The Newton Polytope and Lorentzian Property of Chromatic Symmetric Functions

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UMass, Amherst

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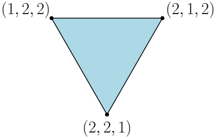
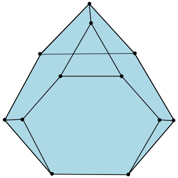
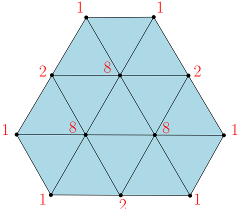
joint with J. Matherne and J. Selover







Outline

Newton polytopes

Saturated Newton Polytopes of CSFs

Lorentzian property of CSFs

“ Polynomials and Power series,
May they forever rule the world!
. . .
You cannot conquer us with the rings of Chow
And shrieks of Chern!
For we too are armed, with *Polygons of Newton*
And Algorithms of Perron!
. . .”

Shreeram S. Abhyankar 1970

Newton Polytopes

- For a polynomial $p = \sum_\alpha c_\alpha \mathbf{x}^\alpha \in \mathbf{R}[x_1, \ldots, x_n]$,

$$\mathrm{supp}(p) = \{\alpha \mid c_\alpha \neq 0\} \subset \mathbf{N}^n$$

- The **Newton polytope** of p is

$$\mathrm{Newton}(P) = conv(\alpha \mid \alpha \in \mathrm{supp}(p)).$$

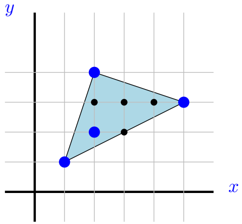
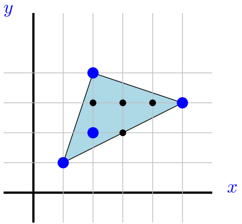


Figure: $x^1y^1 + x^2y^2 + x^2y^4 + x^5y^3$

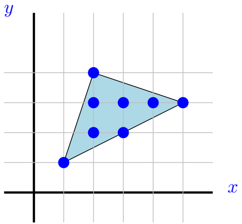
Saturated Newton Polytope (Monical–Tokcan–Yong 2017)

- p has a **saturated Newton polytope** ("is SNP") if

$$\mathrm{supp}(p) = \mathrm{Newton}(p) \cap \mathbf{N}^n$$



(a) $x^1y^1 + x^2y^2 + x^2y^4 + x^5y^3$



(b) $x^1y^1 + x^2y^2 + x^2y^3 + x^2y^4 + x^3y^2 + x^3y^3 + x^4y^3 + x^5y^3$

Figure: An unsaturated and a saturated Newton polytope.

Examples of SNP in Algebraic Combinatorics

Lots of polynomials in algebraic combinatorics are SNP:

- Schur functions s_λ (Rado 1952)
- Stanley symmetric functions (Monical–Tokcan–Yong 2017)
- Schubert polynomials (Fink–Mészáros–St. Dizier 2017)
- Any **Lorentzian** function automatically has SNP (Bränden–Huh)

Why Study SNP?

If p is SNP, $\mathrm{supp}(p)$ has a *hyperplane description* via the Newton polytope, which may give a quick algorithm for deciding if a coefficient of p is zero or nonzero.

$$s_{(2,2,1)}(x_1, x_2, x_3, x_4)$$

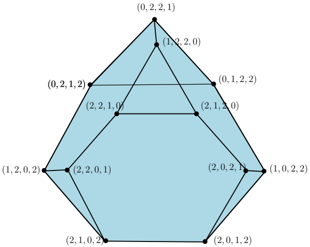


Figure: The permutahedron $P^{(4)}_{(2,2,1)}$.

