Counting Admissible Orderings of a Pinnacle Set

Quinn Minnich

Michigan State University

October 27, 2021

Pinnacle sets

Definition

If $\pi = \pi_1 \ldots \pi_n$ is a permutation in the symmetric group S_n then its *pinnacle set* is

$$\operatorname{Pin} \pi = \{\pi_i \mid \pi_{i-1} < \pi_i > \pi_{i+1}\}.$$

Pinnacle sets

Definition

If $\pi = \pi_1 \ldots \pi_n$ is a permutation in the symmetric group S_n then its *pinnacle set* is

$$\operatorname{Pin} \pi = \{\pi_i \mid \pi_{i-1} < \pi_i > \pi_{i+1}\}.$$

Example: $n = 9$

$$\pi = 297418356$$

Pinnacle sets

Definition

If $\pi = \pi_1 \ldots \pi_n$ is a permutation in the symmetric group S_n then its *pinnacle set* is

$$\operatorname{Pin} \pi = \{\pi_i \mid \pi_{i-1} < \pi_i > \pi_{i+1}\}.$$

Example: $n = 9$

$$\pi = 2{\color{blue}9}741{\color{blue}8}356$$

Pinnacle sets

Definition

If $\pi = \pi_1 \dots \pi_n$ is a permutation in the symmetric group S_n then its *pinnacle set* is

$$\operatorname{Pin} \pi = \{\pi_i \mid \pi_{i-1} < \pi_i > \pi_{i+1}\}.$$

Example: $n = 9$

$$\pi = 297418356$$

So $\operatorname{Pin} \pi = \{8, 9\}$

Pinnacle sets

Definition

If $\pi = \pi_1 \ldots \pi_n$ is a permutation in the symmetric group S_n then its *pinnacle set* is

$$\operatorname{Pin}\pi = \{\pi_i \mid \pi_{i-1} < \pi_i > \pi_{i+1}\}.$$

Example: $n = 9$

$$\pi = 297418356$$

$$\text{So } \operatorname{Pin}\pi = \{8, 9\}$$

It is also possible to go backwards by first specifying a desired pinnacle set $P \subseteq [n]$ and then asking if there exists any $\pi \in S_n$ such that $\operatorname{Pin}\pi = P$. If such a π exists, we say it is a *witness* to the pinnacle set. If P has a witness, we say it is an *admissible pinnacle set*.

Pinnacle sets

Definition

If $\pi = \pi_1 \dots \pi_n$ is a permutation in the symmetric group S_n then its *pinnacle set* is

$$\operatorname{Pin}\pi = \{\pi_i \mid \pi_{i-1} < \pi_i > \pi_{i+1}\}.$$

Example: $n = 9$

$$\pi = 297418356$$

$$\text{So } \operatorname{Pin}\pi = \{8, 9\}$$

It is also possible to go backwards by first specifying a desired pinnacle set $P \subseteq [n]$ and then asking if there exists any $\pi \in S_n$ such that $\operatorname{Pin}\pi = P$. If such a π exists, we say it is a *witness* to the pinnacle set. If P has a witness, we say it is an *admissible pinnacle set*.

Example: $n = 9$ and $P = \{5, 6, 9\}$ has witness $\pi = 152639478$

Pinnacle sets

Definition

If $\pi = \pi_1 \ldots \pi_n$ is a permutation in the symmetric group S_n then its *pinnacle set* is

$$\operatorname{Pin}\pi = \{\pi_i \mid \pi_{i-1} < \pi_i > \pi_{i+1}\}.$$

Example: $n = 9$

$$\pi = 297418356$$
$$\text{So } \operatorname{Pin}\pi = \{8, 9\}$$

It is also possible to go backwards by first specifying a desired pinnacle set $P \subseteq [n]$ and then asking if there exists any $\pi \in S_n$ such that $\operatorname{Pin}\pi = P$. If such a π exists, we say it is a *witness* to the pinnacle set. If P has a witness, we say it is an *admissible pinnacle set*.

Example: $n = 9$ and $P = \{5, 6, 9\}$ has witness $\pi = 152639478$

However, $P = \{3, 4\}$ is not an admissible pinnacle set because there are not enough small non-pinnacles to surround the 3 and the 4.

Pinnacle sets

Pinnacle sets were first introduced in 2018 by Davis et al. and have generated a good deal of recent interest. Among other things in their paper, they noted that some admissible pinnacle sets only have a witness for particular orderings of the pinnacle elements.

Pinnacle sets

Pinnacle sets were first introduced in 2018 by Davis et al. and have generated a good deal of recent interest. Among other things in their paper, they noted that some admissible pinnacle sets only have a witness for particular orderings of the pinnacle elements.

Example: If $n = 7$ then $P = \{4, 5, 7\}$ has multiple witnesses where the pinnacles appear in the order 457, including $\pi = 1425376$.

Pinnacle sets

Pinnacle sets were first introduced in 2018 by Davis et al. and have generated a good deal of recent interest. Among other things in their paper, they noted that some admissible pinnacle sets only have a witness for particular orderings of the pinnacle elements.

Example: If $n = 7$ then $P = \{4, 5, 7\}$ has multiple witnesses where the pinnacles appear in the order 457, including $\pi = 1425376$.

However, there is no witness where the pinnacles appear in the order 475 since this would require four elements less than both 4 and 5 rather than just three.

Pinnacle sets

Pinnacle sets were first introduced in 2018 by Davis et al. and have generated a good deal of recent interest. Among other things in their paper, they noted that some admissible pinnacle sets only have a witness for particular orderings of the pinnacle elements.

Example: If $n = 7$ then $P = \{4, 5, 7\}$ has multiple witnesses where the pinnacles appear in the order 457, including $\pi = 1425376$.

However, there is no witness where the pinnacles appear in the order 475 since this would require four elements less than both 4 and 5 rather than just three.

If a particular ordering of P has a witness, we say that it is an *admissible ordering* of P.

Admissible orderings

Definition

Let ω be an ordering of a set S. Then for any permutation π of a set containing S we say that $\operatorname{Ord}\pi = \omega$ if the elements of S in π appear in order ω.

Admissible orderings

Definition

Let ω be an ordering of a set S. Then for any permutation π of a set containing S we say that $\operatorname{Ord} \pi = \omega$ if the elements of S in π appear in order ω.

Example: Let $S = \{a, b, c\}$ and suppose we are considering permutations of the set $\{1, 2, 3, a, b, c\}$, which contains S. One such permutation is $\pi = 13bc2a$ which has $\operatorname{Ord} \pi = bca$.

Admissible orderings

Definition

Let ω be an ordering of a set S. Then for any permutation π of a set containing S we say that $\operatorname{Ord} \pi = \omega$ if the elements of S in π appear in order ω.

Example: Let $S = \{a, b, c\}$ and suppose we are considering permutations of the set $\{1, 2, 3, a, b, c\}$, which contains S. One such permutation is $\pi = 13bc2a$ which has $\operatorname{Ord} \pi = bca$.

More specifically, given an arbitrary $\pi \in S_n$ such that $\operatorname{Pin} \pi = P$, we may say that $\operatorname{Ord} \pi = \omega$ where ω is the ordering of the elements of P within π.

Admissible orderings

Definition

Let ω be an ordering of a set S. Then for any permutation π of a set containing S we say that $\operatorname{Ord}\pi = \omega$ if the elements of S in π appear in order ω.

Example: Let $S = \{a, b, c\}$ and suppose we are considering permutations of the set $\{1, 2, 3, a, b, c\}$, which contains S. One such permutation is $\pi = 13bc2a$ which has $\operatorname{Ord}\pi = bca$.

More specifically, given an arbitrary $\pi \in S_n$ such that Pin $\pi = P$, we may say that $\operatorname{Ord}\pi = \omega$ where ω is the ordering of the elements of P within π.

Example: Suppose $n = 9$, $P = \{5, 7, 8\}$, and $\pi = 386452719$.

Admissible orderings

Definition

Let ω be an ordering of a set S. Then for any permutation π of a set containing S we say that $\operatorname{Ord}\pi = \omega$ if the elements of S in π appear in order ω.

Example: Let $S = \{a, b, c\}$ and suppose we are considering permutations of the set $\{1, 2, 3, a, b, c\}$, which contains S. One such permutation is $\pi = 13bc2a$ which has $\operatorname{Ord}\pi = bca$.

More specifically, given an arbitrary $\pi \in S_n$ such that $\text{Pin } \pi = P$, we may say that $\operatorname{Ord}\pi = \omega$ where ω is the ordering of the elements of P within π.

