The Quantum k-Bruhat Order

by

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Kennesaw State University

(Joint work with L. Colmenarejo)

Outline

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1. Classical
   - Schubert polynomials
   - k-Bruhat order
   - Monoid structure

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2. Quantum
   - Quantum Schubert polynomials
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   - k-Bruhat order
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2. Quantum
   - Quantum Schubert polynomials
   - Quantum k-Bruhat order
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3. Future work

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$[5,3,2,1,4]s_{2,4} = [5,1,2,3,4]$

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$(\mathcal{P}, \preceq)$

- (Reflexive) $x \preceq x$
- (Antisymmetric) $x \preceq y$ and $y \preceq x$, implies $x = y$
- (Transitive) $x \preceq y$, $y \preceq z$ implies $x \preceq z$

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Example: $\mathcal{P} = [4]$, $1 \prec 2 \prec 3, 4$

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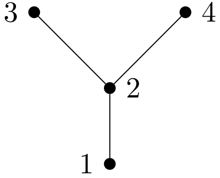

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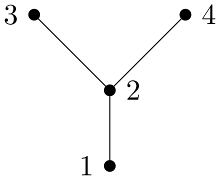

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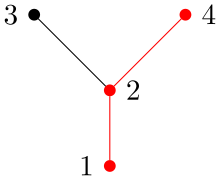

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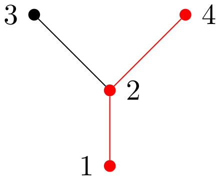

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4. *Chain*

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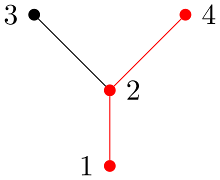

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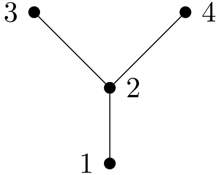

1. *Covering Relation* $\lessdot$
2. *Hasse Diagram*
3. *Interval* $[u, v]$
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5. *Maximal Chain*
6. *Isomorphic*

Schubert Polynomials

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Definitions in terms of

- divided difference operators
- reduced pipe dreams
- bumpless pipe dreams
- compatible sequences
- $\vdots$

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Example: For each Schur polynomial $s_\lambda(x_1, \cdots, x_k)$, there exists $v(\lambda, k) \in S_n$ such that

$$\mathfrak{S}_{v(\lambda,k)} = s_\lambda(x_1, \cdots, x_k).$$

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Example:

$$\mathfrak{S}_{2413} = s_{2,1}(x_1, x_2) = x_1^2 x_2 + x_1 x_2^2$$

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Open Problem: Find a combinatorial construction for the c_{uv}^w.

Schubert Polynomials

Monk's Formula

Monk's Formula: For $u \in S_n$ and $k \in [n-1]$,

$$\mathfrak{S}_u \mathfrak{S}_{s_k} = \sum_{\substack{1 \le a \le k < b \le n \\ \ell(us_{a,b}) = \ell(u)+1}} \mathfrak{S}_{us_{a,b}}.$$

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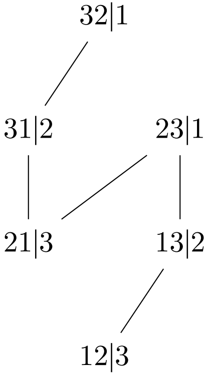
k–Bruhat Order

Poset on S_n defined by covering relations:
$u \lessdot_k w$ if $\ell(w) = \ell(u) + 1$ and $w = us_{a,b}$ where $1 \le a \le k < b \le n$.

k-Bruhat Order

2–Bruhat Order on S_3



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$$c^w_{uv(\lambda,k)} = \left| \left\{ \begin{matrix} \text{maximal chains in } [u,w]_k \text{ satisfying} \\ \textit{some condition} \text{ imposed by } v(\lambda,k) \end{matrix} \right\} \right|.$$

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Proposition (Bergeron-Sottile, 1998)

Let $u, w \in S_n$. Then $u \leq_k w$ if and only if

(i) $a \leq k \leq b$ *implies* $u(a) \leq w(a)$ *and* $u(b) \geq w(b)$.

(ii) *if* $a < b$, $u(a) < u(b)$, *and* $w(a) > w(b)$, *then* $a \leq k < b$.

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Theorem (Bergeron-Sottile, 1998)

Suppose that $u \leq_k w$, $x \leq_l z$, and $wu^{-1} = zx^{-1}$. Then $v \to vu^{-1}x$ induces an isomorphism between $[u, w]_k$ and $[x, z]_l$.

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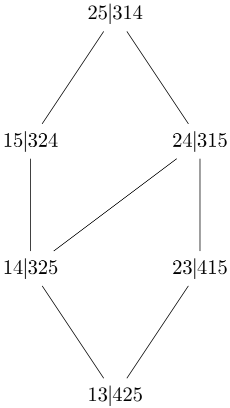
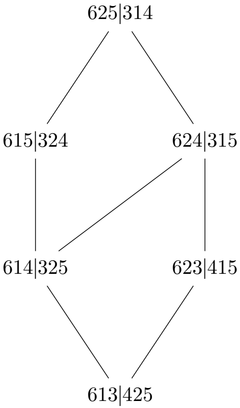


$\leq_2$



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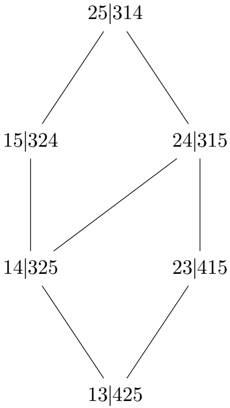

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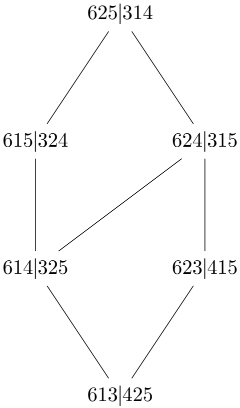