Chromatic Symmetric Function (Stanley 1995)

- For G a graph with vertices $\{1, \ldots, n\}$, the **chromatic symmetric function** is

$$X_G(\mathbf{x}) = \sum_{\substack{f:V(G)\to\mathbf{N} \\ f \text{ proper}}} x_{f(1)} \cdots x_{f(n)}.$$

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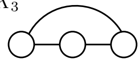
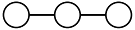
- Example

K_3



$$X_{K_3} = 6(x_1x_2x_3 + x_1x_2x_4 + \cdots) \\ = 6m_{1,1,1}$$

P_3



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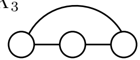

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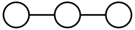

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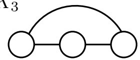

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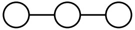

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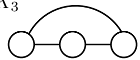

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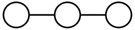

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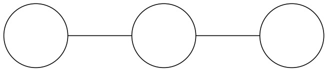
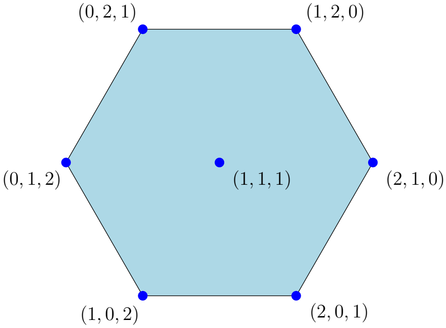
- We can restrict to k variables, setting the rest to 0, $X_G(x_1,\ldots,x_k) \in \mathbf{N}[x_1,\ldots,x_k]$
- $X_G(\underbrace{1,\ldots,1}_{k},0,\ldots) = \chi_G(k)$
- Generalization to noncommuting variables by Gebhard–Sagan 2001

Support of a Chromatic Symmetric Function

- Given a proper coloring f, its **weight** is $\mathrm{wt}(f) = (|f^{-1}(1)|, \dots, |f^{-1}(n)|)$
- $\mathrm{supp}(X_G) = \{\mathrm{wt}(f) \mid f \text{ proper}\}$
- Example



$$\mathrm{supp}(X_{P_3}(x,y,z)) = \{(1,1,1),(2,1,0),(1,2,0),(2,0,1),(1,0,2),(0,2,1),(0,1,2)\}$$



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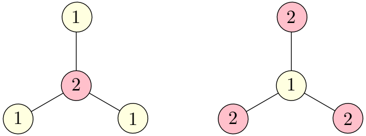
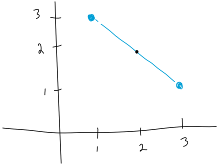
Are Chromatic Symmetric Functions SNP?

Recall $\mathrm{supp}(X_G) = \{\mathrm{wt}(f) \mid f \text{ proper}\}$. Let G be the claw graph.





$$(3,1) \in \mathrm{supp}(X_G(x,y))$$
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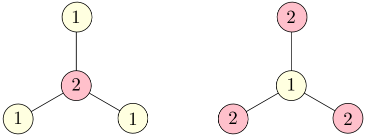

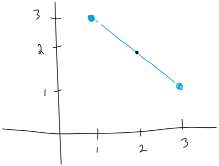

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- The claw graph is not SNP (Monical 2018)

Chromatic symmetric functions of claw-free graphs

- The claw graph is not SNP (Monical 2018)

Conjecture (Stanley 1996)

If G is claw-free then X_G is positive in the Schur basis.

- Conjecture is true for claw-free incomparability graphs (Gasharov 96)

Conjecture (Monical 2018)

If X_G is Schur positive then $X_G(x_1, \ldots, x_k)$ is SNP.

- unlikely both conjectures are true (Adve–Robichaux–Yong 19).
- the latter conjecture has to be tested on G with ≥ 12 vertices.

Claw-free incomparability graphs

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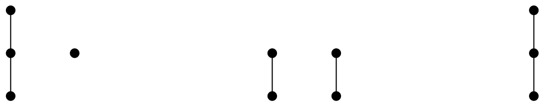
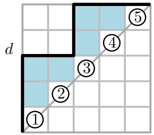
P is 3-**free** if P has no subposet that is a 3-chain, i.e. with $G(P)$ a **co-bipartite graph**.



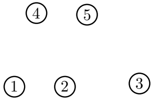
Claw-free Graphs From Dyck Paths

P is **(3+1) and (2+2)-free** if P also has no subposet that is a 3-chain and an incomparable element AND two disjoint 2-chains. In bijection with Dyck paths.

