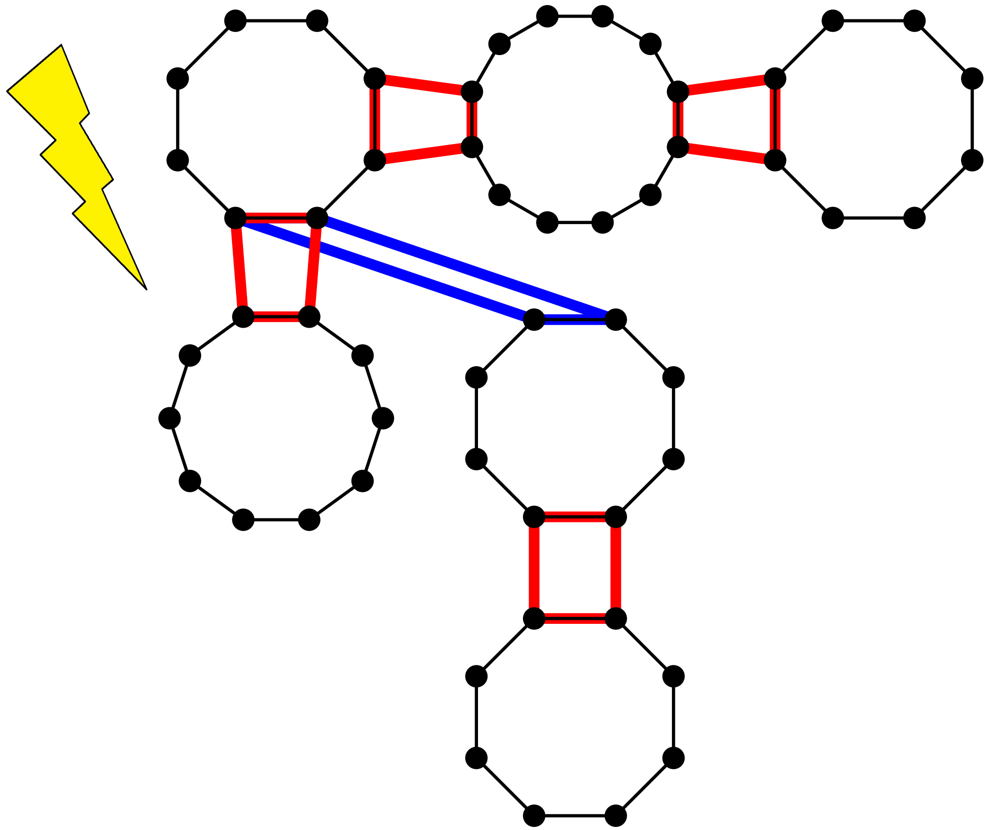

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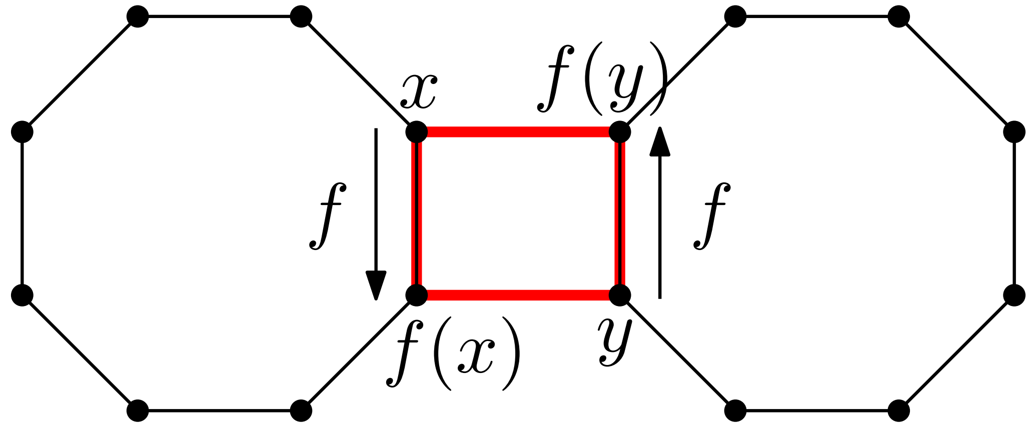

