Matrix Loci, Orbit Harmonics, and Shadow Play

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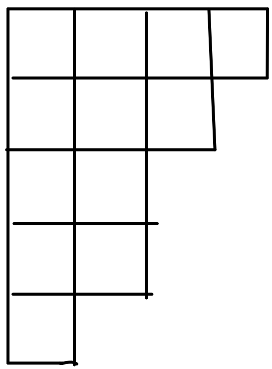
The Symmetric Group

$\lambda \vdash n$ partition

$(4,3,2,2,1) \vdash 12$



Young Diagram

The Symmetric Group

$\lambda \vdash n$ partition

$(4,3,2,2,1) \vdash 12$

$$\begin{array}{|c|c|c|c|}
\hline
1 & 3 & 4 & 8 \\
\hline
2 & 6 & 10 & \multicolumn{1}{c}{} \\
\cline{1-3}
5 & 7 & \multicolumn{2}{c}{} \\
\cline{1-2}
9 & 11 & \multicolumn{2}{c}{} \\
\cline{1-2}
12 & \multicolumn{3}{c}{} \\
\cline{1-1}
\end{array}$$

Standard Young Tableau

The Symmetric Group

Schensted Correspondence

$$\begin{array}{ccc} \mathfrak{S}_n & \longrightarrow & \coprod_{\lambda \vdash n} SYT(\lambda) \times SYT(\lambda) \\ w & \longmapsto & (P(w), Q(w)) \end{array}$$

The Symmetric Group

Schensted Correspondence

$$\mathfrak{S}_n \longrightarrow \coprod_{\lambda \vdash n} SYT(\lambda) \times SYT(\lambda)$$

$$w \longmapsto (P(w), Q(w))$$

$$w = [2, 6, 3, 5, 4, 8, 7, 1] \in \mathfrak{S}_8$$

$$w \longmapsto \left(\begin{array}{llll} 1 & 3 & 4 & 7 \\ 2 & 8 & & \\ 5 & & & \\ 6 & & & \end{array} , \begin{array}{llll} 1 & 2 & 4 & 6 \\ 3 & 7 & & \\ 5 & & & \\ 8 & & & \end{array} \right)$$

The Symmetric Group

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$$w = [\underline{2}, 6, \underline{3}, \underline{5}, 4, \underline{8}, 7, 1] \in \mathfrak{S}_8$$

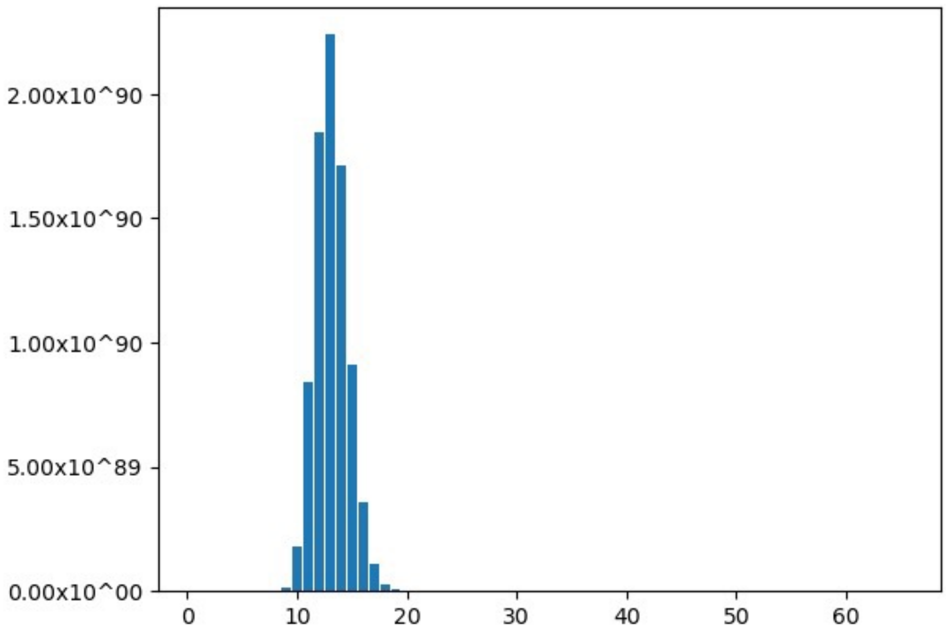
$\mathrm{lis}(w) = 4$

$$w \longmapsto \left(\begin{array}{llll} 1 & 3 & 4 & 7 \\ 2 & 8 & & \\ 5 & & & \\ 6 & & & \end{array} \; , \; \begin{array}{llll} 1 & 2 & 4 & 6 \\ 3 & 7 & & \\ 5 & & & \\ 8 & & & \end{array} \right)$$