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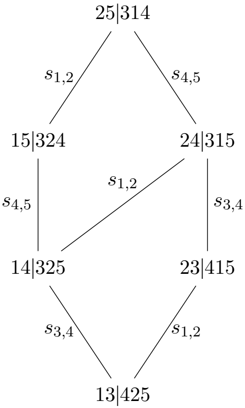

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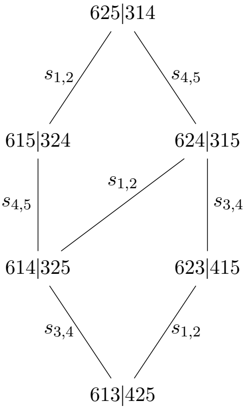

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Grassmannian Bruhat Order

For $\xi, \eta \in S_n$, let $\eta \preceq \xi$ if there exists $u \in S_n$ and $k \in [n-1]$ such that $u \leq_k \eta u \leq_k \xi u$.

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Suppose $u, \zeta \in S_n$. If $u \le_k \zeta u$, then $\nu \to \nu u$ induces an isomorphism between $[e, \zeta]_\le$ and $[u, \zeta u]_k$.

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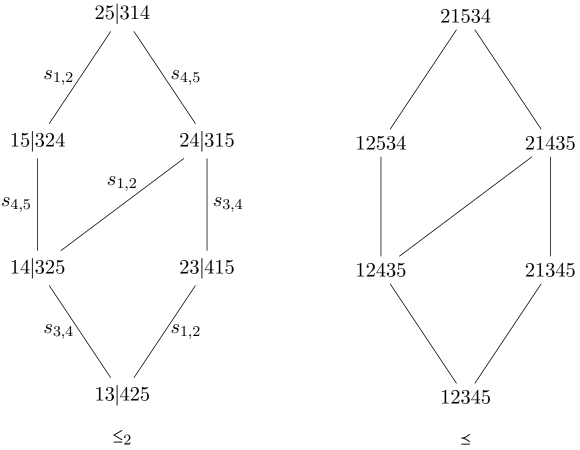

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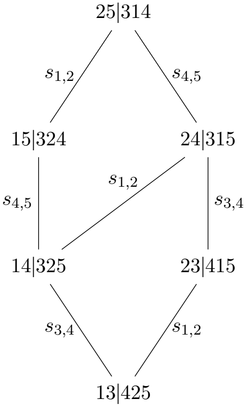

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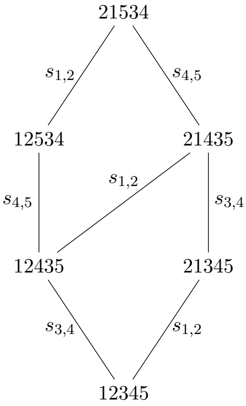

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Monoid Structure

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Let $\mathcal{F}_n$ be the free monoid generated by the symbols

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Define an action of $\mathcal{F}_n$ on $S_n \cup \{0\}$ by

$$\mathbf{0} \cdot u = \mathbf{v}_{a,b} \cdot 0 = 0$$

and

$$\mathbf{v}_{a,b} \cdot u = \begin{cases} s_{a,b}u, & \text{if } u \lessdot s_{a,b}u \\ 0, & \textit{otherwise.} \end{cases}$$

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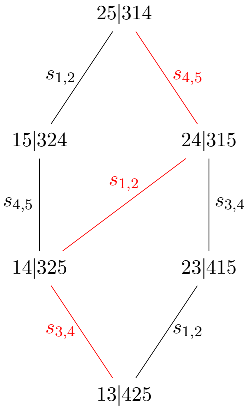

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Grassmannian Bruhat Monoid

$\mathbf{v}_1 \equiv \mathbf{v}_2 \in \mathcal{M}_n$ if and only if $\mathbf{v}_1 \cdot w = \mathbf{v}_2 \cdot w$ for all $w \in S_n$ if and only if $\mathbf{v}_1 \bullet_k w = \mathbf{v}_2 \bullet_k w$ for all $w \in S_n$ and $k \in [n-1]$.

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$\mathcal{M}_n$ is the monoid that has a $\mathbf{0}$ and generators $\mathbf{v}_{a,b}$ indexed by integers $a < b \in [n]$ subject to the relations

(1) $\mathbf{v}_{\beta,\gamma}\mathbf{v}_{\gamma,\delta}\mathbf{v}_{\alpha,\gamma} \equiv \mathbf{v}_{\beta,\delta}\mathbf{v}_{\alpha,\beta}\mathbf{v}_{\beta,\gamma}$ if $\alpha < \beta < \gamma < \delta$,

(2) $\mathbf{v}_{\alpha,\gamma}\mathbf{v}_{\gamma,\delta}\mathbf{v}_{\beta,\gamma} \equiv \mathbf{v}_{\beta,\gamma}\mathbf{v}_{\alpha,\beta}\mathbf{v}_{\beta,\delta}$ if $\alpha < \beta < \gamma < \delta$,

(3) $\mathbf{v}_{\alpha,\beta}\mathbf{v}_{\gamma,\delta} \equiv \mathbf{v}_{\gamma}\mathbf{v}_{\delta}$, if $\beta < \gamma$ or $\alpha < \gamma < \delta < \beta$,

(4) $\mathbf{v}_{\alpha,\gamma}\mathbf{v}_{\beta,\delta} \equiv \mathbf{v}_{\beta,\gamma}\mathbf{v}_{\alpha,\gamma} \equiv \mathbf{0}$, if $\alpha \leq \beta < \gamma \leq \delta$,

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Monoid Structure

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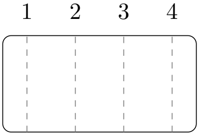

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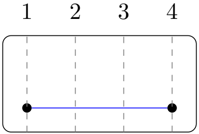

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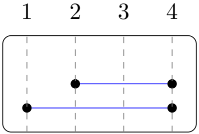