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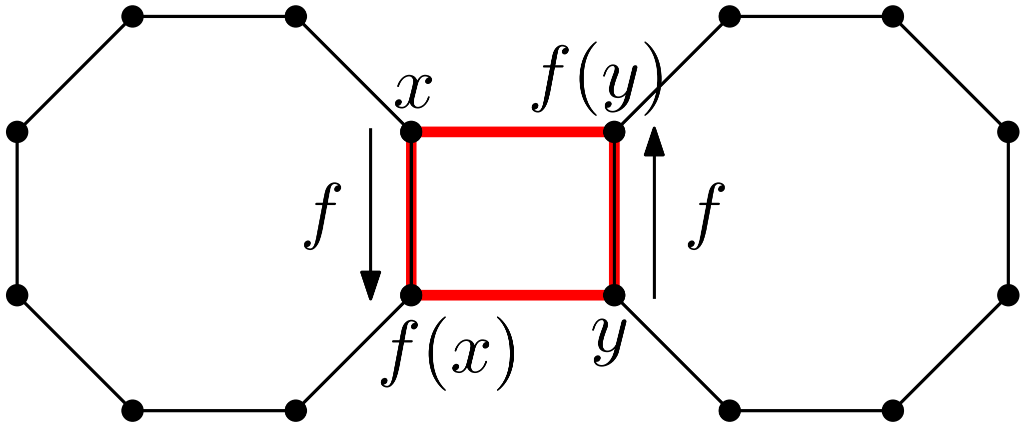
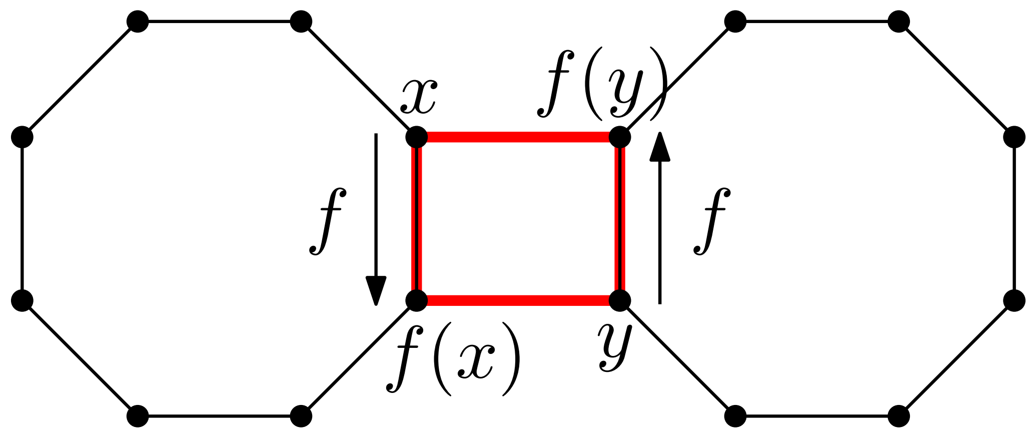


- **Lemma:** If x and y differ in an exchange of one outer matched pair of parenthesis, then $(x, f(x), y, f(y))$ is a gluing 4-cycle in $K(n, k)$.