Example: Suppose $n = 9$, $P = \{5, 7, 8\}$, and $\pi = 386452719$.

Then $\operatorname{Ord}\pi = 857$.

Examples

Definition

Let ω any ordering of P. Then the set of all admissible orderings of P is

$$O(P) = \{\omega \mid \text{there exists } \pi \in S_n \text{ with } \operatorname{Ord}\pi = \omega,\ \operatorname{Pin}\pi = P\ \}.$$

If we denote the cardinality of a set by $\#$, then we also let

$$o(P) = \#O(P).$$

Examples

Definition

Let ω any ordering of P. Then the set of all admissible orderings of P is

$$O(P) = \{\omega \mid \text{there exists } \pi \in S_n \text{ with } \operatorname{Ord}\pi = \omega,\ \operatorname{Pin}\pi = P\ \}.$$

If we denote the cardinality of a set by $\#$, then we also let

$$o(P) = \#O(P).$$

Example: $n = 7$

Examples

Definition

Let ω any ordering of P. Then the set of all admissible orderings of P is

$$O(P) = \{\omega \mid \text{there exists } \pi \in S_n \text{ with } \operatorname{Ord}\pi = \omega,\ \operatorname{Pin}\pi = P\ \}.$$

If we denote the cardinality of a set by $\#$, then we also let

$$o(P) = \#O(P).$$

Example: $n = 7$

- Out of all orderings of $P = \{4, 5, 7\}$, only $457, 547, 745$, and 754 are admissible.

Examples

Definition

Let ω any ordering of P. Then the set of all admissible orderings of P is

$$O(P) = \{\omega \mid \text{there exists } \pi \in S_n \text{ with } \operatorname{Ord}\pi = \omega,\ \operatorname{Pin}\pi = P\ \}.$$

If we denote the cardinality of a set by $\#$, then we also let

$$o(P) = \#O(P).$$

Example: $n = 7$

- Out of all orderings of $P = \{4, 5, 7\}$, only $457, 547, 745$, and 754 are admissible. Therefore, $O(P) = \{457, 547, 745, 754\}$ and $o(P) = 4$.

Examples

Definition

Let ω any ordering of P. Then the set of all admissible orderings of P is

$$O(P) = \{\omega \mid \text{there exists } \pi \in S_n \text{ with } \operatorname{Ord}\pi = \omega,\ \operatorname{Pin}\pi = P\ \}.$$

If we denote the cardinality of a set by $\#$, then we also let

$$o(P) = \#O(P).$$

Example: $n = 7$

- Out of all orderings of $P = \{4, 5, 7\}$, only $457, 547, 745$, and 754 are admissible. Therefore, $O(P) = \{457, 547, 745, 754\}$ and $o(P) = 4$.
- All orderings of $P = \{5, 6, 7\}$ are admissible, and so $O(P) = \{567, 576, 657, 675, 756, 765\}$ and $o(P) = 6$

Examples

Definition

Let ω any ordering of P. Then the set of all admissible orderings of P is

$$O(P) = \{\omega \mid \text{there exists } \pi \in S_n \text{ with } \operatorname{Ord}\pi = \omega,\ \operatorname{Pin}\pi = P\ \}.$$

If we denote the cardinality of a set by $\#$, then we also let

$$o(P) = \#O(P).$$

Example: $n = 7$

- Out of all orderings of $P = \{4, 5, 7\}$, only $457, 547, 745$, and 754 are admissible. Therefore, $O(P) = \{457, 547, 745, 754\}$ and $o(P) = 4$.
- All orderings of $P = \{5, 6, 7\}$ are admissible, and so $O(P) = \{567, 576, 657, 675, 756, 765\}$ and $o(P) = 6$
- No orderings of $P = \{3, 4\}$ are admissible. So $O(P) = \emptyset$ and $o(P) = 0$

Studied Questions

This raises several interesting questions:

Studied Questions

This raises several interesting questions:

Question

For a fixed n, how many admissible pinnacle sets $P \subseteq [n]$ are there?

Studied Questions

This raises several interesting questions:

Question

For a fixed n, how many admissible pinnacle sets $P \subseteq [n]$ are there?

Davis et al. showed that the number of admissible pinnacle sets is

$$\binom{n-1}{\lfloor (n-1)/2 \rfloor}.$$

Studied Questions

This raises several interesting questions:

Question

For a fixed n, how many admissible pinnacle sets $P \subseteq [n]$ are there?

Davis et al. showed that the number of admissible pinnacle sets is

$$\binom{n-1}{\lfloor (n-1)/2 \rfloor}.$$

Question

For a fixed n and desired pinnacle set P, how many permutations in S_n have P as its pinnacle set?

Studied Questions

This raises several interesting questions:

Question

For a fixed n, how many admissible pinnacle sets $P \subseteq [n]$ are there?

Davis et al. showed that the number of admissible pinnacle sets is

$$\binom{n-1}{\lfloor (n-1)/2 \rfloor}.$$

Question

For a fixed n and desired pinnacle set P, how many permutations in S_n have P as its pinnacle set?

This question has received a lot of attention. Currently the fastest known algorithms, one of them given by Fang in 2021 and one by myself, have run time $O(k^2 \log n + k^4)$ where k is the number of elements in P.

Main question

Question

For a fixed n and desired pinnacle set P, what is $o(P)$, that is, how many orderings of P are admissible?

Main question

Question

For a fixed n and desired pinnacle set P, what is $o(P)$, that is, how many orderings of P are admissible?

In 2020 Rusu and Tenner did some work characterizing admissible pinnacle orderings, but they did not give a function to count them.

Main question

Question

For a fixed n and desired pinnacle set P, what is $o(P)$, that is, how many orderings of P are admissible?

In 2020 Rusu and Tenner did some work characterizing admissible pinnacle orderings, but they did not give a function to count them.

For the rest of the talk, we will suppose that our given set P has k elements. We would like to find a formula or algorithm for computing $o(P)$ which, in principle, would depend on n (the length of π) and k.

Main question

Question

For a fixed n and desired pinnacle set P, what is $o(P)$, that is, how many orderings of P are admissible?

In 2020 Rusu and Tenner did some work characterizing admissible pinnacle orderings, but they did not give a function to count them.

For the rest of the talk, we will suppose that our given set P has k elements. We would like to find a formula or algorithm for computing $o(P)$ which, in principle, would depend on n (the length of π) and k.

Actually, this problem is independent of n, which is out first result.

Eliminating n

Definition

Let N_i be the set of the i smallest elements not in P.

Eliminating n

Definition

Let N_i be the set of the i smallest elements not in P.

Example: If $n = 9$ and $P = \{3, 5\}$, then $N_4 = \{1, 2, 4, 6\}$.

Eliminating n

Definition

Let N_i be the set of the i smallest elements not in P.

Example: If $n = 9$ and $P = \{3, 5\}$, then $N_4 = \{1, 2, 4, 6\}$.

Theorem

Suppose $P \subseteq [n]$ with $|P| = k$ and let ω be some ordering of the elements in P. Then there exists a permutation π of the elements $[n]$ with $\mathrm{Ord}\,\pi = \omega$ and $\mathrm{Pin}\,\pi = P$ if and only if there exists a permutation π' of the elements $N_{k+1} \cup P$ with $\mathrm{Ord}\,\pi' = \omega$ and $\mathrm{Pin}\,\pi' = P$.

Eliminating n

Definition

Let N_i be the set of the i smallest elements not in P.

Example: If $n = 9$ and $P = \{3, 5\}$, then $N_4 = \{1, 2, 4, 6\}$.

Theorem

Suppose $P \subseteq [n]$ with $|P| = k$ and let ω be some ordering of the elements in P. Then there exists a permutation π of the elements $[n]$ with $Ord\,\pi = \omega$ and $Pin\,\pi = P$ if and only if there exists a permutation π' of the elements $N_{k+1} \cup P$ with $Ord\,\pi' = \omega$ and $Pin\,\pi' = P$.

In essence, this says that if we have a witness π to an ordering, then we should be able to find a witness π' to that same ordering where the largest non-pinnacles are removed and the remaining non-pinnacles alternate with the pinnacles.

Eliminating n

Definition

Let N_i be the set of the i smallest elements not in P.

Example: If $n = 9$ and $P = \{3, 5\}$, then $N_4 = \{1, 2, 4, 6\}$.

Theorem

Suppose $P \subseteq [n]$ with $|P| = k$ and let ω be some ordering of the elements in P. Then there exists a permutation π of the elements $[n]$ with $Ord\,\pi = \omega$ and $Pin\,\pi = P$ if and only if there exists a permutation π' of the elements $N_{k+1} \cup P$ with $Ord\,\pi' = \omega$ and $Pin\,\pi' = P$.