For d a Dyck path of length $2n$, let $G(d)$ be the graph with vertices $[n]$ and edges (i, j), $i < j$ for each cell (i, j) below the path d.



$G(d)$



P

Stanley–Stembridge conjecture

Conjecture (Stanley–Stembridge 1993)

Let P be a (3+1)-free poset, then $X_{G(P)}$ is positive in the elementary basis.

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Theorem (Guay-Paquet 2013)

Suffices to verify the conjecture on (3+1) and (2+2)-free posets, i.e. for $G(d)$ for Dyck paths d.

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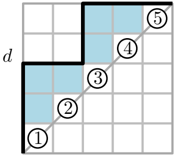

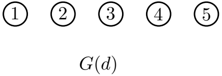

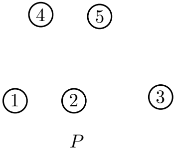

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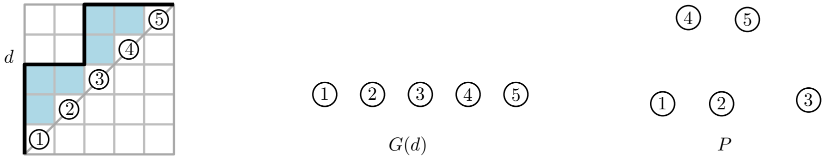

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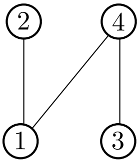
Theorem (Brosnan–Chow 2015, Guay-Paquet 2016)

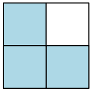
Let d be a Dyck path of length $2n$, then $X_{G(d)}$ encodes a $\mathfrak{S}_n$-representation of Tymoczko on the cohomology of a **regular semisimple Hessenberg variety** *associated to d.*

Greedy coloring co-bipartite graphs

A co-bipartite graph G vertices $\{1,2,\ldots,n_1\} \cup \{n_1+1,\ldots,n_1+n_2\}$ corresponds to a **board** $B \subset [n_1] \times [n_2]$ with (i,j) in B iff (i, n_1+j) *not* in G.



G



B

Proposition (Stanley–Stembridge)

$$X_G = \sum_i i! \cdot (n_1+n_2-2i)! \cdot r_i(B) \cdot m_{2^i 1^{n_1+n_2-2i}},$$

where $r_i(B)$ is the number of placements of i non-attacking rooks on B.

Let $\lambda^{gr}(G) = 2^i 1^{n_1+n_2-2i}$ for max i such that $r_i(B) \neq 0$.

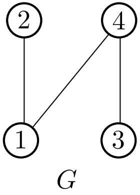
X_G for co-bipartite graphs are SNP

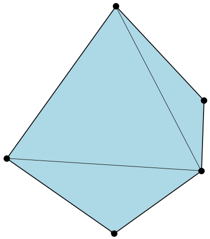
Proposition

Let G be a co-bipartite graph, then $X_G(x_1, \ldots, x_k)$ is SNP, and its Newton polytope is $P^{(k)}_{\lambda^{gr}(G)}$.

Example

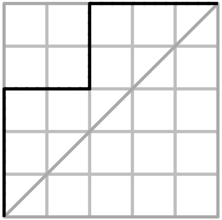
$X_G = 24m_{1111} + 6m_{211} + 2m_{22}$, and Newton$(X_{G(d)}(x_1, x_2, x_3, x_4)) = \mathcal{P}^{(4)}_{22}$.