Gluing cycles



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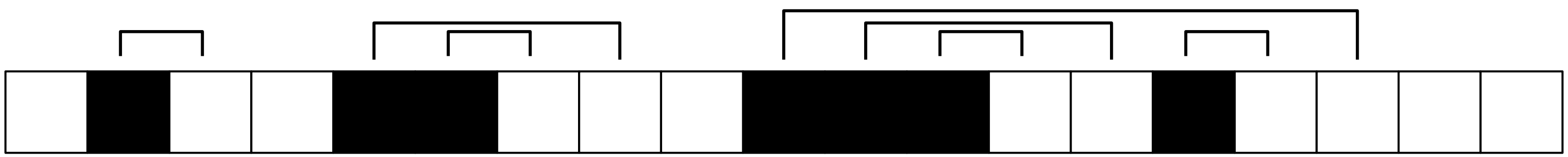
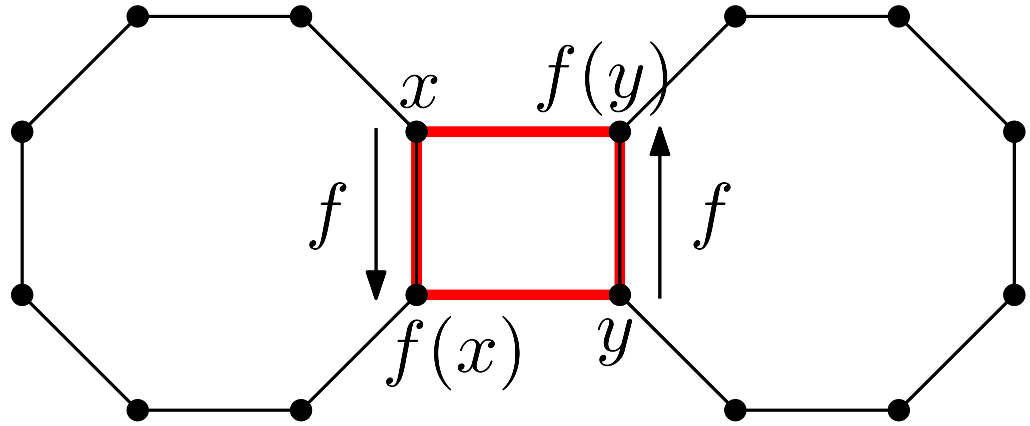
x  speeds 3,2,1,1

Gluing cycles



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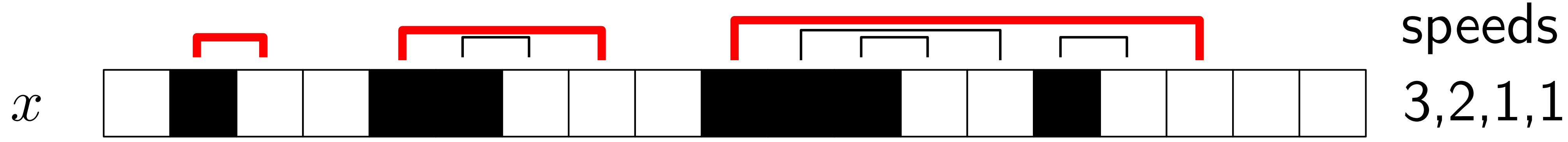

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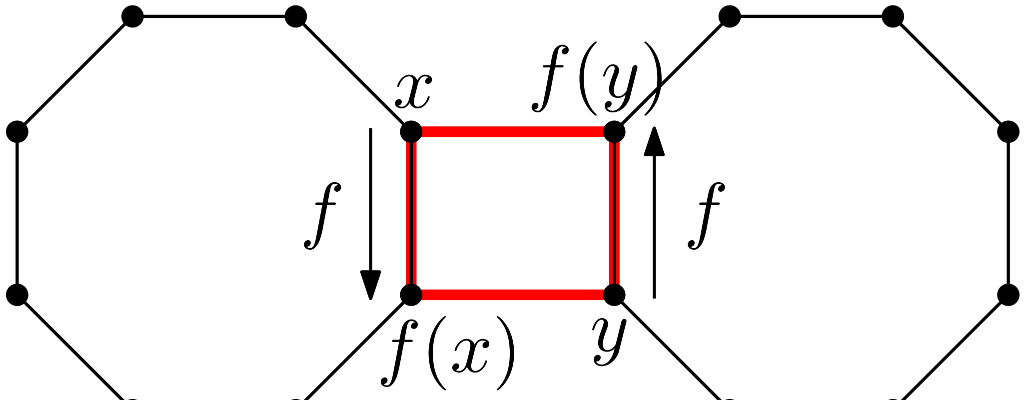