In essence, this says that if we have a witness π to an ordering, then we should be able to find a witness π' to that same ordering where the largest non-pinnacles are removed and the remaining non-pinnacles alternate with the pinnacles.

Example: Suppose $n = 8$ and $P = \{7, 8\}$. Then $\omega = 78$ is admissible because we have $\pi = 43176852$.

Eliminating n

Definition

Let N_i be the set of the i smallest elements not in P.

Example: If $n = 9$ and $P = \{3, 5\}$, then $N_4 = \{1, 2, 4, 6\}$.

Theorem

Suppose $P \subseteq [n]$ with $|P| = k$ and let ω be some ordering of the elements in P. Then there exists a permutation π of the elements $[n]$ with $Ord\,\pi = \omega$ and $Pin\,\pi = P$ if and only if there exists a permutation π' of the elements $N_{k+1} \cup P$ with $Ord\,\pi' = \omega$ and $Pin\,\pi' = P$.

In essence, this says that if we have a witness π to an ordering, then we should be able to find a witness π' to that same ordering where the largest non-pinnacles are removed and the remaining non-pinnacles alternate with the pinnacles.

Example: Suppose $n = 8$ and $P = \{7, 8\}$. Then $\omega = 78$ is admissible because we have $\pi = 43176852$. The theorem states there ought to be a witness using only the non-pinnacles $1, 2$, and 3,

Eliminating n

Definition

Let N_i be the set of the i smallest elements not in P.

Example: If $n = 9$ and $P = \{3, 5\}$, then $N_4 = \{1, 2, 4, 6\}$.

Theorem

Suppose $P \subseteq [n]$ with $|P| = k$ and let ω be some ordering of the elements in P. Then there exists a permutation π of the elements $[n]$ with $Ord\,\pi = \omega$ and $Pin\,\pi = P$ if and only if there exists a permutation π' of the elements $N_{k+1} \cup P$ with $Ord\,\pi' = \omega$ and $Pin\,\pi' = P$.

In essence, this says that if we have a witness π to an ordering, then we should be able to find a witness π' to that same ordering where the largest non-pinnacles are removed and the remaining non-pinnacles alternate with the pinnacles.

Example: Suppose $n = 8$ and $P = \{7, 8\}$. Then $\omega = 78$ is admissible because we have $\pi = 43176852$. The theorem states there ought to be a witness using only the non-pinnacles $1, 2$, and 3, and there is: $\pi' = 17382$.

Eliminating n

The reverse direction of this theorem is easy because given some witness π' on the elements $N_{k+1} \cup P$ to an ordering, we may simply add on all the missing non-pinnacles to the end of π' in increasing order to get a permutation $\pi \in S_n$ with the same ordering.

Eliminating n

The reverse direction of this theorem is easy because given some witness π' on the elements $N_{k+1} \cup P$ to an ordering, we may simply add on all the missing non-pinnacles to the end of π' in increasing order to get a permutation $\pi \in S_n$ with the same ordering.

Example: $n = 9$, $P = \{3, 7\}$, $N_{k+1} = N_3 = \{1, 2, 4\}$, $\omega = 37$.

Eliminating n

The reverse direction of this theorem is easy because given some witness π' on the elements $N_{k+1} \cup P$ to an ordering, we may simply add on all the missing non-pinnacles to the end of π' in increasing order to get a permutation $\pi \in S_n$ with the same ordering.

Example: $n = 9$, $P = \{3, 7\}$, $N_{k+1} = N_3 = \{1, 2, 4\}$, $\omega = 37$.

Then if $\pi' = 13274$, we can define $\pi = 132745689 \in S_9$

Eliminating n

The reverse direction of this theorem is easy because given some witness π' on the elements $N_{k+1} \cup P$ to an ordering, we may simply add on all the missing non-pinnacles to the end of π' in increasing order to get a permutation $\pi \in S_n$ with the same ordering.

Example: $n = 9$, $P = \{3, 7\}$, $N_{k+1} = N_3 = \{1, 2, 4\}$, $\omega = 37$.

Then if $\pi' = 13274$, we can define $\pi = 132745689 \in S_9$

To prove the other direction, we examine the blocks of elements between pinnacles in π and try to replace them with single elements that are sufficiently small.

Eliminating n

The reverse direction of this theorem is easy because given some witness π' on the elements $N_{k+1} \cup P$ to an ordering, we may simply add on all the missing non-pinnacles to the end of π' in increasing order to get a permutation $\pi \in S_n$ with the same ordering.

Example: $n = 9$, $P = \{3, 7\}$, $N_{k+1} = N_3 = \{1, 2, 4\}$, $\omega = 37$.

Then if $\pi' = 13274$, we can define $\pi = 132745689 \in S_9$

To prove the other direction, we examine the blocks of elements between pinnacles in π and try to replace them with single elements that are sufficiently small.

Example: $n = 9$, $P = \{5, 7\}$, $\omega = 57$.

Eliminating n

The reverse direction of this theorem is easy because given some witness π' on the elements $N_{k+1} \cup P$ to an ordering, we may simply add on all the missing non-pinnacles to the end of π' in increasing order to get a permutation $\pi \in S_n$ with the same ordering.

Example: $n = 9$, $P = \{3, 7\}$, $N_{k+1} = N_3 = \{1, 2, 4\}$, $\omega = 37$.

Then if $\pi' = 13274$, we can define $\pi = 132745689 \in S_9$

To prove the other direction, we examine the blocks of elements between pinnacles in π and try to replace them with single elements that are sufficiently small.

Example: $n = 9$, $P = \{5, 7\}$, $\omega = 57$.

$$\pi = 453127689$$

Eliminating n

The reverse direction of this theorem is easy because given some witness π' on the elements $N_{k+1} \cup P$ to an ordering, we may simply add on all the missing non-pinnacles to the end of π' in increasing order to get a permutation $\pi \in S_n$ with the same ordering.

Example: $n = 9$, $P = \{3, 7\}$, $N_{k+1} = N_3 = \{1, 2, 4\}$, $\omega = 37$.

Then if $\pi' = 13274$, we can define $\pi = 132745689 \in S_9$

To prove the other direction, we examine the blocks of elements between pinnacles in π and try to replace them with single elements that are sufficiently small.

Example: $n = 9$, $P = \{5, 7\}$, $\omega = 57$.

$$\pi = 453127689$$
$$\pi' = 453127689$$

First copy the original permutation, and identify each block of consecutive non-pinnacles.

Eliminating n

The reverse direction of this theorem is easy because given some witness π' on the elements $N_{k+1} \cup P$ to an ordering, we may simply add on all the missing non-pinnacles to the end of π' in increasing order to get a permutation $\pi \in S_n$ with the same ordering.

Example: $n = 9$, $P = \{3, 7\}$, $N_{k+1} = N_3 = \{1, 2, 4\}$, $\omega = 37$.

Then if $\pi' = 13274$, we can define $\pi = 132745689 \in S_9$

To prove the other direction, we examine the blocks of elements between pinnacles in π and try to replace them with single elements that are sufficiently small.

Example: $n = 9$, $P = \{5, 7\}$, $\omega = 57$.

$$\pi = 453127689$$
$$\pi' = 453127689$$

First copy the original permutation, and identify each block of consecutive non-pinnacles.

Eliminating n

The reverse direction of this theorem is easy because given some witness π' on the elements $N_{k+1} \cup P$ to an ordering, we may simply add on all the missing non-pinnacles to the end of π' in increasing order to get a permutation $\pi \in S_n$ with the same ordering.

Example: $n = 9$, $P = \{3, 7\}$, $N_{k+1} = N_3 = \{1, 2, 4\}$, $\omega = 37$.

Then if $\pi' = 13274$, we can define $\pi = 132745689 \in S_9$

To prove the other direction, we examine the blocks of elements between pinnacles in π and try to replace them with single elements that are sufficiently small.

Example: $n = 9$, $P = \{5, 7\}$, $\omega = 57$.

$$\pi = 453127689$$
$$\pi' = 453127689$$

Next, delete all but the smallest non-pinnacle in each block.

Eliminating n

The reverse direction of this theorem is easy because given some witness π' on the elements $N_{k+1} \cup P$ to an ordering, we may simply add on all the missing non-pinnacles to the end of π' in increasing order to get a permutation $\pi \in S_n$ with the same ordering.

