# Lattice Walk Enumeration: Analytic, algebraic and geometric aspects

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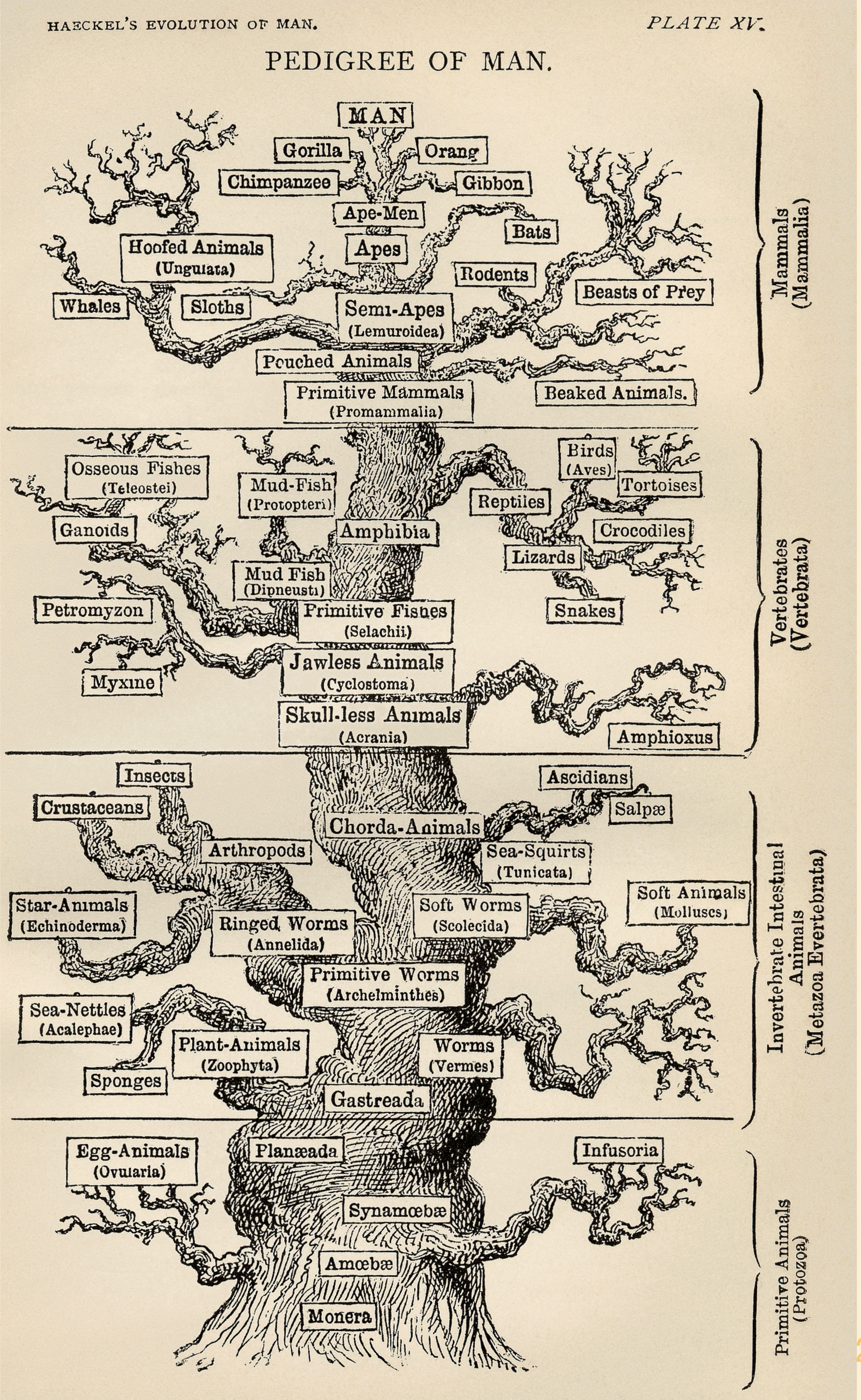
# Humans love a good taxonomy!




The tree of life as seen by Haeckel in *The Evolution of Man* (1879)

Rational numbers

Algebraic Numbers

Periods

Complex Numbers

Rational Functions

Algebraic Functions

Holonomic Functions

Differentiably Algebraic Functions

Complex Analytic Functions

# Holonomic and Differentiably Algebraic functions

| | | |
|---|---|---|
| Holonomic | Satisfy a linear DE with polynomial coefficients. | Rational, Algebraic<br>$e^{p(x)}$ |
| Differentiably Algebraic | Satisfy an algebraic DE. | Holonomic<br>$e^{e^z-1}$ |
| Differentiably Transcendental | NOT differentiably algebraic | $\zeta(z)$<br>$\Gamma(z)$ **(Holder, 1887)** |