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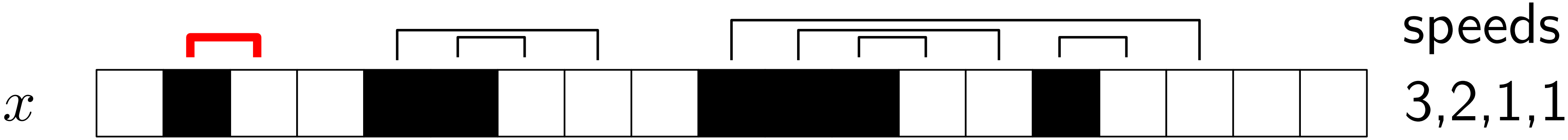

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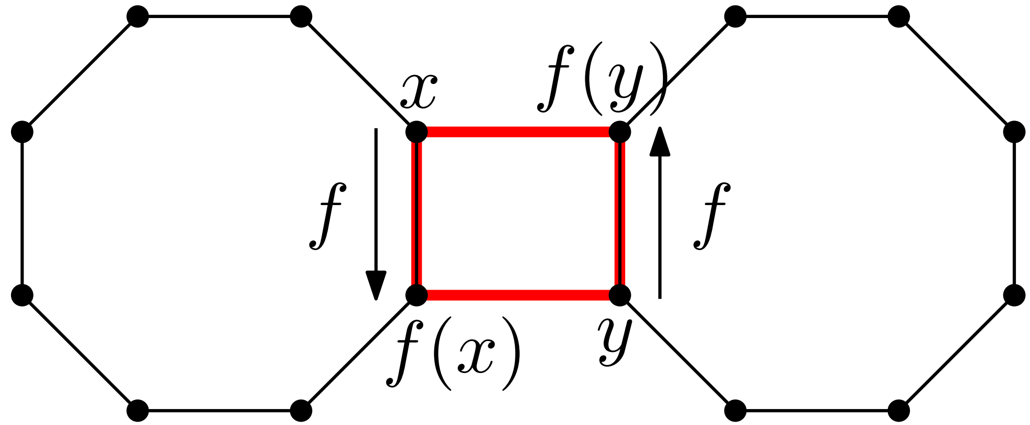

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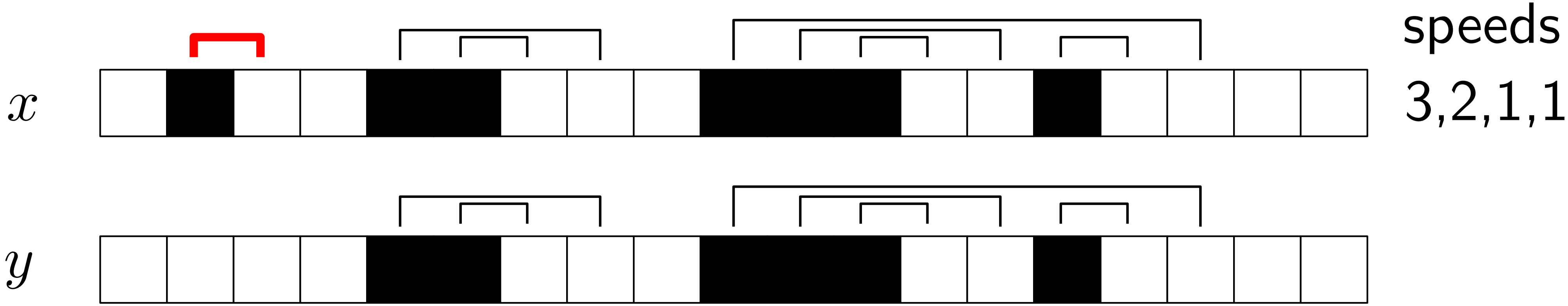