Example: $n = 9$, $P = \{3, 7\}$, $N_{k+1} = N_3 = \{1, 2, 4\}$, $\omega = 37$.

Then if $\pi' = 13274$, we can define $\pi = 132745689 \in S_9$

To prove the other direction, we examine the blocks of elements between pinnacles in π and try to replace them with single elements that are sufficiently small.

Example: $n = 9$, $P = \{5, 7\}$, $\omega = 57$.

$$\pi = 453127689$$
$$\pi' = 45176$$

Next, delete all but the smallest non-pinnacle in each block.

Eliminating n

The reverse direction of this theorem is easy because given some witness π' on the elements $N_{k+1} \cup P$ to an ordering, we may simply add on all the missing non-pinnacles to the end of π' in increasing order to get a permutation $\pi \in S_n$ with the same ordering.

Example: $n = 9$, $P = \{3, 7\}$, $N_{k+1} = N_3 = \{1, 2, 4\}$, $\omega = 37$.

Then if $\pi' = 13274$, we can define $\pi = 132745689 \in S_9$

To prove the other direction, we examine the blocks of elements between pinnacles in π and try to replace them with single elements that are sufficiently small.

Example: $n = 9$, $P = \{5, 7\}$, $\omega = 57$.

$$\pi = 453127689$$
$$\pi' = 45176$$

Finally, within π' standardize just those remaining non-pinnacles to the set N_{k+1}, which in this case is the set $\{1, 2, 3\}$.

Eliminating n

The reverse direction of this theorem is easy because given some witness π' on the elements $N_{k+1} \cup P$ to an ordering, we may simply add on all the missing non-pinnacles to the end of π' in increasing order to get a permutation $\pi \in S_n$ with the same ordering.

Example: $n = 9$, $P = \{3, 7\}$, $N_{k+1} = N_3 = \{1, 2, 4\}$, $\omega = 37$.

Then if $\pi' = 13274$, we can define $\pi = 132745689 \in S_9$

To prove the other direction, we examine the blocks of elements between pinnacles in π and try to replace them with single elements that are sufficiently small.

Example: $n = 9$, $P = \{5, 7\}$, $\omega = 57$.

$$\pi = 453127689$$
$$\pi' = 25173$$

Finally, within π' standardize just those remaining non-pinnacles to the set N_{k+1}, which in this case is the set $\{1, 2, 3\}$.

Eliminating n

The reverse direction of this theorem is easy because given some witness π' on the elements $N_{k+1} \cup P$ to an ordering, we may simply add on all the missing non-pinnacles to the end of π' in increasing order to get a permutation $\pi \in S_n$ with the same ordering.

Example: $n = 9$, $P = \{3, 7\}$, $N_{k+1} = N_3 = \{1, 2, 4\}$, $\omega = 37$.

Then if $\pi' = 13274$, we can define $\pi = 132745689 \in S_9$

To prove the other direction, we examine the blocks of elements between pinnacles in π and try to replace them with single elements that are sufficiently small.

Example: $n = 9$, $P = \{5, 7\}$, $\omega = 57$.

$$\pi = 453127689$$
$$\pi' = 25173$$

Eliminating n

By standardizing our result, we can always reduce to a case where the pinnacles not only alternate with the non-pinnacles, but where the witness is in S_{2k+1}.

Eliminating n

By standardizing our result, we can always reduce to a case where the pinnacles not only alternate with the non-pinnacles, but where the witness is in S_{2k+1}.

Example: when considering the ordering 57 for $P = \{5, 7\}$ and $n = 9$, we may reduce

Eliminating n

By standardizing our result, we can always reduce to a case where the pinnacles not only alternate with the non-pinnacles, but where the witness is in S_{2k+1}.

Example: when considering the ordering 57 for $P = \{5, 7\}$ and $n = 9$, we may reduce

$$\pi = 453127689 \longrightarrow \pi' = 25173 \longrightarrow \pi'' = 24153.$$

Eliminating n

By standardizing our result, we can always reduce to a case where the pinnacles not only alternate with the non-pinnacles, but where the witness is in S_{2k+1}.

Example: when considering the ordering 57 for $P = \{5, 7\}$ and $n = 9$, we may reduce

$$\pi = 453127689 \longrightarrow \pi' = 25173 \longrightarrow \pi'' = 24153.$$

Then π'' is a witness to the ordering 45 of $P' = \{4, 5\}$ where $n = 5$. In π'' we have that 4 in P' plays the role of 5 in P, and 5 in P' plays the role of 7 in P.

Eliminating n

By standardizing our result, we can always reduce to a case where the pinnacles not only alternate with the non-pinnacles, but where the witness is in S_{2k+1}.

Example: when considering the ordering 57 for $P = \{5, 7\}$ and $n = 9$, we may reduce

$$\pi = 453127689 \longrightarrow \pi' = 25173 \longrightarrow \pi'' = 24153.$$

Then π'' is a witness to the ordering 45 of $P' = \{4, 5\}$ where $n = 5$. In π'' we have that 4 in P' plays the role of 5 in P, and 5 in P' plays the role of 7 in P.

Therefore, when trying to find $o(P)$, it is enough to find $o(P')$ since there is a bijection between their orderings.

Eliminating n

By standardizing our result, we can always reduce to a case where the pinnacles not only alternate with the non-pinnacles, but where the witness is in S_{2k+1}.

Example: when considering the ordering 57 for $P = \{5, 7\}$ and $n = 9$, we may reduce

$$\pi = 453127689 \longrightarrow \pi' = 25173 \longrightarrow \pi'' = 24153.$$

Then π'' is a witness to the ordering 45 of $P' = \{4, 5\}$ where $n = 5$. In π'' we have that 4 in P' plays the role of 5 in P, and 5 in P' plays the role of 7 in P.

Therefore, when trying to find $o(P)$, it is enough to find $o(P')$ since there is a bijection between their orderings.

In what follows, we will assume the case where $n = 2k + 1$ so that pinnacles and non-pinnacles alternate.

Reducing P

Since the complexity of the problem ultimately depends on k alone, we would like a way of reducing k.

Reducing P

Since the complexity of the problem ultimately depends on k alone, we would like a way of reducing k.

Definition

Given some $P = \{p_1 < p_2 < \cdots < p_k\}$ and complement set $N = \{n_1 < n_2 < \cdots < n_{k+1}\}$ so that $P \cup N = [2k+1]$, we consider the set $P \cup N \setminus \{p_1, n_1\}$ and standardize. We define the *reduction operator* $r(P) = P'$ to be the set to which $P \setminus \{p_1\}$ standardizes.

Reducing P

Since the complexity of the problem ultimately depends on k alone, we would like a way of reducing k.

Definition

Given some $P = \{p_1 < p_2 < \cdots < p_k\}$ and complement set $N = \{n_1 < n_2 < \cdots < n_{k+1}\}$ so that $P \cup N = [2k+1]$, we consider the set $P \cup N \setminus \{p_1, n_1\}$ and standardize. We define the *reduction operator* $r(P) = P'$ to be the set to which $P \setminus \{p_1\}$ standardizes.

In other words, $r(P)$ is calculated by removing the smallest element in P and the smallest element not in P, and then shifting the remaining elements in P down.

Reducing P

Since the complexity of the problem ultimately depends on k alone, we would like a way of reducing k.

Definition

Given some $P = \{p_1 < p_2 < \cdots < p_k\}$ and complement set $N = \{n_1 < n_2 < \cdots < n_{k+1}\}$ so that $P \cup N = [2k+1]$, we consider the set $P \cup N \setminus \{p_1, n_1\}$ and standardize. We define the *reduction operator* $r(P) = P'$ to be the set to which $P \setminus \{p_1\}$ standardizes.

In other words, $r(P)$ is calculated by removing the smallest element in P and the smallest element not in P, and then shifting the remaining elements in P down.

- If $P = \{1, 2, 5\}$, then $N = \{3, 4, 6, 7\}$, $p_1 = 1$, $n_1 = 3$, and $P \cup N \setminus \{p_1, n_1\} = \{2, 4, 5, 6, 7\}$. So $r(P) = \{1, 3\}$.

Reducing P

Since the complexity of the problem ultimately depends on k alone, we would like a way of reducing k.

Definition

Given some $P = \{p_1 < p_2 < \cdots < p_k\}$ and complement set $N = \{n_1 < n_2 < \cdots < n_{k+1}\}$ so that $P \cup N = [2k+1]$, we consider the set $P \cup N \setminus \{p_1, n_1\}$ and standardize. We define the *reduction operator* $r(P) = P'$ to be the set to which $P \setminus \{p_1\}$ standardizes.