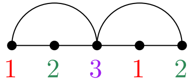




Greedy coloring of a Dyck path

- If d is a Dyck path, let $\mathrm{gr}(d)$ be the greedy coloring of $G(d)$
- Define $\lambda^{gr}(d) = \mathrm{wt}(\mathrm{gr}(d))$





$$\lambda^{gr}(d) = (2, 2, 1)$$

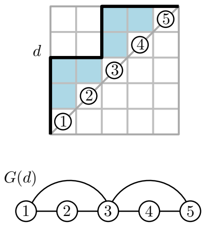
X_G for Dyck paths are SNP

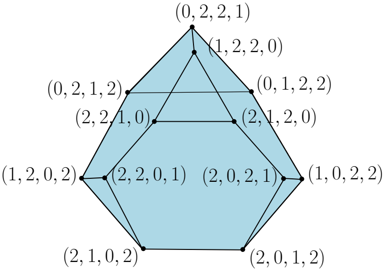
Theorem (Matherne–M–Selover 2022)

Let d be a Dyck path, then $X_{G(d)}(x_1, \ldots, x_k)$ is SNP, and its Newton polytope is $P^{(k)}_{\lambda^{gr}(d)}$.

Example

$X_{G(d)} = 36s_{(1,1,1,1,1)} + 16s_{(2,1,1,1)} + 4s_{(2,2,1)}$





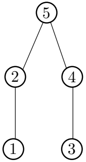
X_G for (3+1)-free posets is SNP

Theorem (Matherne–M–Selover 2022)

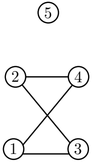
Let P be a (3+1)-free poset, then $X_{G(P)}(x_1,\ldots,x_k)$ is SNP, and its Newton polytope is $P^{(k)}_{\lambda^{gr}(P)}$.

Example

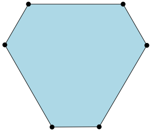
$X_{G(P)} = 120m_{1^5} + 36 * m_{21^3} + 10m_{2^21} + 4m_{31^2} + 2m_{32}$ and $\text{Newton}(X_{G(P)}(x_1,x_2,x_3)) = \mathcal{P}^{(3)}_{32}$



P



$G(P)$



About the proof for Dyck paths

- Since $G(d)$ is claw-free, whenever $\lambda \in \mathrm{supp}(X_{G(d)}(x_1, \ldots, x_k))$ then integerPoints$(\mathcal{P}_{\lambda}^{(k)}) \subset \mathrm{supp}(X_{G(d)}(x_1, \ldots, x_k))$ (Stanley 1998)

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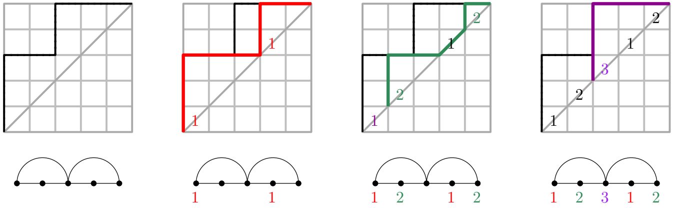
- Main tool: **Bounce path** characterization of greedy coloring

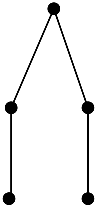


About the proof for (3+1)-free posets: listings

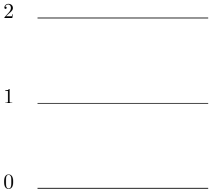
Theorem (Guay-Paquet–M–Rowland 2013)

- *A (3+1)-free poset P can be represent by a* **part listing***: certain word L on alphabet $\{v_0, v_1, \ldots,\} \cup \{b_{i,i+1}(H) \mid H \text{ bicolored graph}\}$.*
- *If L has no $b_{i,i+1}(H)$ then P is (3+1) and (2+2)-free.*

Example

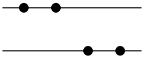


P

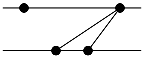


About the proof for (3+1)-free posets: modular law

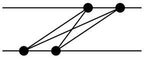
The **basic bicolored graphs** for $j = 0, 1, \ldots, s$ are $U_j^{(i)} := v_{i+1}^{s-j} v_i^r v_{i+1}^j$.