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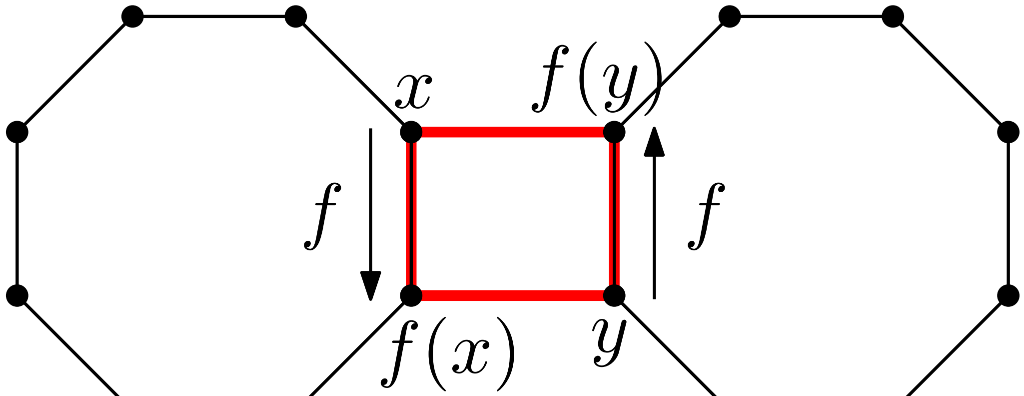

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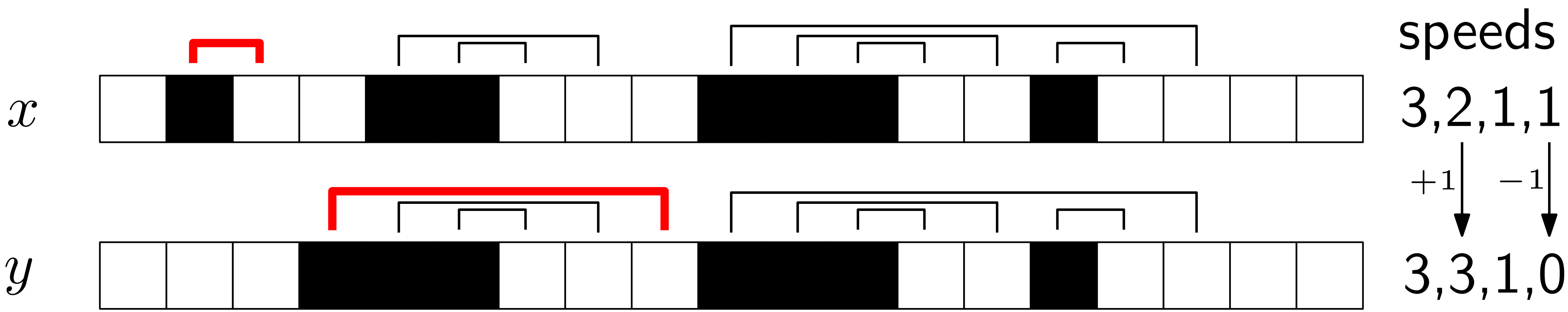
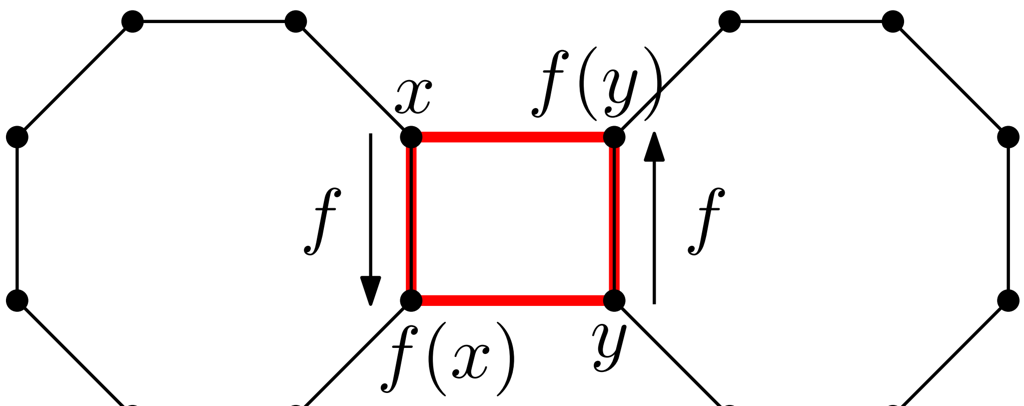