In other words, $r(P)$ is calculated by removing the smallest element in P and the smallest element not in P, and then shifting the remaining elements in P down.

- If $P = \{1, 2, 5\}$, then $N = \{3, 4, 6, 7\}$, $p_1 = 1$, $n_1 = 3$, and $P \cup N \setminus \{p_1, n_1\} = \{2, 4, 5, 6, 7\}$. So $r(P) = \{1, 3\}$.
- If $P = \{3, 5, 7\}$, then $N = \{1, 2, 4, 6\}$, $p_1 = 3$, $n_1 = 1$, and $P \cup N \setminus \{p_1, n_1\} = \{2, 4, 5, 6, 7\}$. So $r(P) = \{3, 5\}$.

Reducing P

Since the complexity of the problem ultimately depends on k alone, we would like a way of reducing k.

Definition

Given some $P = \{p_1 < p_2 < \cdots < p_k\}$ and complement set $N = \{n_1 < n_2 < \cdots < n_{k+1}\}$ so that $P \cup N = [2k+1]$, we consider the set $P \cup N \setminus \{p_1, n_1\}$ and standardize. We define the *reduction operator* $r(P) = P'$ to be the set to which $P \setminus \{p_1\}$ standardizes.

In other words, $r(P)$ is calculated by removing the smallest element in P and the smallest element not in P, and then shifting the remaining elements in P down.

- If $P = \{1, 2, 5\}$, then $N = \{3, 4, 6, 7\}$, $p_1 = 1$, $n_1 = 3$, and $P \cup N \setminus \{p_1, n_1\} = \{2, 4, 5, 6, 7\}$. So $r(P) = \{1, 3\}$.
- If $P = \{3, 5, 7\}$, then $N = \{1, 2, 4, 6\}$, $p_1 = 3$, $n_1 = 1$, and $P \cup N \setminus \{p_1, n_1\} = \{2, 4, 5, 6, 7\}$. So $r(P) = \{3, 5\}$.

Note that $n_1 < p_1$ implies $r(P) = \{p_2 - 2 < \cdots < p_k - 2\}$

Expanded orderings

Recall that N_i is the set of the i smallest elements not in P.

Expanded orderings

Recall that N_i is the set of the i smallest elements not in P.

Definition

Let ω be an ordering of the elements $P \cup N_i$ for some i. Then we let

$$O_i(P) = \{\omega \mid \text{there exists } \pi \in S_{2k+1} \text{ with } \operatorname{Ord}\pi = \omega, \operatorname{Pin}\pi = P\}$$

and let

$$o_i(P) = \#O_i(P).$$

Expanded orderings

Recall that N_i is the set of the i smallest elements not in P.

Definition

Let ω be an ordering of the elements $P \cup N_i$ for some i. Then we let

$$O_i(P) = \{\omega \mid \text{there exists } \pi \in S_{2k+1} \text{ with } \operatorname{Ord}\pi = \omega, \operatorname{Pin}\pi = P\}$$

and let

$$o_i(P) = \#O_i(P).$$

In essence, $O_i(P)$ is a set of orderings in which we keep track of not only the desired pinnacle values in P, but also some of the smallest elements not in P.

Examples

Suppose that $P = \{3, 5\}$. Then...

Examples

Suppose that $P = \{3, 5\}$. Then...

- $O_0(P) = O(P) = \{35, 53\}$ (with witnesses 13254 and 45231).

Examples

Suppose that $P = \{3, 5\}$. Then...

- $O_0(P) = O(P) = \{35, 53\}$ (with witnesses 13254 and 45231).
- $O_1(P) = \{135, 153, 315, 513, 351, 531\}$ since each has a witness. For instance, the ordering 135 has witness $\pi = 13254$.

Examples

Suppose that $P = \{3, 5\}$. Then...

- $O_0(P) = O(P) = \{35, 53\}$ (with witnesses 13254 and 45231).
- $O_1(P) = \{135, 153, 315, 513, 351, 531\}$ since each has a witness. For instance, the ordering 135 has witness $\pi = 13254$.
- $O_2(P) = \{1325, 2315, 5132, 5231\}$. For instance, 1325 has witness $\pi = 13254$, but the ordering 1352 does not have a witness because there would be no place to insert the 4 to create the necessary pinnacles.

Examples

Suppose that $P = \{3, 5\}$. Then...

- $O_0(P) = O(P) = \{35, 53\}$ (with witnesses 13254 and 45231).
- $O_1(P) = \{135, 153, 315, 513, 351, 531\}$ since each has a witness. For instance, the ordering 135 has witness $\pi = 13254$.
- $O_2(P) = \{1325, 2315, 5132, 5231\}$. For instance, 1325 has witness $\pi = 13254$, but the ordering 1352 does not have a witness because there would be no place to insert the 4 to create the necessary pinnacles.
- $O_3(P) = \{13254, 23154, 45132, 45231\}$ which is every permutation that has pinnacle set P. In general however, we will not need to consider $O_i(P)$ where i is larger than the smallest element in P.

Expanded orderings

We can further refine $O_i(P)$ as follows:

Expanded orderings

We can further refine $O_i(P)$ as follows:

Definition

Let $j \in \{0, 1, 2\}$ and p_1 be the smallest element in P. Then we define

$$O_i^j(P) = \{\omega \in O_i(P) \mid p_1 \text{ is adjacent to exactly } j \text{ elements in } N_i\}.$$

Also we let

$$o_i^j(P) = \#O_i^j(P).$$

Expanded orderings

We can further refine $O_i(P)$ as follows:

Definition

Let $j \in \{0, 1, 2\}$ and p_1 be the smallest element in P. Then we define

$$O_i^j(P) = \{\omega \in O_i(P) \mid p_1 \text{ is adjacent to exactly } j \text{ elements in } N_i\}.$$

Also we let

$$o_i^j(P) = \#O_i^j(P).$$

The $O_i^j(P)$ clearly partition $O_i(P)$ for $j \in \{0, 1, 2\}$.

Expanded orderings

We can further refine $O_i(P)$ as follows:

Definition

Let $j \in \{0, 1, 2\}$ and p_1 be the smallest element in P. Then we define

$$O_i^j(P) = \{\omega \in O_i(P) \mid p_1 \text{ is adjacent to exactly } j \text{ elements in } N_i\}.$$

Also we let

$$o_i^j(P) = \#O_i^j(P).$$

The $O_i^j(P)$ clearly partition $O_i(P)$ for $j \in \{0, 1, 2\}$.

Example: if $P = \{4, 5\}$ and $i = 2$ so that ω is an ordering of $\{1, 2, 4, 5\}$, then...

Expanded orderings

We can further refine $O_i(P)$ as follows:

Definition

Let $j \in \{0, 1, 2\}$ and p_1 be the smallest element in P. Then we define

$$O_i^j(P) = \{\omega \in O_i(P) \mid p_1 \text{ is adjacent to exactly } j \text{ elements in } N_i\}.$$

Also we let

$$o_i^j(P) = \#O_i^j(P).$$

The $O_i^j(P)$ clearly partition $O_i(P)$ for $j \in \{0, 1, 2\}$.

Example: if $P = \{4, 5\}$ and $i = 2$ so that ω is an ordering of $\{1, 2, 4, 5\}$, then...

- $O_2^2(P)$

Expanded orderings

We can further refine $O_i(P)$ as follows:

Definition

Let $j \in \{0, 1, 2\}$ and p_1 be the smallest element in P. Then we define

$$O_i^j(P) = \{\omega \in O_i(P) \mid p_1 \text{ is adjacent to exactly } j \text{ elements in } N_i\}.$$

Also we let

$$o_i^j(P) = \#O_i^j(P).$$

The $O_i^j(P)$ clearly partition $O_i(P)$ for $j \in \{0, 1, 2\}$.

Example: if $P = \{4, 5\}$ and $i = 2$ so that ω is an ordering of $\{1, 2, 4, 5\}$, then...

- $O_2^2(P) = \{1425, 2415, 5241, 5142\}$

Expanded orderings

We can further refine $O_i(P)$ as follows:

Definition

Let $j \in \{0, 1, 2\}$ and p_1 be the smallest element in P. Then we define

$$O_i^j(P) = \{\omega \in O_i(P) \mid p_1 \text{ is adjacent to exactly } j \text{ elements in } N_i\}.$$

Also we let

$$o_i^j(P) = \#O_i^j(P).$$

The $O_i^j(P)$ clearly partition $O_i(P)$ for $j \in \{0, 1, 2\}$.