U_0



U_1



U_2

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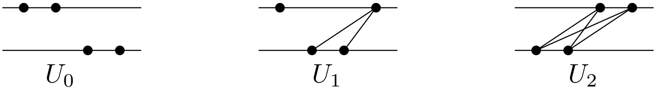

Theorem (Guay-Paquet 2013)

Let L be part listing of poset P_L with bicolored graph $b_{i,i+1}(H)$, then $X_{G(L)}$ is a convex combination *of $X_{G(L_j)}$ where L_i is obtained from L by replacing $b_{i,i+1}(H)$ by $U_j^{(i)}$.*

$$X_{G(L)} = \sum_{j=0}^{r} q_j X_{G(L_j)}$$

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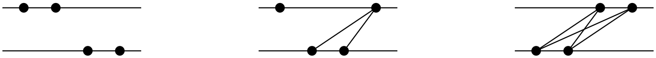

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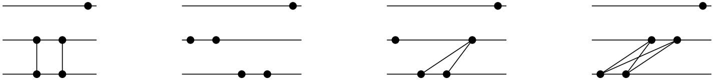

L L_0 L_1 L_2

About the proof for (3+1)-free posets: greedy weight

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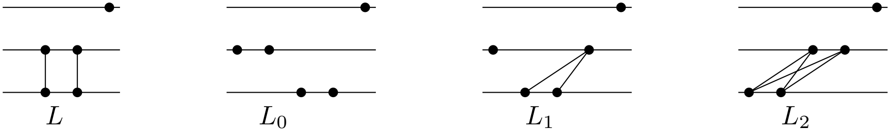

About the proof for (3+1)-free posets: dominance lemma

Lemma (Matherne–M–Selover)

Let $j < k$ and part listings $L_j = Ab_{i,i+1}(U_j)B$ and $L_k = Ab_{i,i+1}(U_k)B$ then $supp(X_{G(L_j)}) \subseteq supp(X_{G(L_k)})$.

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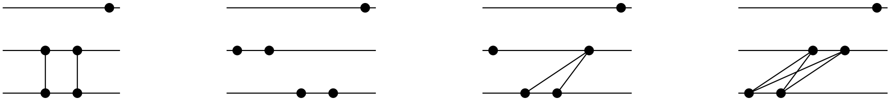
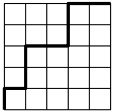


L $\quad L_0 \quad L_1 \quad L_2$



Outline

Newton polytopes

Saturated Newton Polytopes of CSFs

Lorentzian property of CSFs

M-convexity

$I \subset \mathbb{Z}^k$ is **M-convex** if it has the **exchange property**: for any i and $\alpha, \beta \in I$ with $\alpha_i > \beta_i$ there is j such that

$$\alpha_j < \beta_j \quad \text{and} \quad \alpha - e_i + e_j \in I \quad \text{and} \quad \beta - e_j + e_i \in I.$$

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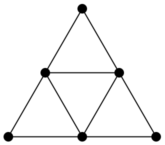
Example

supp$(X_G(x_1, \ldots, x_6))$ is not M-convex since $(1,1,1,3,0,0)$ and $(0,0,2,2,2,0)$ are in support but not

$$(0,0,2,2,2,0) - e_i + e_4.$$



Lorentzian property

Definition (Brändén–Huh 2020)

A homogeneous polynomial $f \in \mathbb{R}_{\geq 0}[x_1, \dots, x_k]$ of degree n is **Lorentzian** if

- $\mathrm{supp}(f)$ is M-convex, and
- the quadratic form of
$$\frac{\partial}{\partial x_{i_1}} \circ \cdots \circ \frac{\partial}{\partial x_{i_{n-2}}}(f)$$
has at most one positive eigenvalue for all $i_1, i_2, \dots, i_{n-2} \in [k]$.

Lorentzian property and log-concavity

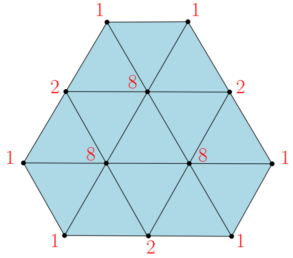
Theorem (Brändén–Huh 2020)

Let $f = \sum_{\alpha \in \Delta_k^n} c_\alpha \mathbf{x}^\alpha$ *be a Lorentzian polynomial. Then*

$$(\alpha!)^2 c_\alpha^2 \geq (\alpha + e_i - e_j)!(\alpha - e_i + e_j)! \cdot c_{\alpha+e_i-e_j} c_{\alpha-e_i+e_j} \text{ for all } i, j \text{ in } [k] \text{ and } \alpha \text{ in } \Delta_k^n,$$

and thus

$$c_\alpha^2 \geq c_{\alpha+e_i-e_j} c_{\alpha-e_i+e_j} \text{ for all } i, j \text{ in } [k] \text{ and } \alpha \text{ in } \Delta_k^n.$$



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Example

$s_2(x_1, x_2) = x_1^2 + x_1 x_2 + x_2^2$ has matrix $\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1 \end{bmatrix}$.