Gluing cycles



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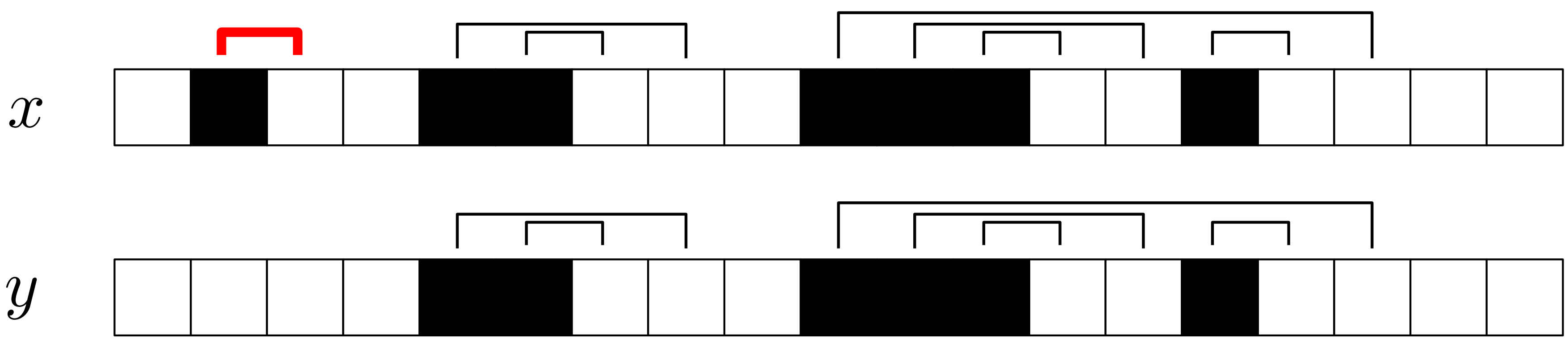
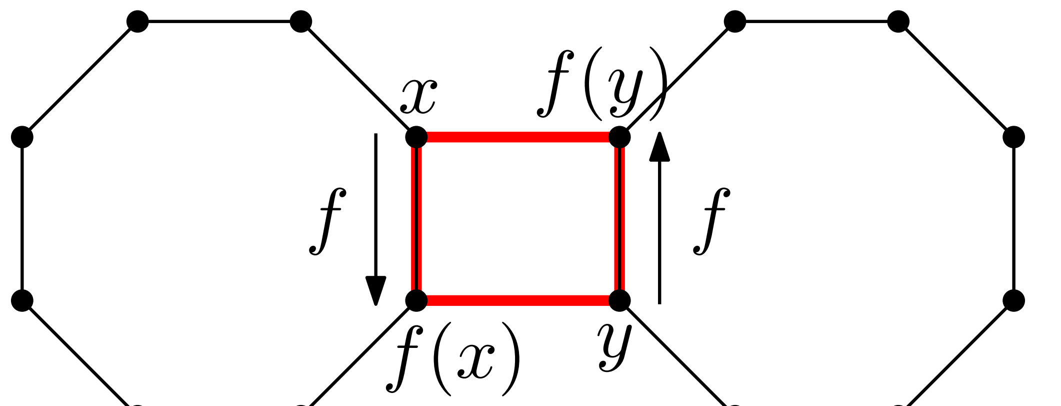