Example: if $P = \{4, 5\}$ and $i = 2$ so that ω is an ordering of $\{1, 2, 4, 5\}$, then...

- $O_2^2(P) = \{1425, 2415, 5241, 5142\}$
- $O_2^1(P)$

Expanded orderings

We can further refine $O_i(P)$ as follows:

Definition

Let $j \in \{0,1,2\}$ and p_1 be the smallest element in P. Then we define

$$O_i^j(P) = \{\omega \in O_i(P) \mid p_1 \text{ is adjacent to exactly } j \text{ elements in } N_i\}.$$

Also we let

$$o_i^j(P) = \#O_i^j(P).$$

The $O_i^j(P)$ clearly partition $O_i(P)$ for $j \in \{0,1,2\}$.

Example: if $P = \{4,5\}$ and $i = 2$ so that ω is an ordering of $\{1,2,4,5\}$, then...

- $O_2^2(P) = \{1425, 2415, 5241, 5142\}$
- $O_2^1(P) = \{4251, 4152, 1524, 2514\}$

Expanded orderings

We can further refine $O_i(P)$ as follows:

Definition

Let $j \in \{0, 1, 2\}$ and p_1 be the smallest element in P. Then we define

$$O_i^j(P) = \{\omega \in O_i(P) \mid p_1 \text{ is adjacent to exactly } j \text{ elements in } N_i\}.$$

Also we let

$$o_i^j(P) = \#O_i^j(P).$$

The $O_i^j(P)$ clearly partition $O_i(P)$ for $j \in \{0, 1, 2\}$.

Example: if $P = \{4, 5\}$ and $i = 2$ so that ω is an ordering of $\{1, 2, 4, 5\}$, then...

- $O_2^2(P) = \{1425, 2415, 5241, 5142\}$
- $O_2^1(P) = \{4251, 4152, 1524, 2514\}$
- $O_2^0(P)$

Expanded orderings

We can further refine $O_i(P)$ as follows:

Definition

Let $j \in \{0,1,2\}$ and p_1 be the smallest element in P. Then we define

$$O_i^j(P) = \{\omega \in O_i(P) \mid p_1 \text{ is adjacent to exactly } j \text{ elements in } N_i\}.$$

Also we let

$$o_i^j(P) = \#O_i^j(P).$$

The $O_i^j(P)$ clearly partition $O_i(P)$ for $j \in \{0,1,2\}$.

Example: if $P = \{4,5\}$ and $i = 2$ so that ω is an ordering of $\{1,2,4,5\}$, then...

- $O_2^2(P) = \{1425, 2415, 5241, 5142\}$
- $O_2^1(P) = \{4251, 4152, 1524, 2514\}$
- $O_2^0(P) = \emptyset$ since in any witness 4 has to be surrounded by two elements in $\{1,2,3\}$ to be a pinnacle.

Major result

These ideas will allow us to recurse on $k = \#P$ thanks to the following result.

Major result

These ideas will allow us to recurse on $k = \#P$ thanks to the following result.

Theorem

Suppose $i \geq 0$, P is non-empty, and $i < p_1$. Suppose further that for some $j \in \{0, 1, 2\}$ we have $O_i^j(P) \neq \emptyset$. Then if we let $P' = r(P)$, we have the following

$$o_i^j(P) = \begin{cases} i(i-1)o_{i-1}(P') & \text{if } j = 2, \\ 2io_i(P') & \text{if } j = 1, \\ o_{i+1}(P') & \text{if } j = 0. \end{cases}$$

Major result

These ideas will allow us to recurse on $k = \#P$ thanks to the following result.

Theorem

Suppose $i \geq 0$, P is non-empty, and $i < p_1$. Suppose further that for some $j \in \{0, 1, 2\}$ we have $O_i^j(P) \neq \emptyset$. Then if we let $P' = r(P)$, we have the following

$$o_i^j(P) = \begin{cases} i(i-1)o_{i-1}(P') & \text{if } j = 2, \\ 2io_i(P') & \text{if } j = 1, \\ o_{i+1}(P') & \text{if } j = 0. \end{cases}$$

Basically, this means that every ordering in $O_i(P)$ corresponds to some ordering in one of $O_{i-1}(P'), O_i(P')$, or $O_{i+1}(P')$ depending on the value of j.

Major result

These ideas will allow us to recurse on $k = \#P$ thanks to the following result.

Theorem

Suppose $i \geq 0$, P is non-empty, and $i < p_1$. Suppose further that for some $j \in \{0, 1, 2\}$ we have $O_i^j(P) \neq \emptyset$. Then if we let $P' = r(P)$, we have the following

$$o_i^j(P) = \begin{cases} i(i-1)o_{i-1}(P') & \text{if } j = 2, \\ 2io_i(P') & \text{if } j = 1, \\ o_{i+1}(P') & \text{if } j = 0. \end{cases}$$

Basically, this means that every ordering in $O_i(P)$ corresponds to some ordering in one of $O_{i-1}(P'), O_i(P')$, or $O_{i+1}(P')$ depending on the value of j.

The proof effectively deletes the smallest element in P together with some element smaller than all elements in P, and then standardizes the remaining elements of P to be in P'.

Proof of major result

We'll illustrate one case of the proof by example.

Proof of major result

We'll illustrate one case of the proof by example.

Suppose that $j = 2$ and $i = 3$; we must show that

$$o_3^2(P) = 3(3-1)o_{3-1}(P') = 3 \cdot 2o_2(P').$$

Proof of major result

We'll illustrate one case of the proof by example.

Suppose that $j = 2$ and $i = 3$; we must show that

$$o_3^2(P) = 3(3-1)o_{3-1}(P') = 3 \cdot 2 o_2(P').$$

Consider $P = \{4, 6, 8, 9\}$ so that $P' = \{4, 6, 7\}$, and
$\omega = 3416829 \in O_3^2(P)$.

Proof of major result

We'll illustrate one case of the proof by example.

Suppose that $j = 2$ and $i = 3$; we must show that

$$o_3^2(P) = 3(3-1)o_{3-1}(P') = 3 \cdot 2 o_2(P').$$

Consider $P = \{4, 6, 8, 9\}$ so that $P' = \{4, 6, 7\}$, and $\omega = 3416829 \in O_3^2(P)$.

To get the ordering in $O_2(P')$, we replace p_1 and its adjacent elements with the element 0 and standardize to the set $P' \cup [i-1]$.

Proof of major result

We'll illustrate one case of the proof by example.

Suppose that $j = 2$ and $i = 3$; we must show that

$$o_3^2(P) = 3(3-1)o_{3-1}(P') = 3 \cdot 2 o_2(P').$$

Consider $P = \{4, 6, 8, 9\}$ so that $P' = \{4, 6, 7\}$, and $\omega = 3416829 \in O_3^2(P)$.

To get the ordering in $O_2(P')$, we replace p_1 and its adjacent elements with the element 0 and standardize to the set $P' \cup [i-1]$.

$$3416829 \longrightarrow 06829 \longrightarrow 14627$$

Proof of major result

We'll illustrate one case of the proof by example.

Suppose that $j = 2$ and $i = 3$; we must show that

$$o_3^2(P) = 3(3-1)o_{3-1}(P') = 3 \cdot 2 o_2(P').$$

Consider $P = \{4, 6, 8, 9\}$ so that $P' = \{4, 6, 7\}$, and
$\omega = 3416829 \in O_3^2(P)$.

To get the ordering in $O_2(P')$, we replace p_1 and its adjacent elements with the element 0 and standardize to the set $P' \cup [i-1]$.

$$3416829 \longrightarrow 06829 \longrightarrow 14627$$

The swap has effectively lost us one element in N_i along with the smallest pinnacle.

Proof of major result

We'll illustrate one case of the proof by example.

Suppose that $j = 2$ and $i = 3$; we must show that

$$o_3^2(P) = 3(3-1)o_{3-1}(P') = 3 \cdot 2o_2(P').$$

Consider $P = \{4, 6, 8, 9\}$ so that $P' = \{4, 6, 7\}$, and $\omega = 3416829 \in O_3^2(P)$.

To get the ordering in $O_2(P')$, we replace p_1 and its adjacent elements with the element 0 and standardize to the set $P' \cup [i-1]$.

$$3416829 \longrightarrow 06829 \longrightarrow 14627$$

The swap has effectively lost us one element in N_i along with the smallest pinnacle.

We must now show the result is admissible. Take any witness to ω and preform the same process as above to get a witness to the new ordering, except now we standardize to $[2k-1]$.

Proof of major result

We'll illustrate one case of the proof by example.