# Classification of Combinatorial Classes

A **combinatorial class** is a set equipped with a size function. Ordinary Generating Functions **(OGF)** encode enumerative data in the coefficients of formal power series.

$$\mathscr{C} \implies C(z) := \sum_{n=0}^{\infty} \left|\mathscr{C}_n\right| z^n$$

**Problem: Characterize combinatorial classes with transcendental OGF.**

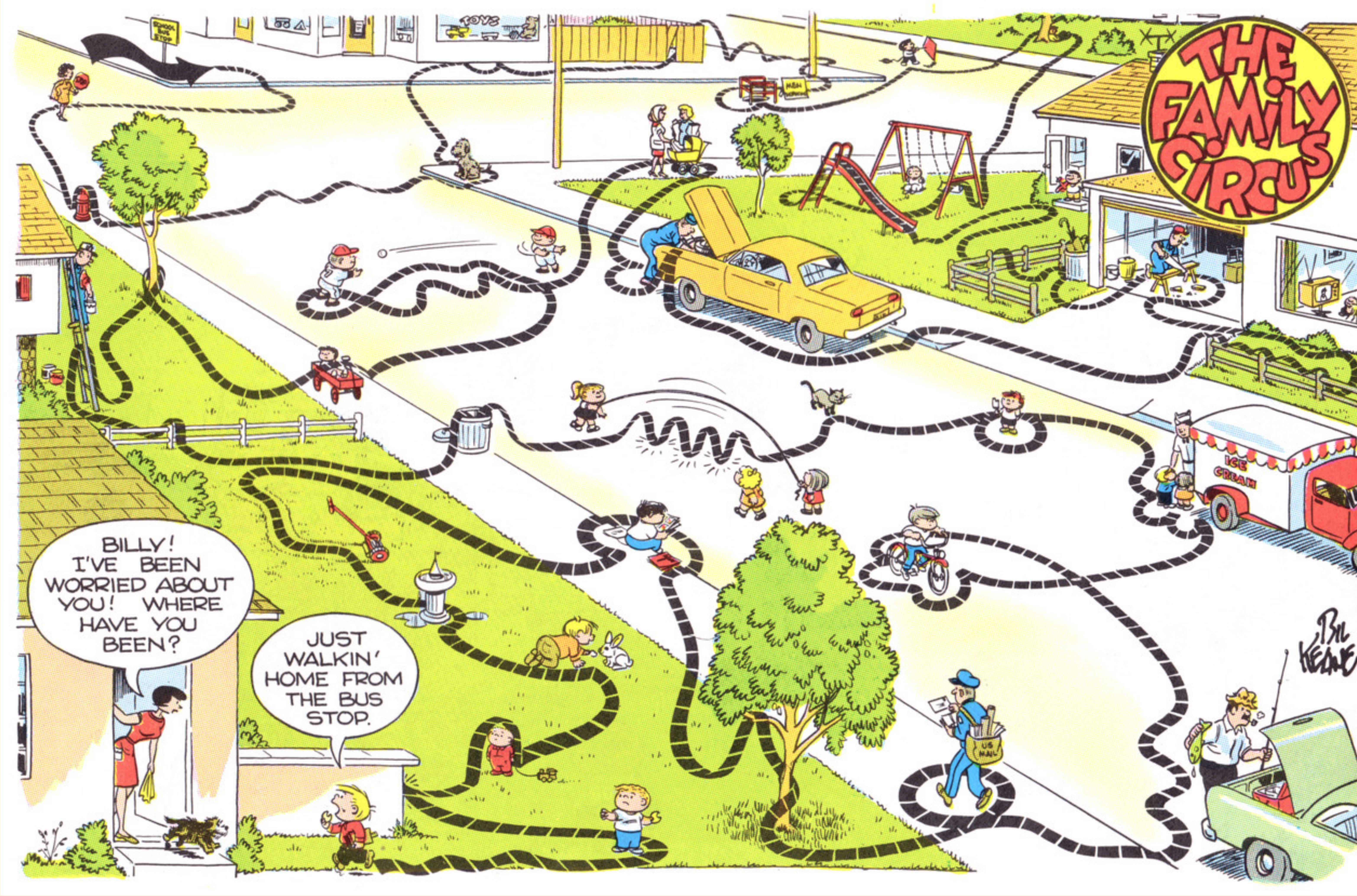
| Type of | | | eferences |
|---|---|---|---|
| Finite c | | | |
| Iterative | | | |
| Recursively specifiable class<br>Algebraic class | Trees<br>CFL (not inherently ambiguous) | Algebraic function | Chomsky & Schutzenberger (1960s) |
| ? | Shuffles of Dyck Paths<br>k-regular labelled graphs<br>SYT of bounded height | Holonomic | Stanley 1980 |
| ? | Set partitions | Differentiably algebraic | Rubel 1987 |

# Why are holonomic functions interesting to study from the combinatorial perspective?

"Almost anything is non-holonomic unless it is holonomic by design."
– Flajolet Gerhold & Salvy

- Closure properties mirror combinatorial actions
- The DE is a useful data structure for both reasoning and computation
- Clear proof strategies (singularity arguments, asymptotic arguments)
- **Conjecture (Christol, 1990):** If a series with positive integer coefficients has positive, finite, radius of convergence is holonomic, then it can be written as the diagonal of a multivariate rational function.