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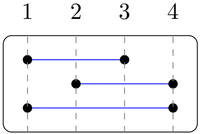

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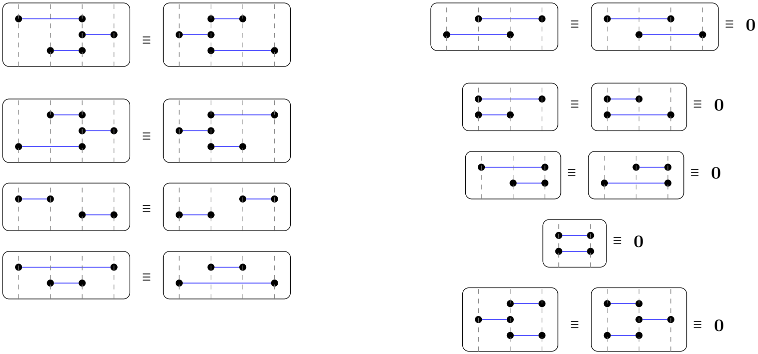

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$\mathbf{v}_1 \equiv \mathbf{v}_2 \in \mathcal{M}_n$ if and only if $\mathbf{v}_1 \cdot w = \mathbf{v}_2 \cdot w$ for all $w \in S_n$ if and only if $\mathbf{v}_1 \bullet_k w = \mathbf{v}_2 \bullet_k w$ for all $w \in S_n$ and $k \in [n-1]$.

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Poset Level:

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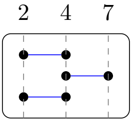

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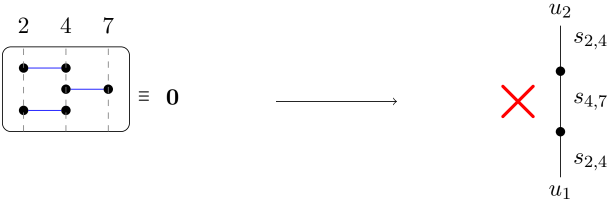

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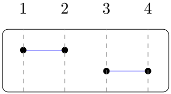

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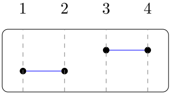

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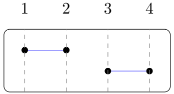

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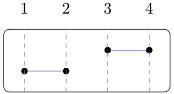

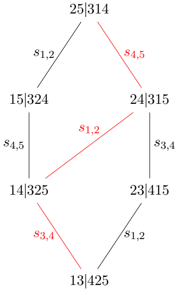

$\leq_2$

Monoid Structure

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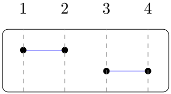

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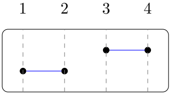

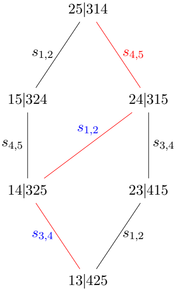

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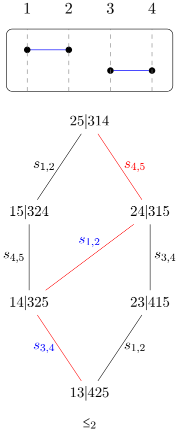

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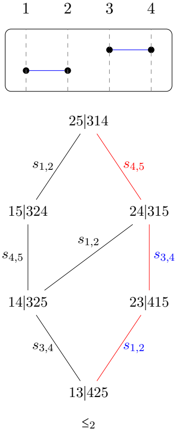

Classical to Quantum

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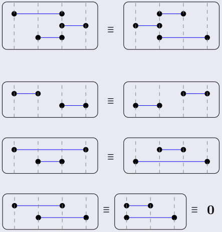
$$\{\mathbf{v}_{a,b} \mid a, b \in [n],\ a < b\} \cup \{\mathbf{0}\}$$

Theorem (Bergeron-Sottile, 1999)



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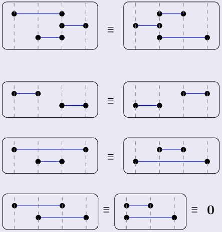

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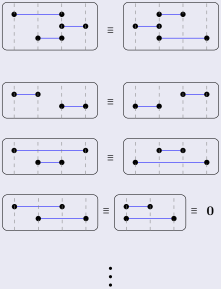

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Theorem

?

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The Quantum Case

Quantum Schubert Polynomials

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Quantum Schubert polynomials are indexed by permutations:

$$u \in S_n \to \mathfrak{S}_u^q \in \mathbb{Z}[q_1, \cdots, q_{n-1}][x_1, \cdots, x_n].$$

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Definitions in terms of

- Schubert polynomials (SEM basis)
- Quantum bumpless pipe dreams (2024)

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Example: Quantum Schur polynomials $s_\lambda^q(x_1, \cdots, x_k) = \mathfrak{S}_{v(\lambda,k)}^q$.

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Example:

$$\mathfrak{S}_{2413}^q = s_{2,1}^q(x_1, x_2) = x_1^2 x_2 + x_1 x_2^2 + q_1 x_1 + q_1 x_2 - q_2 x_1$$

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For geometric reasons, the expansion $\mathfrak{S}_u^q * \mathfrak{S}_v^q = \sum_w c_{uv}^{w\mathbf{d}} q^{\mathbf{d}} \mathfrak{S}_w^q$ has coefficients $c_{uv}^{w\mathbf{d}} \in \mathbb{Z}_{\geq 0}$.

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Open Problem: Find a combinatorial construction for the $c_{uv}^{w\mathbf{d}}$.

Quantum Schubert Polynomials

Quantum Monk's Formula

Quantum Monk's Formula: For $u \in S_n$ and $k \in [n-1]$,

$$\mathfrak{S}_u^q * \mathfrak{S}_{s_k}^q = \sum_{\substack{1 \le a \le k < b \le n \\ \ell(us_{a,b}) = \ell(u)+1}} \mathfrak{S}_{us_{a,b}}^q + \sum_{\substack{1 \le a \le k < b \le n \\ \ell(us_{a,b})+2(b-a) = \ell(u)+1}} \mathbf{q_{a,b}} \mathfrak{S}_{us_{a,b}}^q$$

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where $\mathbf{q_{a,b}} = q_a q_{a+1} \cdots q_{b-1}$.