Suppose that $j = 2$ and $i = 3$; we must show that

$$o_3^2(P) = 3(3-1)o_{3-1}(P') = 3 \cdot 2o_2(P').$$

Consider $P = \{4, 6, 8, 9\}$ so that $P' = \{4, 6, 7\}$, and $\omega = 3416829 \in O_3^2(P)$.

To get the ordering in $O_2(P')$, we replace p_1 and its adjacent elements with the element 0 and standardize to the set $P' \cup [i-1]$.

$$3416829 \longrightarrow 06829 \longrightarrow 14627$$

The swap has effectively lost us one element in N_i along with the smallest pinnacle.

We must now show the result is admissible. Take any witness to ω and preform the same process as above to get a witness to the new ordering, except now we standardize to $[2k-1]$.

$$\pi = 341658397 \longrightarrow 0658397 \longrightarrow 1436275$$

Proof of major result

Want to show:

$$o_3^2(P) = 3(3-1)o_{3-1}(P') = 3 \cdot 2 o_2(P')$$

where $P = \{4, 6, 8, 9\}$ so that $P' = \{4, 6, 7\}$

Proof of major result

Want to show:

$$o_3^2(P) = 3(3-1)o_{3-1}(P') = 3 \cdot 2o_2(P')$$

where $P = \{4, 6, 8, 9\}$ so that $P' = \{4, 6, 7\}$

To reverse this process, we first choose an ordered pair of distinct elements $x, y \in N_i$ which gives us $i(i-1)$ choices. We then form the factor xp_1y. For this example, to get back to the original ordering we will use 341.

Proof of major result

Want to show:

$$o_3^2(P) = 3(3-1)o_{3-1}(P') = 3 \cdot 2 o_2(P')$$

where $P = \{4, 6, 8, 9\}$ so that $P' = \{4, 6, 7\}$

To reverse this process, we first choose an ordered pair of distinct elements $x, y \in N_i$ which gives us $i(i-1)$ choices. We then form the factor xp_1y. For this example, to get back to the original ordering we will use 341.

Next, we start with our ordering in $O_2(P')$ and standardize it to the set $(P \setminus \{p_1\}) \cup (N_i \setminus \{x, y\}) \cup \{0\}$. Then we replace the 0 with the factor we formed.

Proof of major result

Want to show:

$$o_3^2(P) = 3(3-1)o_{3-1}(P') = 3 \cdot 2o_2(P')$$

where $P = \{4, 6, 8, 9\}$ so that $P' = \{4, 6, 7\}$

To reverse this process, we first choose an ordered pair of distinct elements $x, y \in N_i$ which gives us $i(i-1)$ choices. We then form the factor xp_1y. For this example, to get back to the original ordering we will use 341.

Next, we start with our ordering in $O_2(P')$ and standardize it to the set $(P \setminus \{p_1\}) \cup (N_i \setminus \{x, y\}) \cup \{0\}$. Then we replace the 0 with the factor we formed.

$$14627 \longrightarrow 06829 \longrightarrow 3416829$$

Proof of major result

Want to show:

$$o_3^2(P) = 3(3-1)o_{3-1}(P') = 3 \cdot 2o_2(P')$$

where $P = \{4, 6, 8, 9\}$ so that $P' = \{4, 6, 7\}$

To reverse this process, we first choose an ordered pair of distinct elements $x, y \in N_i$ which gives us $i(i-1)$ choices. We then form the factor xp_1y. For this example, to get back to the original ordering we will use 341.

Next, we start with our ordering in $O_2(P')$ and standardize it to the set $(P \setminus \{p_1\}) \cup (N_i \setminus \{x, y\}) \cup \{0\}$. Then we replace the 0 with the factor we formed.

$$14627 \longrightarrow 06829 \longrightarrow 3416829$$

The witness that shows this ordering is admissible is gotten by doing these same steps on some original witness, except now we standardize to $[2k+1] \setminus \{p_1, x, y\} \cup \{0\}$.

Proof of major result

Want to show:

$$o_3^2(P) = 3(3-1)o_{3-1}(P') = 3 \cdot 2 o_2(P')$$

where $P = \{4, 6, 8, 9\}$ so that $P' = \{4, 6, 7\}$

To reverse this process, we first choose an ordered pair of distinct elements $x, y \in N_i$ which gives us $i(i-1)$ choices. We then form the factor xp_1y. For this example, to get back to the original ordering we will use 341.

Next, we start with our ordering in $O_2(P')$ and standardize it to the set $(P \setminus \{p_1\}) \cup (N_i \setminus \{x, y\}) \cup \{0\}$. Then we replace the 0 with the factor we formed.

$$14627 \longrightarrow 06829 \longrightarrow 3416829$$

The witness that shows this ordering is admissible is gotten by doing these same steps on some original witness, except now we standardize to $[2k+1] \setminus \{p_1, x, y\} \cup \{0\}$.

$$\pi = 1436275 \longrightarrow 0658297 \longrightarrow 341658297$$

Major result

Theorem

Suppose $i \geq 0$, P is non-empty, and $i < p_1$. Suppose further that for some $j \in \{0, 1, 2\}$ we have $O_i^j(P) \neq \emptyset$. Then if we let $P' = r(P)$, we have the following

$$o_i^j(P) = \begin{cases} i(i-1)o_{i-1}(P') & \text{if } j = 2, \\ 2io_i(P') & \text{if } j = 1, \\ o_{i+1}(P') & \text{if } j = 0. \end{cases}$$

Main recursion

Definition

Given some set P with smallest element p_1, we make the following definition:

$$\delta_j = \begin{cases} 1 & \text{if } p_1 > j, \\ 0 & \text{otherwise.} \end{cases}$$

Main recursion

Definition

Given some set P with smallest element p_1, we make the following definition:

$$\delta_j = \begin{cases} 1 & \text{if } p_1 > j, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem

Let $|P| \geq 1$, $P' = r(P)$ *and* $0 \leq i < p_1$. *Then we have the following recursion.*

$$o_i(P) = i(i-1)o_{i-1}(P') + (2i)\delta_{i+1}o_i(P') + \delta_{i+2}o_{i+1}(P').$$

Compare

Theorem

Suppose $i \geq 0$, P is non-empty, and $i < p_1$. Suppose further that for some $j \in \{0, 1, 2\}$ we have $O_i^j(P) \neq \emptyset$. Then if we let $P' = r(P)$, we have the following

$$o_i^j(P) = \begin{cases} i(i-1)o_{i-1}(P') & \text{if } j = 2, \\ 2io_i(P') & \text{if } j = 1, \\ o_{i+1}(P') & \text{if } j = 0. \end{cases}$$

Theorem

Let $|P| \geq 1$, $P' = r(P)$ and $0 \leq i < p_1$. Then we have the following recursion.

$$o_i(P) = i(i-1)o_{i-1}(P') + (2i)\delta_{i+1}o_i(P') + \delta_{i+2}o_{i+1}(P').$$

Main recursion

Theorem

Let $|P| \geq 1$, $P' = r(P)$ and $0 \leq i < p_1$. Then we have the following recursion.

$$o_i(P) = i(i-1)o_{i-1}(P') + (2i)\delta_{i+1}o_i(P') + \delta_{i+2}o_{i+1}(P').$$

Why do we need the δ_i's?

Main recursion

Theorem

Let $|P| \geq 1$, $P' = r(P)$ and $0 \leq i < p_1$. Then we have the following recursion.

$$o_i(P) = i(i-1)o_{i-1}(P') + (2i)\delta_{i+1}o_i(P') + \delta_{i+2}o_{i+1}(P').$$

Why do we need the δ_i's?

Consider the case $P = \{3, 5\}$ and $i = 2$. Then
$O_2(P) = O_2^2(P) = \{1325, 2315, 5132, 5231\}$.

Main recursion

Theorem

Let $|P| \geq 1$, $P' = r(P)$ and $0 \leq i < p_1$. Then we have the following recursion.

$$o_i(P) = i(i-1)o_{i-1}(P') + (2i)\delta_{i+1}o_i(P') + \delta_{i+2}o_{i+1}(P').$$

Why do we need the δ_i's?

Consider the case $P = \{3, 5\}$ and $i = 2$. Then $O_2(P) = O_2^2(P) = \{1325, 2315, 5132, 5231\}$.

Therefore only the first term in the recursion, which corresponds to $j = 2$, should contribute to the sum.

Main recursion

Theorem

Let $|P| \geq 1$, $P' = r(P)$ and $0 \leq i < p_1$. Then we have the following recursion.

$$o_i(P) = i(i-1)o_{i-1}(P') + (2i)\delta_{i+1}o_i(P') + \delta_{i+2}o_{i+1}(P').$$

Why do we need the δ_i's?