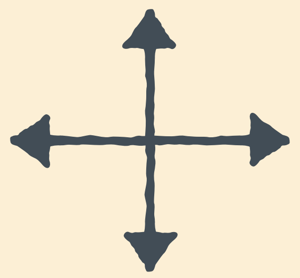
The universe of all combinatorial classes is too large. Let us consider the world of lattice paths

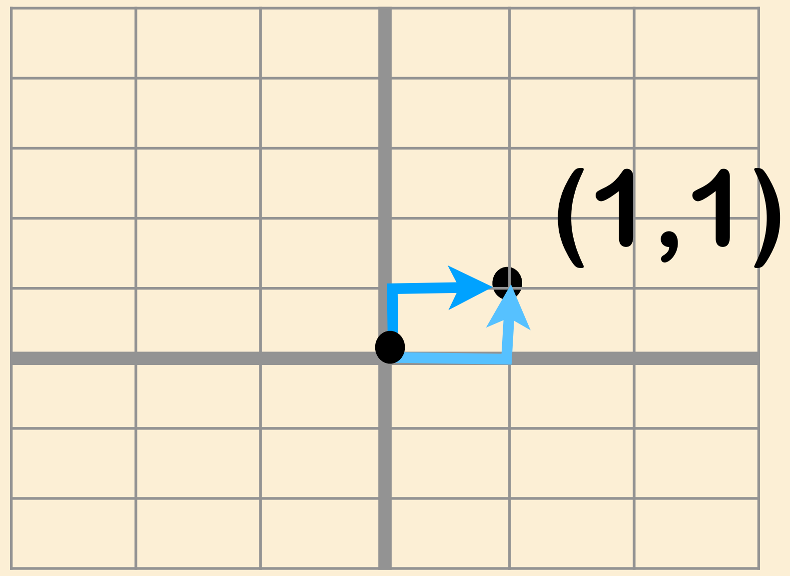


# A walk is a sequence of steps

**We consider fixed, finite sets of possible steps**

**Strategy:** **Encode each walk with a monomial marking its endpoint.**



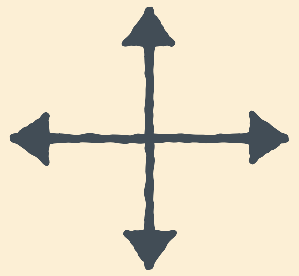



Multiplying monomials captures what happens when we take steps in sequence.

$$(x + 1/x + y + 1/y) \times (x + 1/x + y + 1/y)$$

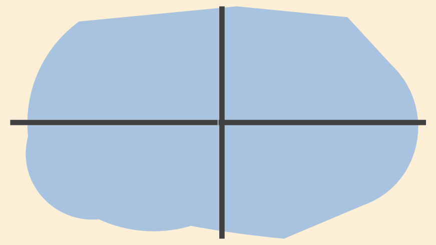
$$= x^2 + 1/x^2 + y^2 + 1/y^2 + 4 + 2xy + 2x/y + 2y/x + 2/xy$$

$$\frac{1}{1 - z\left(x + 1/x + y + 1/y\right)}$$



# NESW walks in various regions

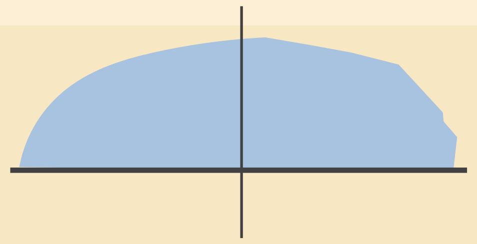
$$F(z) = \sum_{n \geq 0} \left(\text{\#walks of length } n \text{ that stay in the blue region}\right) z^n$$



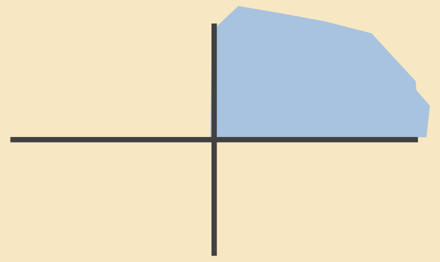
**Rational**

$$\frac{1}{1-4z}$$

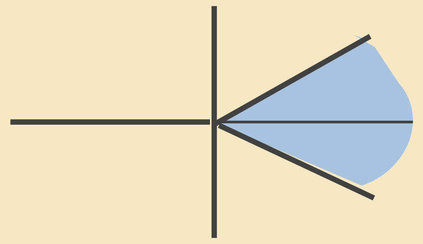
We can classify the nature of generating functions of NESW walks in some regions