Quantum k–Bruhat Order

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Quantum k–Bruhat Order: Poset on

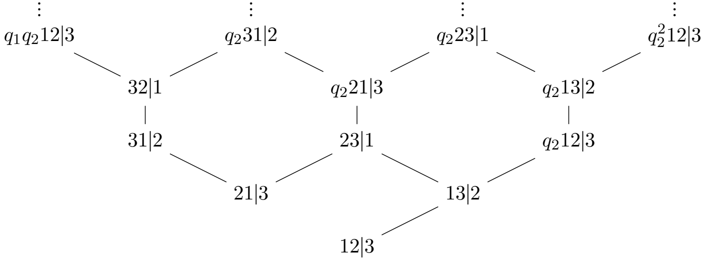
$$S_n[\mathbf{q}] = \{\mathbf{q}^\alpha w \mid w \in S_n,\ \alpha \in \mathbb{N}^{n-1}\}$$

with relation $\leq_k^q$ defined by covering relations:

1. $w \lessdot_k^q ws_{a,b}$ if $a \leq k < b$ and $\ell(w) + 1 = \ell(ws_{a,b})$;
2. $w \lessdot_k^q \mathbf{q_{ab}} ws_{a,b}$ if $a \leq k < b$ and $\ell(w) + 1 = \ell(ws_{a,b}) + 2(b-a)$; and
3. extend q-multiplicatively: $u \leq_k^q v$ if and only if $\mathbf{q}^\alpha u \leq_k^q \mathbf{q}^\alpha v$ for $u, v \in S_n[\mathbf{q}]$ and any $\alpha \in \mathbb{N}^{n-1}$.

Quantum k-Bruhat Order

Quantum 2-Bruhat Order on $S_3[\mathbf{q}]$



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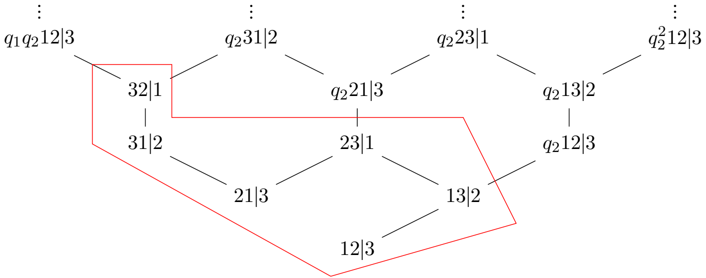

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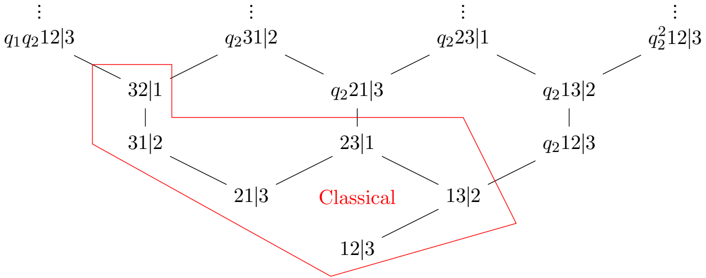

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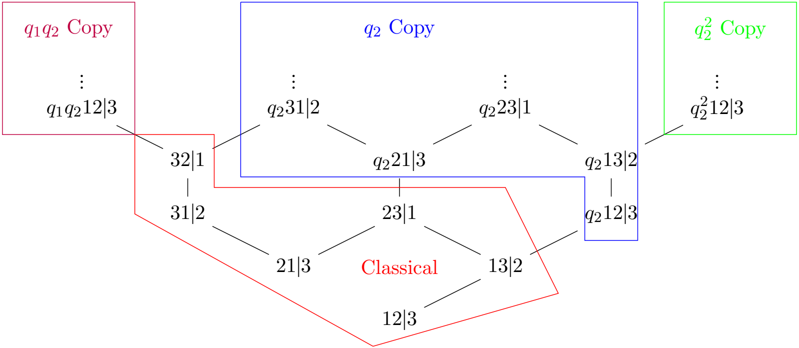

Structure Constants

Quantum Structure Constant Result

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Theorem (Benedetti et al., 2024)

Let $u \in S_n$, $a \leq k$, and $b \leq n-k$. Then

$$\mathfrak{S}_u^q * s^q_{(b,1^{a-1})}(x_1, \cdots, x_k) = \sum \binom{s(wu^{-1}) - 1}{ht(wu^{-1}) - a} q^{\alpha} \mathfrak{S}_w^q$$

where the sum is over all minimal intervals $[u, q^{\alpha} w]_k^q$ such that $l(q^{\alpha} w) - l(u) = a + b - 1$.

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Quantum Structure Constant Result

Theorem (Benedetti et al., 2024)

Let $u \in S_n$. Then

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where the sum is over all minimal intervals $[u, q^\alpha w]_k^q$ with $l(q^\alpha w) - l(u) = r$ such that wu^{-1} is a cycle.

Quantum k-Bruhat Order

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Quantum k–Bruhat Order

Problem: Given $\mathbf{q}^{\alpha}u, \mathbf{q}^{\beta}w \in S_n[\mathbf{q}]$, establish a method to verify whether or not $\mathbf{q}^{\alpha}u <_k^q \mathbf{q}^{\beta}w$ utilizing only α, β, u, w, and no other elements of $S_n[\mathbf{q}]$.