speeds
3,2,1,1

Gluing cycles



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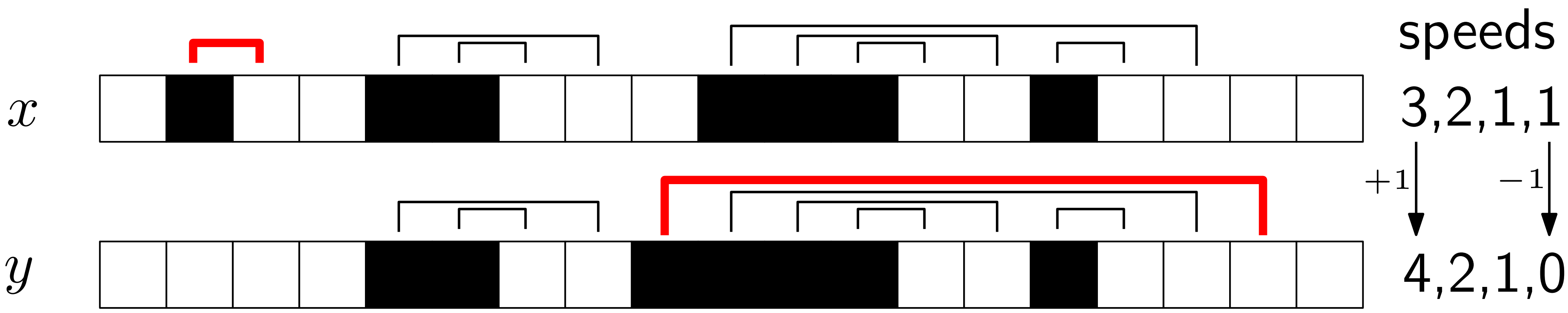

Gluing cycles

- decrease speed of slowest glider in x by 1, increase speed of another glider by 1

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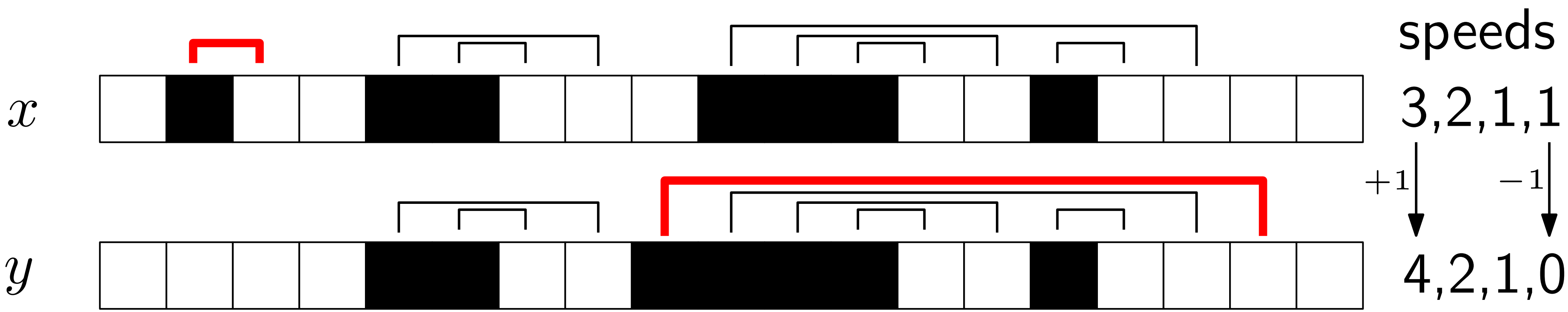

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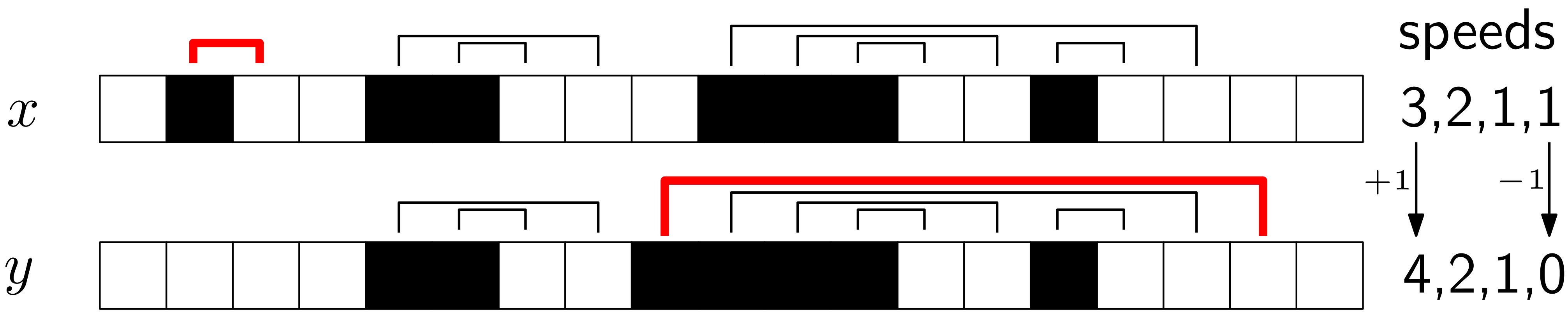