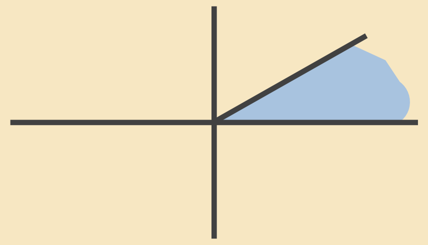
**Algebraic**



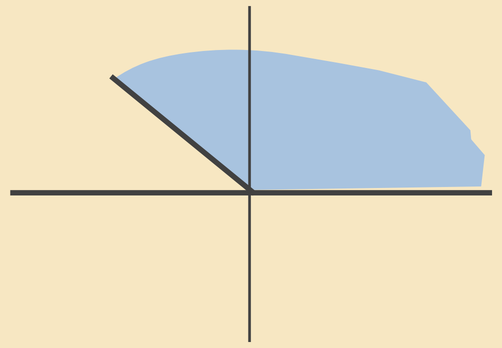
**Holonomic**



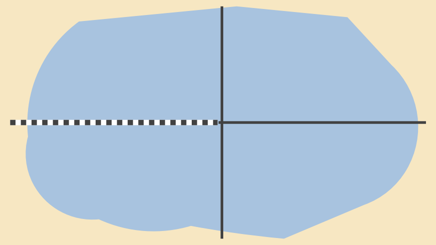
**Holonomic**



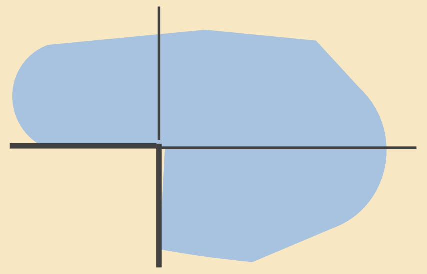
**Holonomic**
**(Gouyou-Beauchamps)**



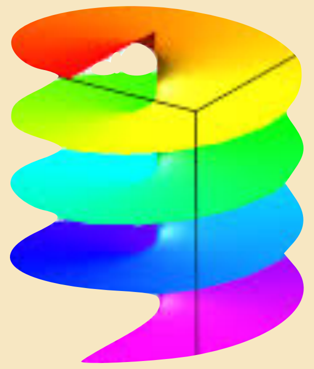
**Nasty algebraic**
**(Bostan & Kauers…)**



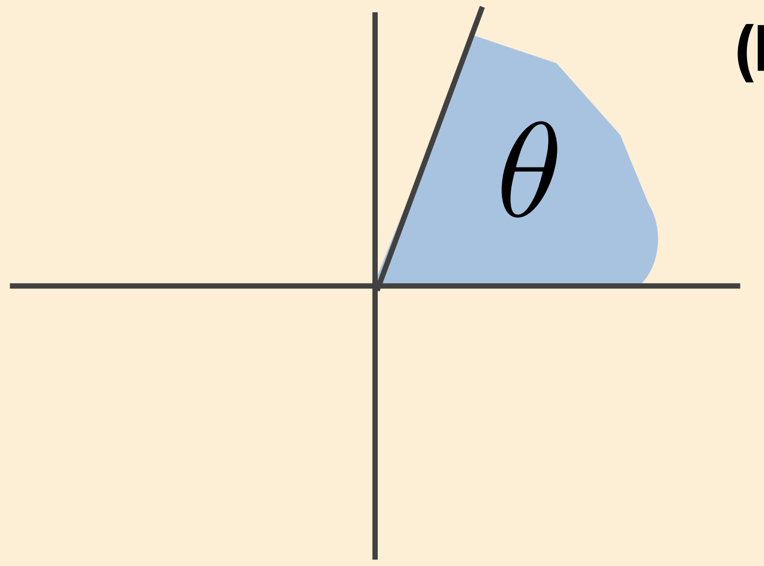
**Slit plane model**
**Algebraic**
**(Bousquet-Mélou & Schaeffer)**