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Monoid Structure

Monoid Structure

Let $\mathcal{F}_n^q$ be the free monoid generated by the symbols

$$\{\mathbf{v}_{a,b} \mid a \neq b \text{ with } a, b \in [n]\} \cup \{\mathbf{0}\}.$$

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$$\{\mathbf{v}_{a,b} \mid a \neq b \text{ with } a, b \in [n]\} \cup \{\mathbf{0}\}.$$

Define an action of $\mathcal{F}_n^q$ on $S_n[\mathbf{q}] \cup \{0\}$ by $\mathbf{0} \bullet_k w = \mathbf{v}_{a,b} \bullet_k 0 = 0$,

$$\mathbf{v}_{a,b} \bullet_k w = \begin{cases} s_{a,b}w, & a < b, \ w \lessdot_k^q s_{a,b}w \\ \mathbf{q_{ij}}s_{a,b}w, & a > b, \ w \lessdot_k^q \mathbf{q_{ij}}s_{a,b}w \ (w(i) = a, u(j) = b) \\ 0, & otherwise, \end{cases}$$

and extend to $S_n[\mathbf{q}] \cup \{\mathbf{0}\}$ by setting $\mathbf{v}_{a,b} \bullet_k (\mathbf{q}^\alpha u) = \mathbf{q}^\alpha \mathbf{v}_{a,b} \bullet_k u$.

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$$\mathbf{v}_{1,3}\mathbf{v}_{3,1}\mathbf{v}_{1,2}\mathbf{v}_{2,3} \bullet_2 123 = q_2 231$$

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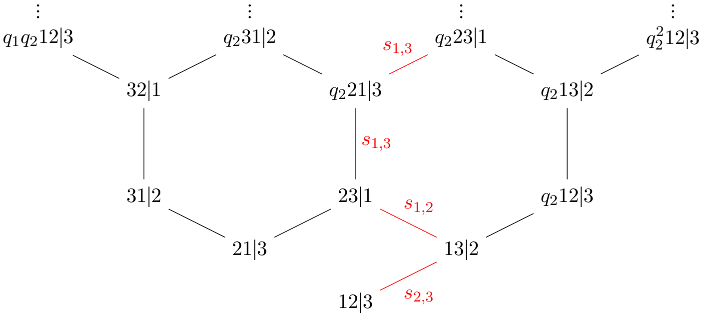

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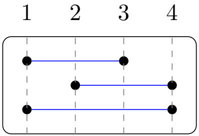

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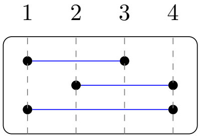

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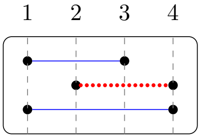

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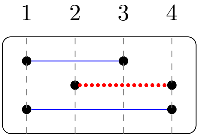
$\mathbf{v}_{1,3}\mathbf{v}_{4,2}\mathbf{v}_{1,4} \longrightarrow$ 

Elements $\mathbf{u}, \mathbf{v} \in \mathcal{F}_n^q$ are *equivalent*, denoted $\mathbf{v} \equiv \mathbf{u}$, if and only if $\mathbf{v} \bullet_k w = \mathbf{u} \bullet_k w$ for all $w \in S_n$ and all $k \in [n-1]$.

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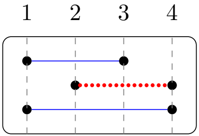

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Problem: Characterize the set of equivalences $\mathbf{u} \equiv \mathbf{v}$ and $\mathbf{u} \equiv \mathbf{0}$ satisfied by the elements of $\mathcal{F}_n^q$.

Monoid Relations

Classical Equivalences

Equivalences from classical case carry over.

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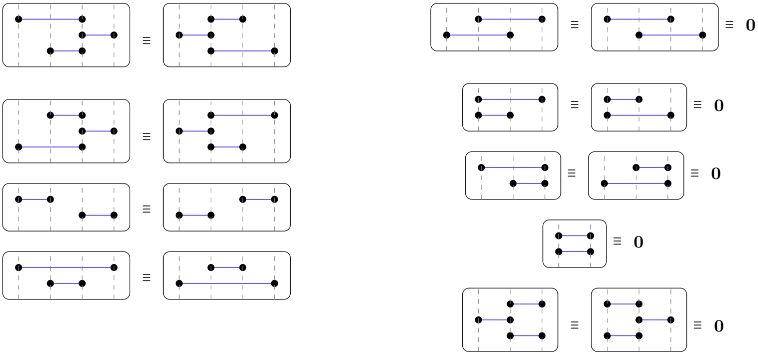

Equivalence Preserving Transformations

Flattening

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Define $Supp(\mathbf{v}_{a_1,b_1}\cdots\mathbf{v}_{a_n,b_n}) = \{a_1, b_1, \cdots, a_n, b_n\}$.

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Lemma (Colmenarejo-M., 2025+)

$\mathbf{v}_1 \equiv \mathbf{v}_2 \not\equiv 0$ *implies* $Supp(\mathbf{v}_1) = Supp(\mathbf{v}_2)$.

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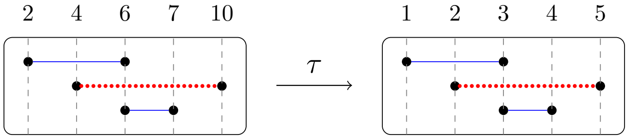

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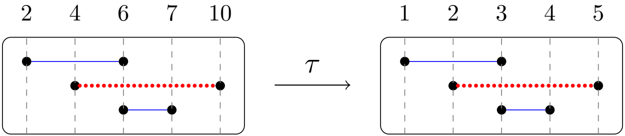

Theorem (Colmenarejo-M., 2025+)

1. $\mathbf{v}_1 \equiv \mathbf{v}_2$ *if and only if* $\tau(\mathbf{v}_1) \equiv \tau(\mathbf{v}_2)$.
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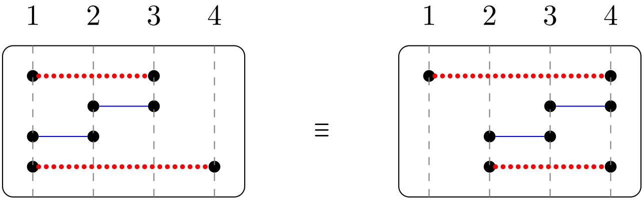