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$s_2(x_1, x_2) = \frac{1}{2}x_1^2 + x_1 x_2 + \frac{1}{2}x_2^2$ has matrix $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$.

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- Schubert polynomials $N(\mathfrak{S}_w(x_1, \ldots, x_k))$ (conjecture Huh–Matherne–Mészáros–St. Dizier 2019)

Main conjecture

Conjecture (Matherne–M–Selover 2022)

Let d be a Dyck path. Then $X_{G(d)}(x_1, \ldots, x_k)$ is Lorentzian.

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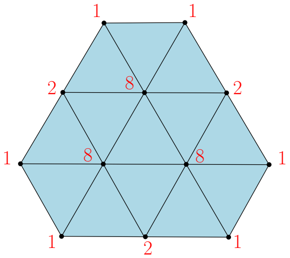
- Verified for Dyck paths of size $n \leq 7$ with $k \leq 8$ variables.

Example

For $d = nneneene$, $\lambda^{gr}(d) = (3, 1)$, $X_{G(d)} = 24m_{1111} + 8m_{211} + 2m_{22} + m_{31}$, and Newton$(X_{G(d)}(x_1, \ldots, x_k)) = \mathcal{P}_{31}^{(k)}$.



Main conjecture

Conjecture (Matherne–M–Selover 2022)

Let d be a Dyck path. Then $X_{G(d)}(x_1, \ldots, x_k)$ is Lorentzian.

- Verified for Dyck paths of size $n \leq 7$ with $k \leq 8$ variables.
- Not true for other incomparability graphs.

Example

Let P be the (2+2)-poset so $G(P) = C_4$ and $X_{C_4} = 24m_{1111} + 4m_{211} + 2m_{22}$.

Now $f = X_{C_4}(x_1, \ldots, x_5)$ is not Lorentzian. Quadratic form of $\frac{\partial}{\partial x_1} \circ \frac{\partial}{\partial x_2} f$ has matrix

$$A = \begin{pmatrix} 0 & 8 & 8 & 8 & 8 \\ 8 & 0 & 8 & 8 & 8 \\ 8 & 8 & 8 & 24 & 24 \\ 8 & 8 & 24 & 8 & 24 \\ 8 & 8 & 24 & 24 & 8 \end{pmatrix}, \quad \text{eigenvalues } 32 + 8\sqrt{15}, 32 - 8\sqrt{15}, \ldots$$

Abelian Dyck paths

Definition (Harada–Precup 2019)

An **Abelian Dyck path** is a path d such that $G(d)$ is co-bipartite.

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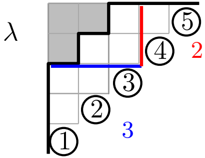
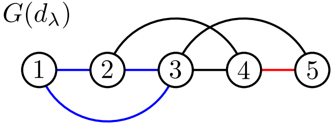
Encoded by $\lambda \subset n_1 \times n_2$.





$X_{G(d)}$ of Abelian Dyck paths are Lorentzian

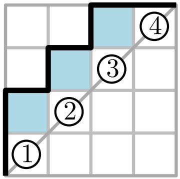
Theorem (Matherne–M–Selover 2022)

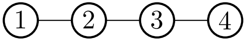
Let d be an abelian Dyck path. Then $X_{G(d)}(x_1, \ldots, x_k)$ is Lorentzian.