**Holonomic**
**(Bousquet-Mélou… )**



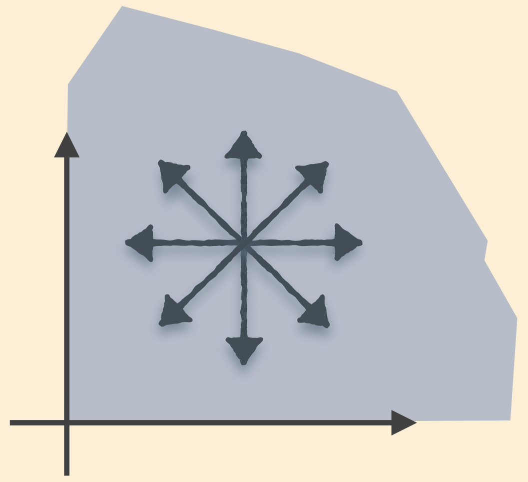
**Classification via winding angle**
**(Excursions: Budd 2017; Elvey-Price)**




**(Excursions: Denisov & Wachtel 2016)**

$$e(n) \sim \alpha \rho^{-n} n^{f(\theta)}$$

# Small step walks in the quarter plane



- **Fix set of vectors** $\mathcal{S} \subseteq \{(i,j) \mid i,j \in \{0,1,-1\}\} \setminus \{(0,0)\}$. **Define the generating function which marks the endpoint.**

$$Q_{\mathcal{S}}(x,y;z) = Q_{\mathcal{S}}(x,y) := \sum_{n\geq 0} \sum_{(i,j)\in\mathbb{N}^2} \#\text{walks}_{\mathcal{S}}\, (0,0) \xrightarrow{n} (i,j)\, x^i y^j z^n$$

- **Theorem** **(M.&Rechnitzer, 2009) There exist NON holonomic models**

- **Theorem** **(Bousquet-Mélou&M., 2010) There are 79 nontrivial, distinct models, accounting for models in bijection**

- **Conjecture** **(Bousquet-Mélou&M., 2010)** $Q_{\mathcal{S}}(1,1)$ **and** $Q_{\mathcal{S}}(x,y)$ **are holonomic if a certain group is well defined, and of finite order.**

# A "certain group"

Bousquet-Mélou & M. 2010, M. 2007: Holonomy appears to be correlated with the finiteness of a **certain group**.

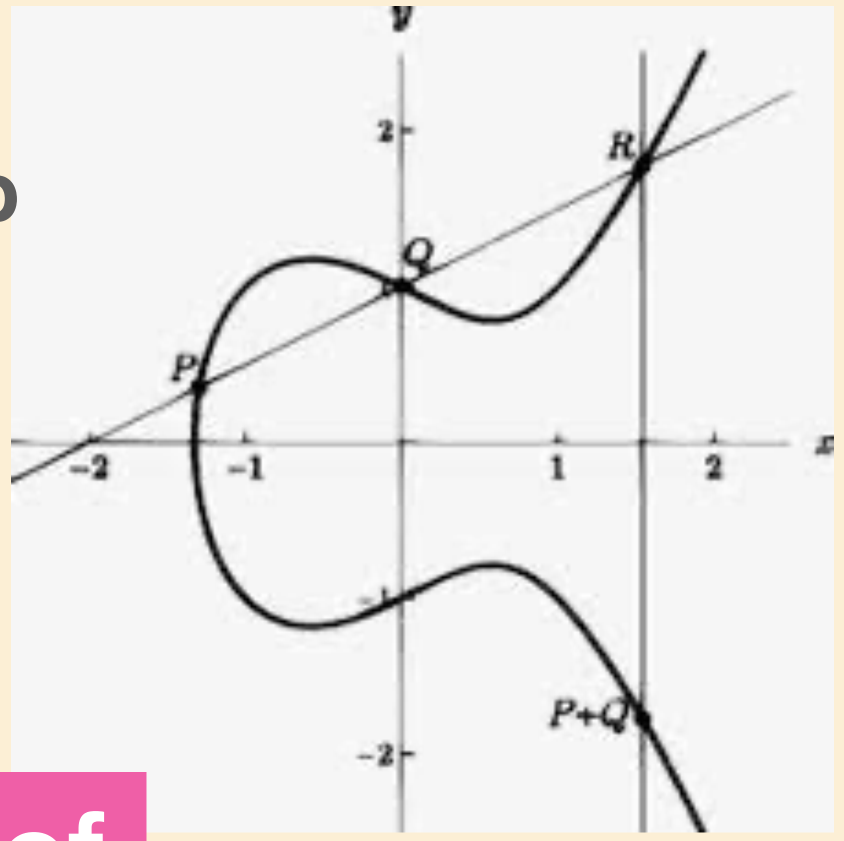
It is a particular group of **bi-rational transformations** that fix the Laurent polynomial inventory of the step set (the kernel):

$$K_{\mathcal{S}}(x, y) = 1 - z \sum_{(i,j)\in\mathcal{S}} x^i y^j$$

Eg. NESW: $K_+(x, y) = 1 - z(x + 1/x + y + 1/y)$ generating involutions: $\sigma_1 : (x, y) \mapsto (x, 1/y), \sigma_2 : (x, y) \mapsto (1/x, y)$, generates a group of order 4.