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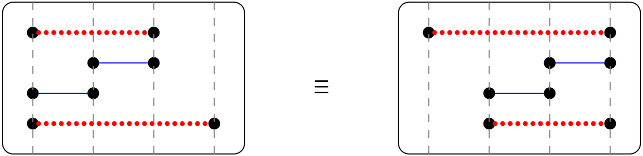

Equivalence Preserving Transformations

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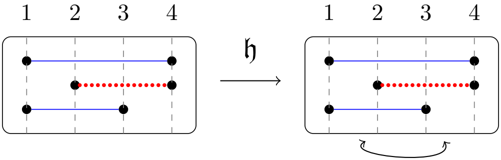

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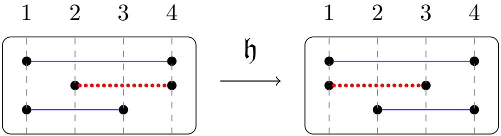

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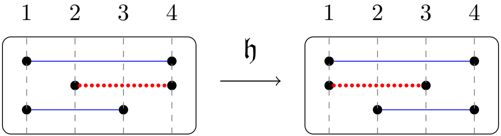

Theorem (Colmenarejo-M., 2025+)

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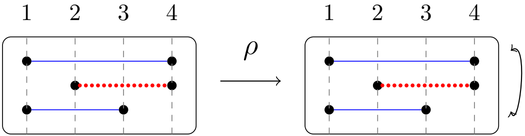

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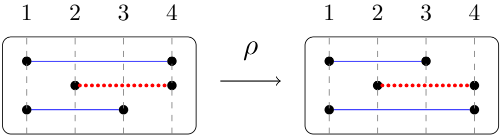

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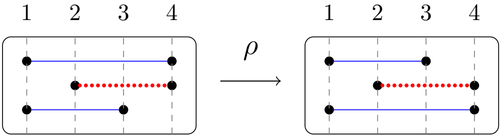

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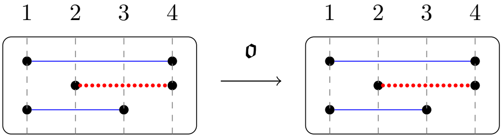
Equivalence Preserving Transformations

Cyclic Shift



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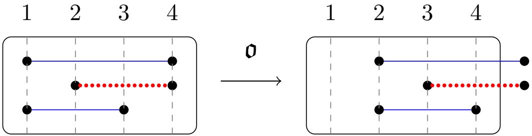

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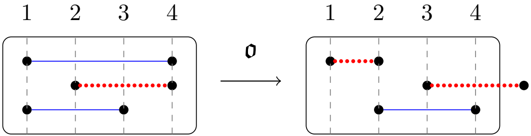

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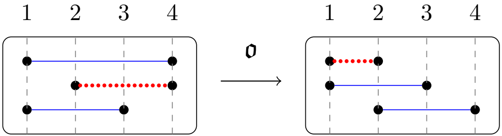

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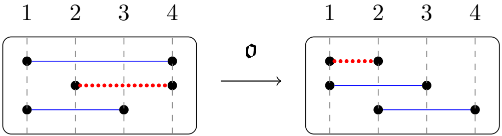

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Finding Equivalences

Characterizing Low Order Zero Equivalences

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- Consider only elements with first generator quantum and indices as small as possible.

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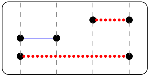
$\mathbf{0} \equiv$ 

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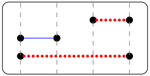
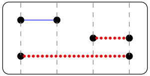
$\mathbf{0} \equiv$  $\equiv$ 

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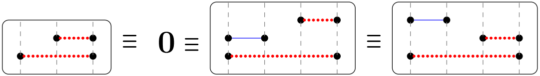

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- Consider only elements with first generator quantum and indices as small as possible.
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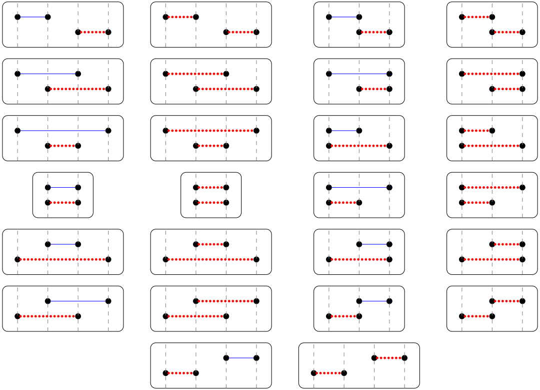
- Throw out any resulting from a lower order zero equivalence.
- Apply equivalence preserving transformations to those found to generate all of fixed order.

Order 2 Zero

Characterizing Order 2 Zero Equivalences



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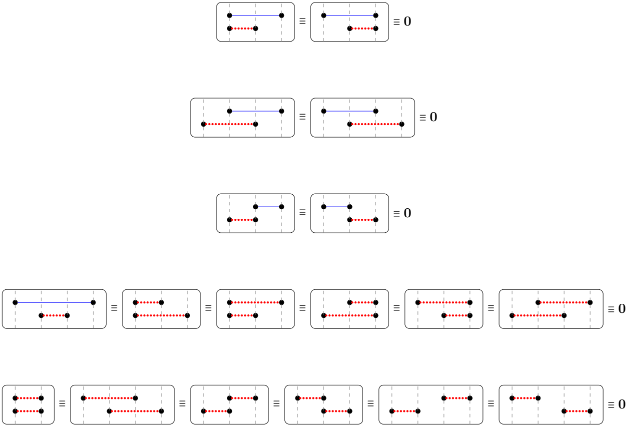

Order 2 Zero

Characterizing Order 2 Zero Equivalences

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Characterizing Low Order Nonzero Equivalences