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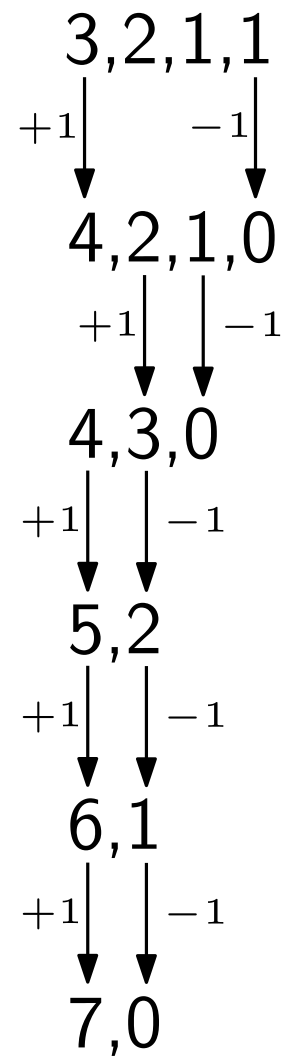

Gluing cycles

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- number partition of $x <_{\text{lex}}$ number partition of y
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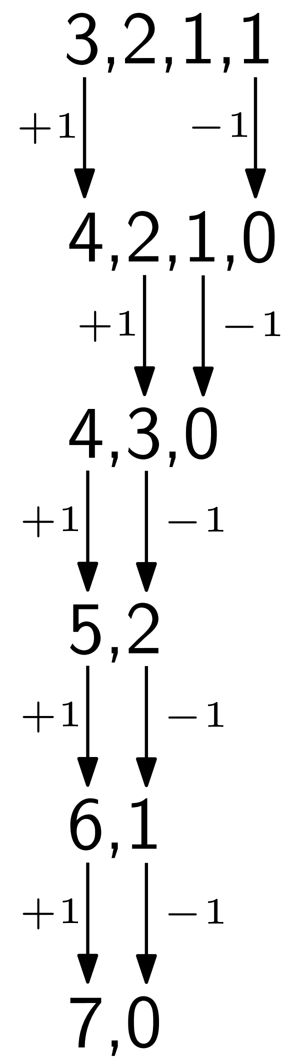
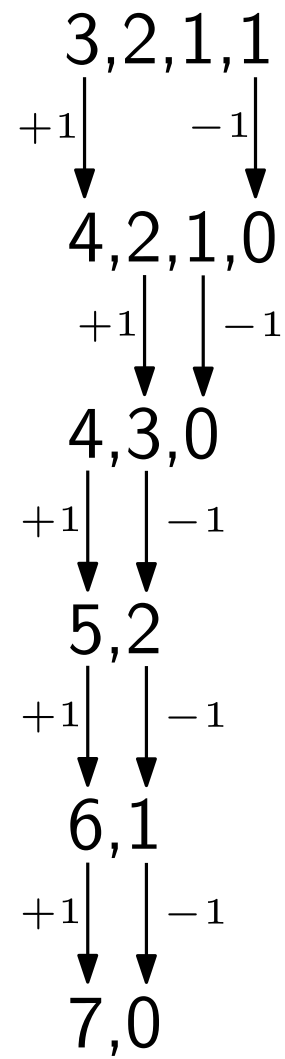


Gluing cycles

- decrease speed of slowest glider in x by 1, increase speed of another glider by 1
- number partition of $x <_{\text{lex}}$ number partition of y
- sequence of gluing cycles to connect to cycle with lex. largest number partition k
- proves connectivity



Open questions

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- stronger Hamiltonicity properties: Hamilton-connectedness, factorization into HCs

Thank you!