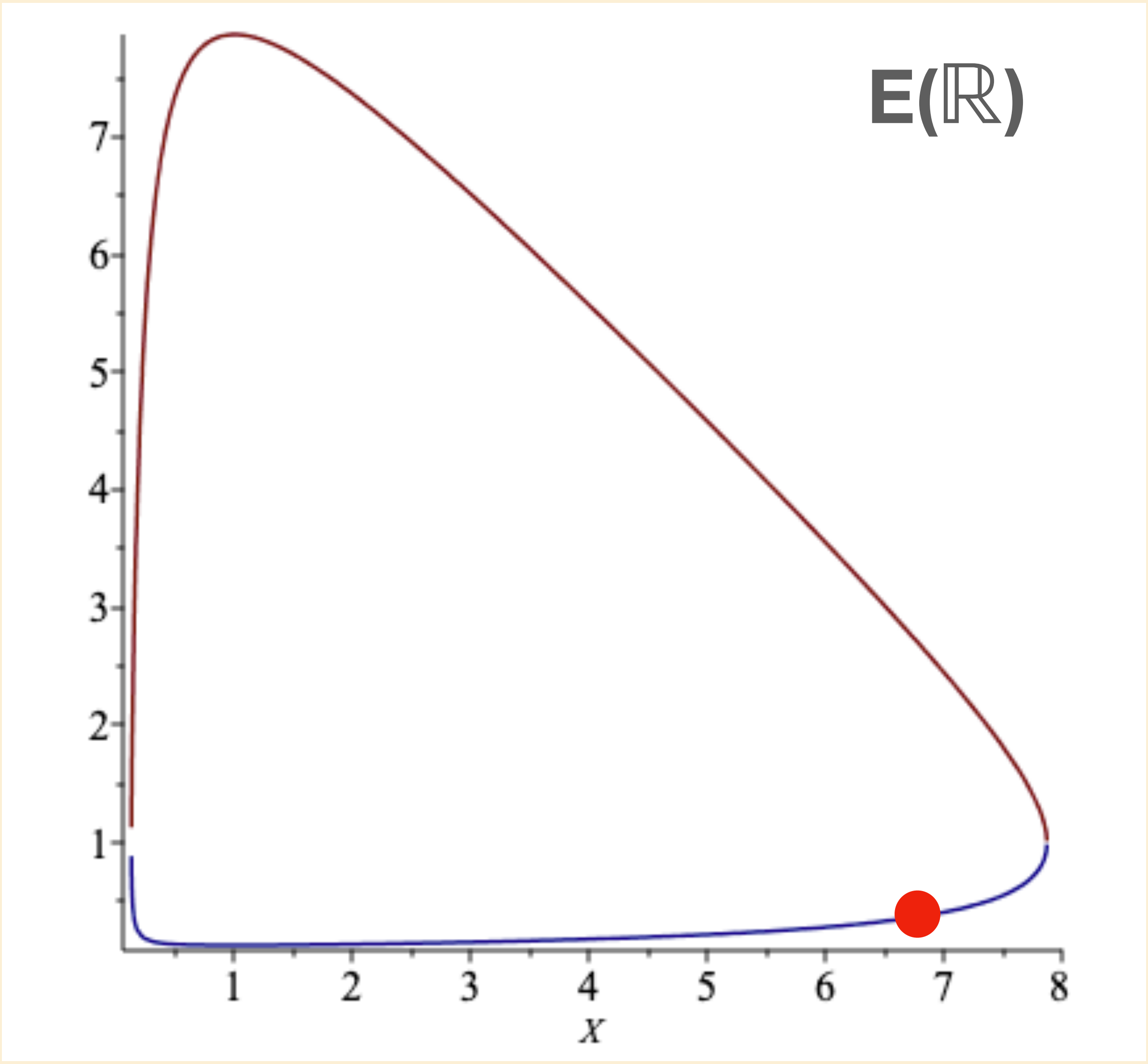
Define E as the compactification of the zero set of K in $\mathbb{P}^1 \times \mathbb{P}^1$ (for an evaluation of z). **E is an algebraic curve of genus 1 or 0.** Transformations move around the curve.