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Theorem (Colmenarejo-M., 2025+)

Suppose that $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *satisfy* $S = Supp(\mathbf{v_1}) = Supp(\mathbf{v_2})$ *with* $N = |S|$*. Then* $\mathbf{v}_1 \equiv \mathbf{v}_2$ *if and only if* $flat(\mathbf{v}_1) \bullet_k u = flat(\mathbf{v}_2) \bullet_k u$ *for all* $u \in S_N$ *and* $k \in [N-1]$*.*

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Characterizing Low Order Nonzero Equivalences

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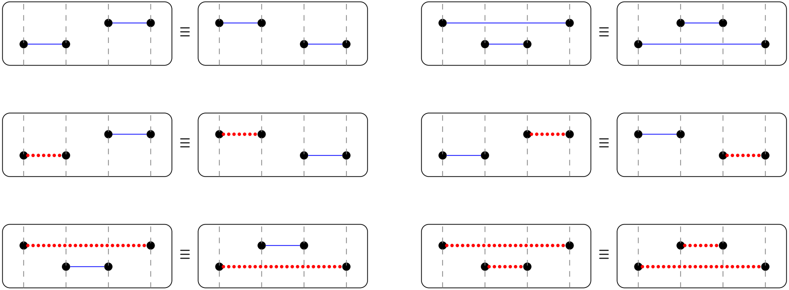
- Throw out any which are a consequence of an equivalence of smaller order.

Order 2 Nonzero

Order 2 Nonzero Equivalences



Order 2 Equivalences

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All order 2 equivalences can be formed from the classical ones by applying sequences of equivalence preserving transformations.

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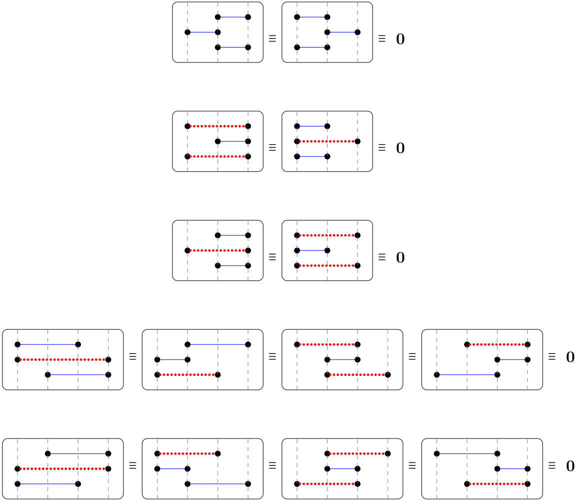
Theorem (Colmenarejo-M., 2025+)

Suppose $\mathbf{v} = \mathbf{v}_{a_1,b_1} \cdots \mathbf{v}_{a_n,b_n}$ *with* $|Supp(\mathbf{v})| = 2n - 1$ *or* $2n$*. If* $\mathbf{v} \equiv \mathbf{0}$*, then the equivalence is a result of an order 2 zero equivalence.*

Order 3 Zero

Order 3 Zero Equivalences



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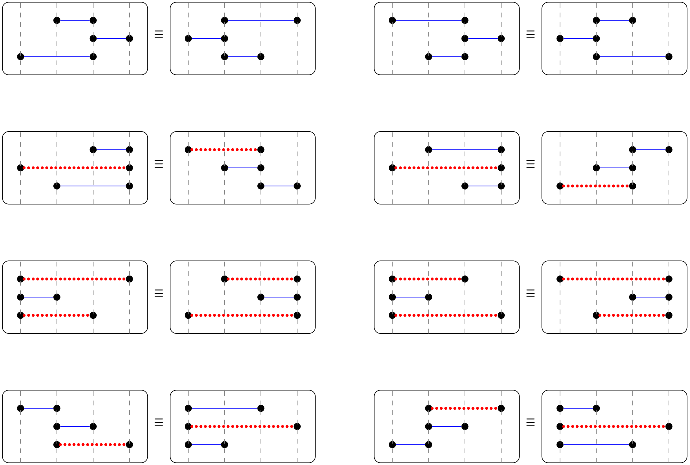
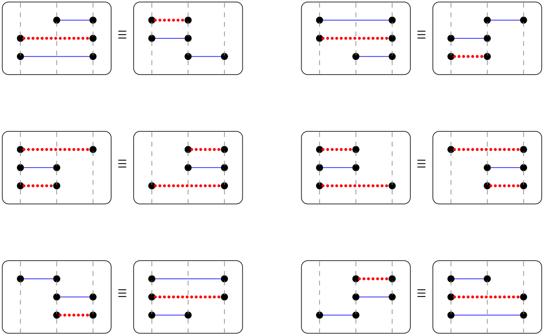
Order 3 Nonzero Equivalences



Order 3 Nonzero

Order 3 Nonzero Equivalences



Order 4 Zero

Order 4 Zero Equivalences

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Theorem (Colmenarejo-M., 2025+)

Suppose $\mathbf{v} = \mathbf{v}_{a_1,b_1} \cdots \mathbf{v}_{a_n,b_n}$ *with* $|Supp(\mathbf{v})| = 2n-2, 2n-1,$ *or* $2n$. *If* $\mathbf{v} \equiv \mathbf{0}$, *then the equivalence is a result of an order 2 or 3 zero equivalence.*

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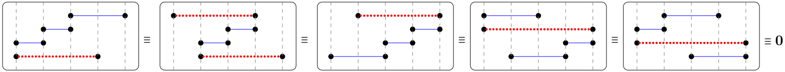