Example

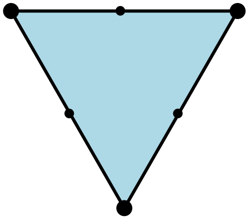
$X_G = 24m_{1111} + 6m_{211} + 2m_{22}$. For $f = X_G(x_1, x_2, x_3)$, quadratic form of $\frac{\partial}{\partial x_3} f = 4x_1^2 + 12x_1x_2 + 4x_2^2$ has matrix $\begin{bmatrix} 8 & 12 \\ 12 & 8 \end{bmatrix}$.

d





$G(d)$



About proof: block matrix

For abelian Dyck paths d_λ

$$X_G = \sum_{\alpha} c_\alpha \mathbf{x}^\alpha = \sum_i i! \cdot (n-2i)! \cdot r_i(\lambda) \cdot m_{2^i 1^{n-2i}},$$

where $r_i(\lambda)$ is the number of placements of i non-attacking rooks on λ.
For each $\alpha \in \Delta_k^{n-2}$ we check $k \times k$ matrix $H_\alpha = ((\alpha + e_r + e_s)! \cdot c_{\alpha+e_r+e_s})_{r,s=1}^k$ has at most one positive eigenvalue.
For $\alpha = (2^{i-1}, 1^{n-2i}, 0^{k-n+i+1})$:

$$H_\alpha = \begin{array}{c} \left(\begin{array}{ccc|ccc} 0 & & a & & & \\ & \ddots & & & b & \\ a & & 0 & & & \\ \hline & & & b & & c \\ & b & & & \ddots & \\ & & & c & & b \end{array}\right) \\ \qquad n-2i \qquad\qquad k-n+i+1 \end{array}, \qquad \begin{aligned} a &= 2^{i+1} \cdot (i+1)! \cdot (n-2i-2)! \cdot r_{i+1} \\ b &= 2^i \cdot i! \cdot (n-2i)! \cdot r_i \\ c &= 2^{i-1} \cdot (i-1)! \cdot (n-2i+2)! \cdot r_{i-1}, \end{aligned}$$

About proof: ultra log-concavity of rook numbers

For $\lambda \subset n_1 \times n_2$

$$\begin{aligned} a &= 2^{i+1} \cdot (i+1)! \cdot (n-2i-2)! \cdot r_{i+1} \\ b &= 2^{i} \cdot i! \cdot (n-2i)! \cdot r_{i} \\ c &= 2^{i-1} \cdot (i-1)! \cdot (n-2i+2)! \cdot r_{i-1}, \end{aligned}$$

$X_{G(d_\lambda)}(x_1, \ldots, x_k)$ is Lorentzian iff

$$b - c \leq 0, \qquad (1)$$

$$-(n-2i)(k-n+i+1)b^2 + (n-2i-1)a(b+(k-n+i)c) \leq 0, \qquad (2)$$

About proof: ultra log-concavity of rook numbers

For $\lambda \subset n_1 \times n_2$

$$a = 2^{i+1} \cdot (i+1)! \cdot (n-2i-2)! \cdot r_{i+1}$$
$$b = 2^{i} \cdot i! \cdot (n-2i)! \cdot r_{i}$$
$$c = 2^{i-1} \cdot (i-1)! \cdot (n-2i+2)! \cdot r_{i-1},$$

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- (1) follows from

$$(i+1) \cdot r_{i+1} \leq (n_1 - i)(n_2 - i) \cdot r_i$$

About proof: ultra log-concavity of rook numbers

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- (1) follows from
$$(i+1) \cdot r_{i+1} \leq (n_1 - i)(n_2 - i) \cdot r_i$$
- (2) follows from
$$r_i^2 \geq \left(1 + \frac{1}{i}\right)\left(1 + \frac{1}{\ell - i}\right)\left(1 + \frac{1}{\lambda_1 - i}\right) r_{i-1} r_{i+1}.$$

About proof: ultra log-concavity of rook numbers

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  A consequence of real-rootedness of **hit polynomial** for *Ferrers boards*:
  $\sum_{i=0}^{N} (N-i)! \cdot r_i(\lambda) \cdot (x-1)^i$ (Haglund–Ono–Wagner 1999)

Thank you

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