In some cases E is an elliptic curve, and the group is the group law of the elliptic curve




**The GEOMETRIC understanding of E gives COMBINATORIAL results**

Eg. NESW
E=The zero locus of $K_+(x, y)$ evaluated at z=0.1



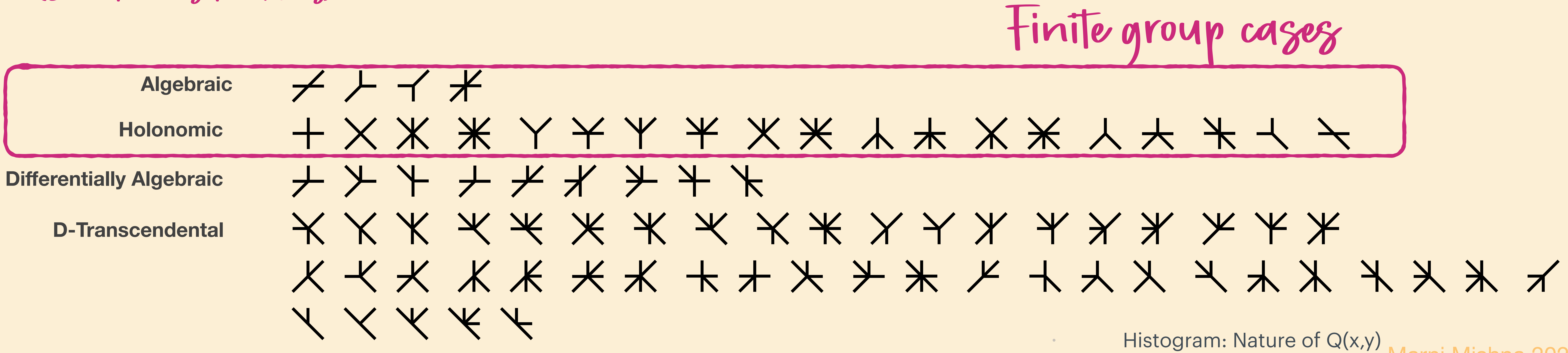
# Small step walks in the quarter plane

Bousquet-Mélou & M. 2010, M. 2007: Holonomy appears to be correlated with the finiteness of a certain group.

$$Q_{\mathcal{S}}(x,y) = \sum_{n\geq 0} \sum_{(i,j)\in\mathbb{N}^2} \#\text{walks}_{\mathcal{S}}\,(0,0) \xrightarrow{\;n\;} (i,j)\, x^i y^j z^n$$

A decade long, international collaboration determined the classification of $Q_{\mathcal{S}}(x,y)$

(See references for details)




Histogram: Nature of Q(x,y)

# Key idea to show Holonomic

- Use reflection principle / kernel method to write $Q(x,y)$ as the diagonal of a rational function.
  (=> Integral of a rational=> Holonomic)

- If the model is (essentially) a walk in the Weyl chamber, the "group" is the corresponding Coxeter group.

# Key Lemma to show Differentially Transcendental

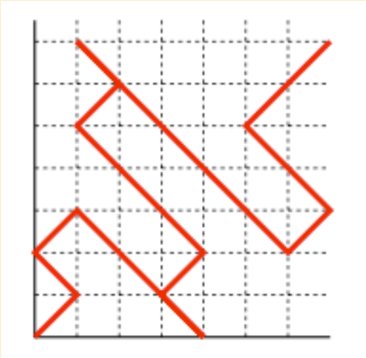
- **Lemma (Ishizaki-Ozawara)** Let $q$ be a complex number, but not a root of unity. Given $f(z) \in \mathbb{C}[[z]][1/z]$, if there are $a(z)$ and $b(z)$ that are rational functions over $\mathbb{C}$ such that $f(qz) = a(z)f(z) + b(z)$, then $f(z)$ is either rational or differentiably transcendental.

Principle: If F(z) satisfies a (q-) shift equation AND a DE then it is rational.