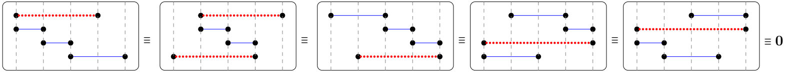

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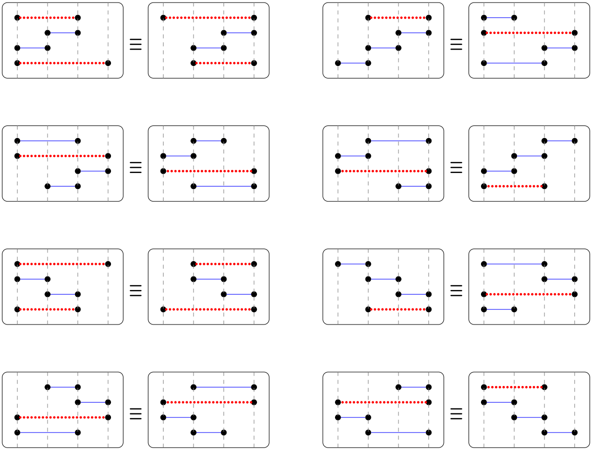
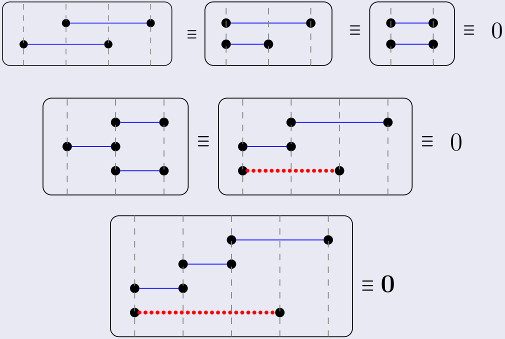


Low Order

Theorem (Colmenarejo-M., 2025+)

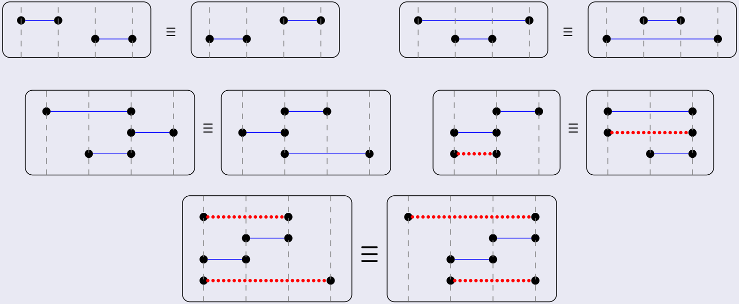
The order 2, 3, and 4 zero equivalences are those of the following forms along with those related to them by sequences of equivalence preserving operators.



Low Order

Theorem (Colmenarejo-M., 2025+)

The order 2, 3, and 4 nonzero equivalences are those of the following forms along with those related to them by sequences of equivalence preserving transformations.



Arbitrary Order Equivalences

Equivalences of Arbitrary Order

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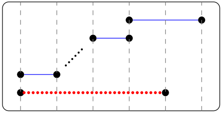

$\equiv \mathbf{0}$

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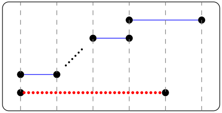

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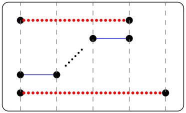
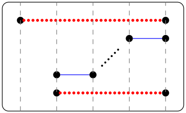


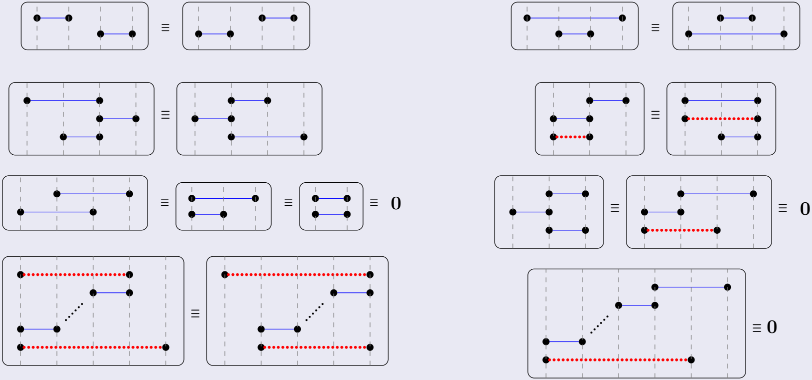
$\equiv$



Results

Theorem (Colmenarejo-M., 2025+)

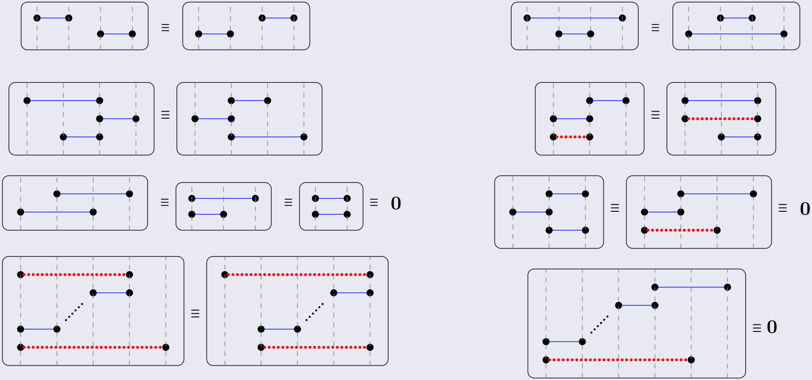
The quantum monoid satisfies the following equivalences along along with those related to them by sequences of equivalence preserving transformations.



Results

Conjecture (Colmenarejo-M., 2025+)

The quantum monoid is defined by the following equivalences along with those related to them by sequences of equivalence preserving transformations.



Future Work

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Problem: Given $\mathbf{q}^{\alpha}u, \mathbf{q}^{\beta}w \in S_n[\mathbf{q}]$, establish a method to verify whether or not $\mathbf{q}^{\alpha}u <_k^q \mathbf{q}^{\beta}w$ utilizing only α, β, u, w, and no other elements of $S_n[\mathbf{q}]$.

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Problem: Complete characterization of equivalences defining the quantum version of $\mathcal{M}_n$.

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Problem: Determine structure constant consequences.