Consider the case $P = \{3, 5\}$ and $i = 2$. Then $O_2(P) = O_2^2(P) = \{1325, 2315, 5132, 5231\}$.

Therefore only the first term in the recursion, which corresponds to $j = 2$, should contribute to the sum.

But $P' = \{3\}$ and $O_2(P') = \{132, 231\}$ is nonempty, and therefore will contribute to the sum if the $\delta_{i+1} = \delta_3$ didn't kill it.

Main recursion

Theorem

Let $|P| \geq 1$, $P' = r(P)$ *and* $0 \leq i < p_1$. *Then we have the following recursion.*

$$o_i(P) = i(i-1)o_{i-1}(P') + (2i)\delta_{i+1}o_i(P') + \delta_{i+2}o_{i+1}(P').$$

So we're done!

Main recursion

Theorem

Let $|P| \geq 1$, $P' = r(P)$ and $0 \leq i < p_1$. Then we have the following recursion.

$$o_i(P) = i(i-1)o_{i-1}(P') + (2i)\delta_{i+1}o_i(P') + \delta_{i+2}o_{i+1}(P').$$

So we're done!

We can find the number of admissible orderings by starting with $o_0(P) = o(P)$. Each time we apply the recursion the number of elements in P goes down by one until we hit our base case when $P = \emptyset$, where we will always have only one ordering.

Run time analysis

Let us denote m applications of the reduction operator by P^m.

Run time analysis

Let us denote m applications of the reduction operator by P^m.

For instance, $r(P) = P' = P^1$ and $r(r(r(P))) = P^3$.

Run time analysis

Let us denote m applications of the reduction operator by P^m.

For instance, $r(P) = P' = P^1$ and $r(r(r(P))) = P^3$.

- All calculated terms are of the form $o_i(P^m)$.

Run time analysis

Let us denote m applications of the reduction operator by P^m.

For instance, $r(P) = P' = P^1$ and $r(r(r(P))) = P^3$.

- All calculated terms are of the form $o_i(P^m)$.
- i and m start at zero.

Run time analysis

Let us denote m applications of the reduction operator by P^m.

For instance, $r(P) = P' = P^1$ and $r(r(r(P))) = P^3$.

- All calculated terms are of the form $o_i(P^m)$.
- i and m start at zero.
- i and m can increase by at most one each time the recursion is applied.

Run time analysis

Let us denote m applications of the reduction operator by P^m.

For instance, $r(P) = P' = P^1$ and $r(r(r(P))) = P^3$.

- All calculated terms are of the form $o_i(P^m)$.
- i and m start at zero.
- i and m can increase by at most one each time the recursion is applied.
- m can be at most k.

Run time analysis

Let us denote m applications of the reduction operator by P^m.

For instance, $r(P) = P' = P^1$ and $r(r(r(P))) = P^3$.

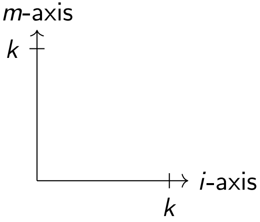
- All calculated terms are of the form $o_i(P^m)$.
- i and m start at zero.
- i and m can increase by at most one each time the recursion is applied.
- m can be at most k.
- We always have $i \leq k + 1 - m$, which is the number of elements not in P after m applications of the recursion.

Run time analysis

Let us denote m applications of the reduction operator by P^m.

For instance, $r(P) = P' = P^1$ and $r(r(r(P))) = P^3$.

- All calculated terms are of the form $o_i(P^m)$.
- i and m start at zero.
- i and m can increase by at most one each time the recursion is applied.
- m can be at most k.
- We always have $i \leq k + 1 - m$, which is the number of elements not in P after m applications of the recursion.



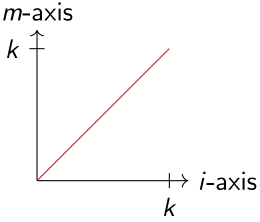
The terms $o_i(P^m)$ we need to compute can be thought of as points in the plane.

Run time analysis

Let us denote m applications of the reduction operator by P^m.

For instance, $r(P) = P' = P^1$ and $r(r(r(P))) = P^3$.

- All calculated terms are of the form $o_i(P^m)$.
- i and m start at zero.
- i and m can increase by at most one each time the recursion is applied.
- m can be at most k.
- We always have $i \leq k + 1 - m$, which is the number of elements not in P after m applications of the recursion.



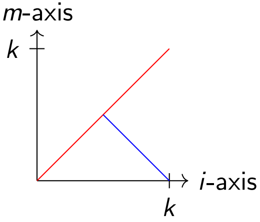
m must increase whenever i increases, i can never exceed m.

Run time analysis

Let us denote m applications of the reduction operator by P^m.

For instance, $r(P) = P' = P^1$ and $r(r(r(P))) = P^3$.

- All calculated terms are of the form $o_i(P^m)$.
- i and m start at zero.
- i and m can increase by at most one each time the recursion is applied.
- m can be at most k.
- We always have $i \leq k + 1 - m$, which is the number of elements not in P after m applications of the recursion.



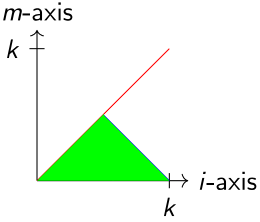
$i \leq k + 1 - m.$

Run time analysis

Let us denote m applications of the reduction operator by P^m.

For instance, $r(P) = P' = P^1$ and $r(r(r(P))) = P^3$.

- All calculated terms are of the form $o_i(P^m)$.
- i and m start at zero.
- i and m can increase by at most one each time the recursion is applied.
- m can be at most k.
- We always have $i \leq k + 1 - m$, which is the number of elements not in P after m applications of the recursion.



So the total number of terms is asymptotically $k^2/4$.

Additional work

There are still some open questions about pinnacle sets.

Additional work

There are still some open questions about pinnacle sets.

Question

Is there a faster way of computing $o(P)$, or a closed formula that can give the number of admissible orderings of P without having to keep track of elements not in P?

Additional work

There are still some open questions about pinnacle sets.

Question

Is there a faster way of computing $o(P)$, or a closed formula that can give the number of admissible orderings of P without having to keep track of elements not in P?

Let $p_n(P)$ be the number of permutations $\pi \in S_n$ with pinnacle set P.

Additional work

There are still some open questions about pinnacle sets.

Question

Is there a faster way of computing $o(P)$, or a closed formula that can give the number of admissible orderings of P without having to keep track of elements not in P?

Let $p_n(P)$ be the number of permutations $\pi \in S_n$ with pinnacle set P.

In addition to this recursion, I was also able to give a combinatorial proof of a weighted sum of $p_n(Q)$ over all $Q \subset P$ originally proven by Fang in 2021, and I was also able to come up with a recursion to compute $p_n(P)$ directly.

Additional work

There are still some open questions about pinnacle sets.

Question

Is there a faster way of computing $o(P)$, or a closed formula that can give the number of admissible orderings of P without having to keep track of elements not in P?

Let $p_n(P)$ be the number of permutations $\pi \in S_n$ with pinnacle set P.

In addition to this recursion, I was also able to give a combinatorial proof of a weighted sum of $p_n(Q)$ over all $Q \subset P$ originally proven by Fang in 2021, and I was also able to come up with a recursion to compute $p_n(P)$ directly.

Question

Does there exist a way of computing $p_n(P)$ that has a faster run time than $O(k^2 \log(n) + k^4)$?

Thank you!

$P \cup N \in [10]$-ded

(pun intended)

References

Robert Davis, Sarah A. Nelson, T. Kyle Petersen, and Bridget E. Tenner. The pinnacle set of a permutation. *Discrete Mathematics*, 341(11):3249–3270, 2018.

Irena Rusu and Bridget Tenner. Admissible pinnacle orderings. *Graphs and Combinatorics*, 37:1205–1214, 2021.

Diaz-Lopez, Alexander and Harris, Pamela E. and Huang, Isabella and Insko, Erik and Nilsen,Lars. A formula for enumerating permutations with a fixed pinnacle set. *Discrete Mathematics*, 344(6):112375, 2021

Justice Falque, Jean-Christophe Novelli, and Jean-Yves Thibon. Pinnacle Sets Revisited. arXiv:2106.05248v1. 2021

Wenjie Fang. Efficient recurrence for the enumeration of permutations with a fixed pinnacle set. arXiv:2106.09147v1. 2021