- **Example:** $\Gamma(x+1) = x\Gamma(x)$

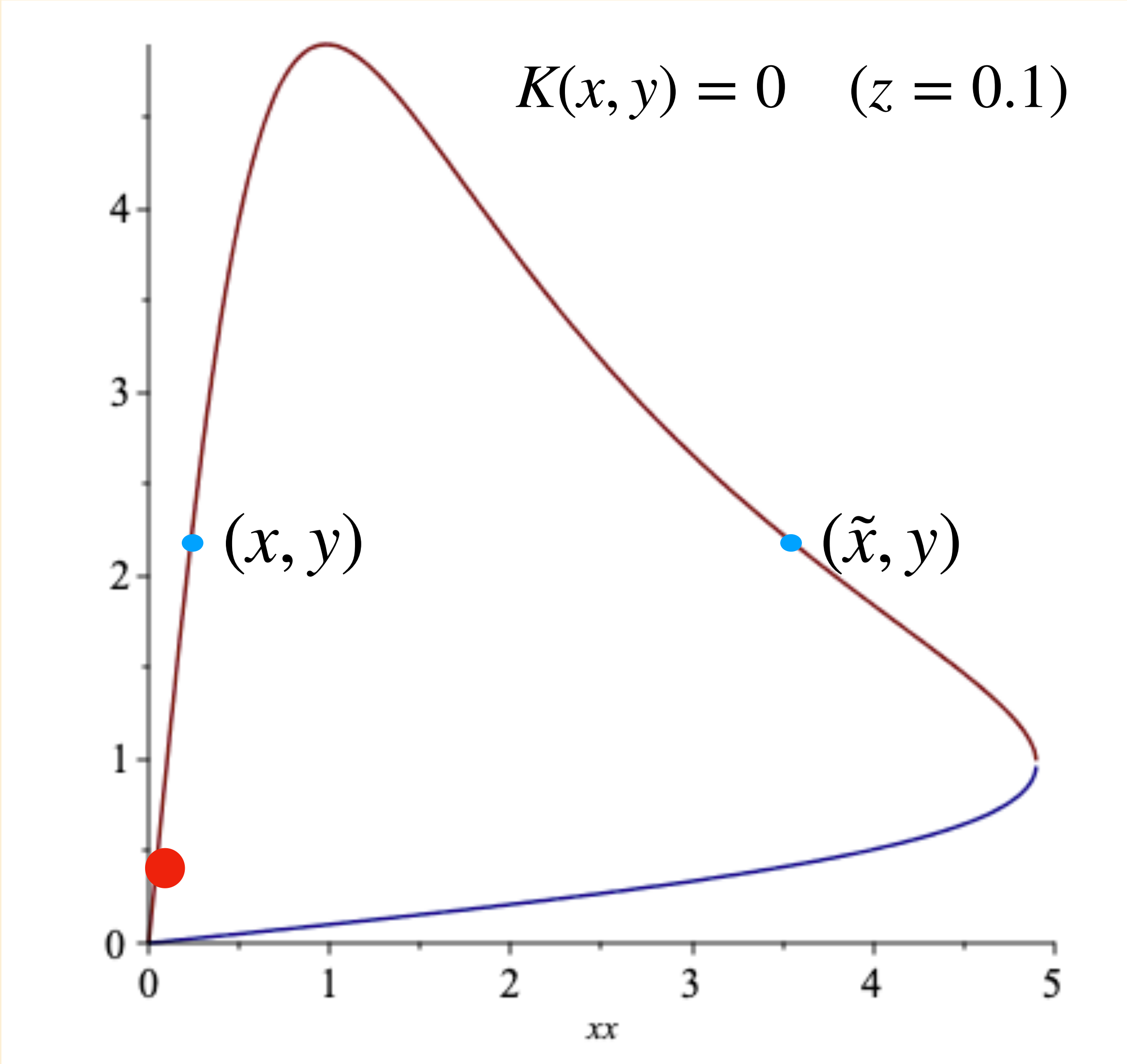
  Since $\Gamma(x)$ is not rational, it is differentially transcendental.

- **Strategy:** Find such a rational parametrization of E (the related elliptic curve) and use this to find such an equation for the generating function.



# A genus 0 non-holonomic model

M. & Rechnitzer
Dreyfus, Hardouin, Roques
Bostan




- **Functional equation**
  $Q(x,y) = 1 + z(x/y + y/x + xy)Q(x,y) - z(x/y)Q(x,0) - z(y/x)Q(0,y)$
- **Kernel version of the equation**
  $xy(1 - z(x/y + y/x + xy))Q(x,y) = xy - R(x) - R(y)$
  $K(x,y)$
- **For every** $(x,y) \in E$ **there is a second** $(\tilde{x},y) \in E$
  $0 = (x - \tilde{x})y - R(x) + R(\tilde{x}) \quad \Longrightarrow \quad R(x(s)) - R(x(\underbrace{v^2}_{q} s)) = \text{Rational}(s,v)$
- **Rational parametrization of E** $K(x(s),y(s)) = K(\tilde{x}(s),y(s)) = 0$
  $x(s) = \frac{v(1-v^2)s}{(s^2+1)}, y(s) = \frac{(1-v^2)s}{v^2s^2+1}, z = \frac{v}{v^2+1}$ **other root:** $\tilde{x}(v^2s) = x(s)$

Since the R(x) is not rational, it is differentiably transcendental in x.

# Summary

- Lattice walks provide a natural context for studying combinatorial classes with transcendental OGF

- Most known classes with holonomic OGF in bijection with some kind of lattice walks model => Evidence for Christol’s conjecture.

- Differentiably transcendental classes occur when the OGF satisfy a kind of (q-)shift equation

# Open Questions & Future Work

- Complete the classification of univariate OGF
- Automate various aspects of the parametrization
- Mine the geometry further
    - Configuration of base points of E correlates with DA (Hardouin and Singer 2020) Is there a combinatorial source?
- Examine other non-holonomic classes to determine if they are DA
- Conjecture (Pak, Yeliussizov 2019) DA OGF & DA EGF => HOLONOMY

# Lattice path classification references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