$\lambda_1 = \mathrm{lis}(w)$

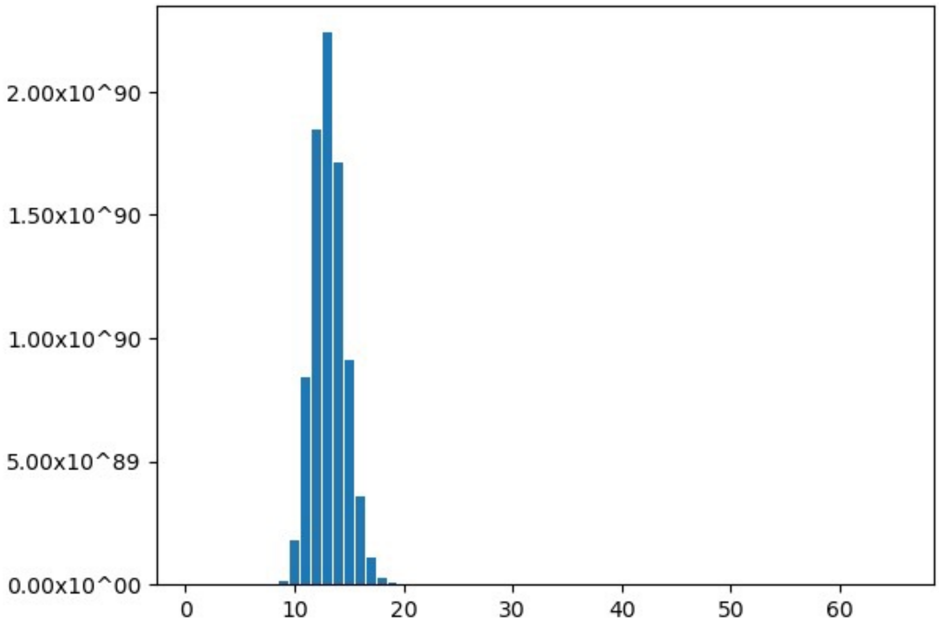
The Symmetric Group

$$a_{n,k} := \#\{w \in \mathfrak{S}_n : \mathrm{lis}(w) = k\}$$



$a_{n,k}$ when $n = 65$

The Symmetric Group



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Chen's Conjecture $\{a_{n,k}\}$ is log-concave : $a_{n,k-1} \cdot a_{n,k+1} \leq a_{n,k}^2$

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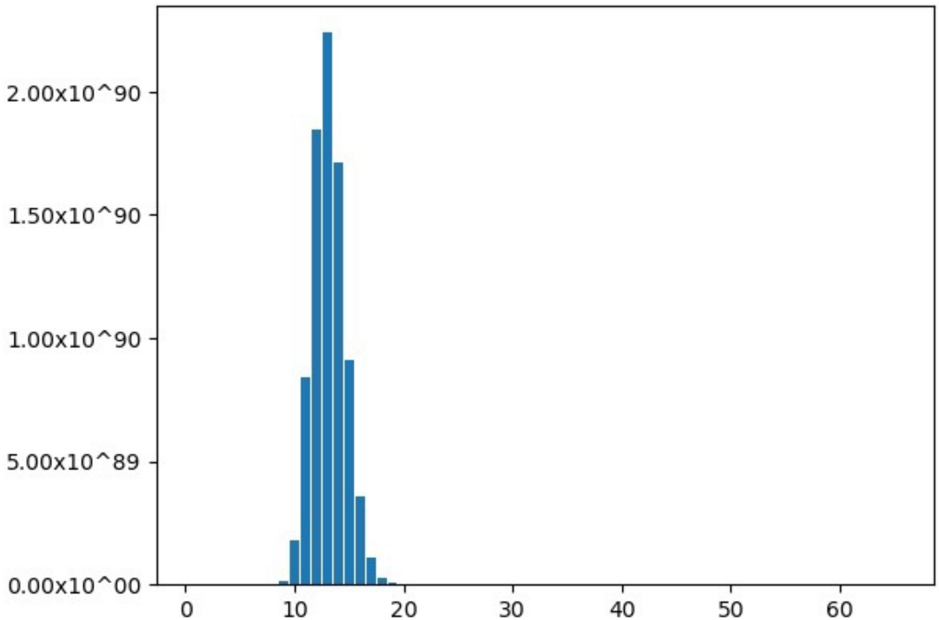

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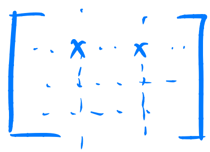
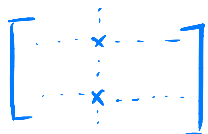
Baik-Deift-Johansson $\{a_{n,k}\}$ converges to the Tracy-Widom distribution of random GUE matrices as $n \to \infty$.

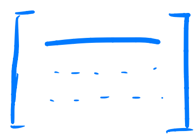
$$X_{n\times n} = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{bmatrix}$$ variables

$I_n \subseteq \mathbb{C}[X_{n\times n}]$ is the ideal generated by

$x_{i,j}^2$

$x_{i,j} \cdot x_{i,j'}$ 

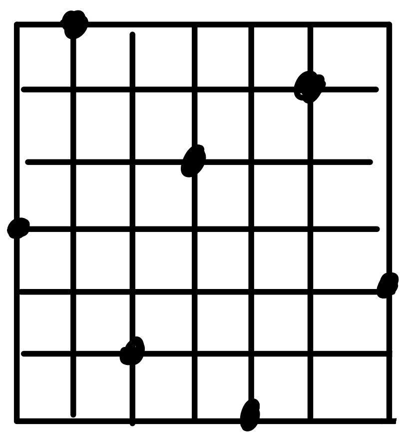
$x_{i,j} \cdot x_{i',j}$ 

$x_{i,1} + x_{i,2} + \cdots + x_{i,n}$ 

$x_{1,j} + x_{2,j} + \cdots + x_{n,j}$ 

Viennot Shadow

$w = [4, 7, 2, 5, 1, 6, 3] \in \mathfrak{S}_7$



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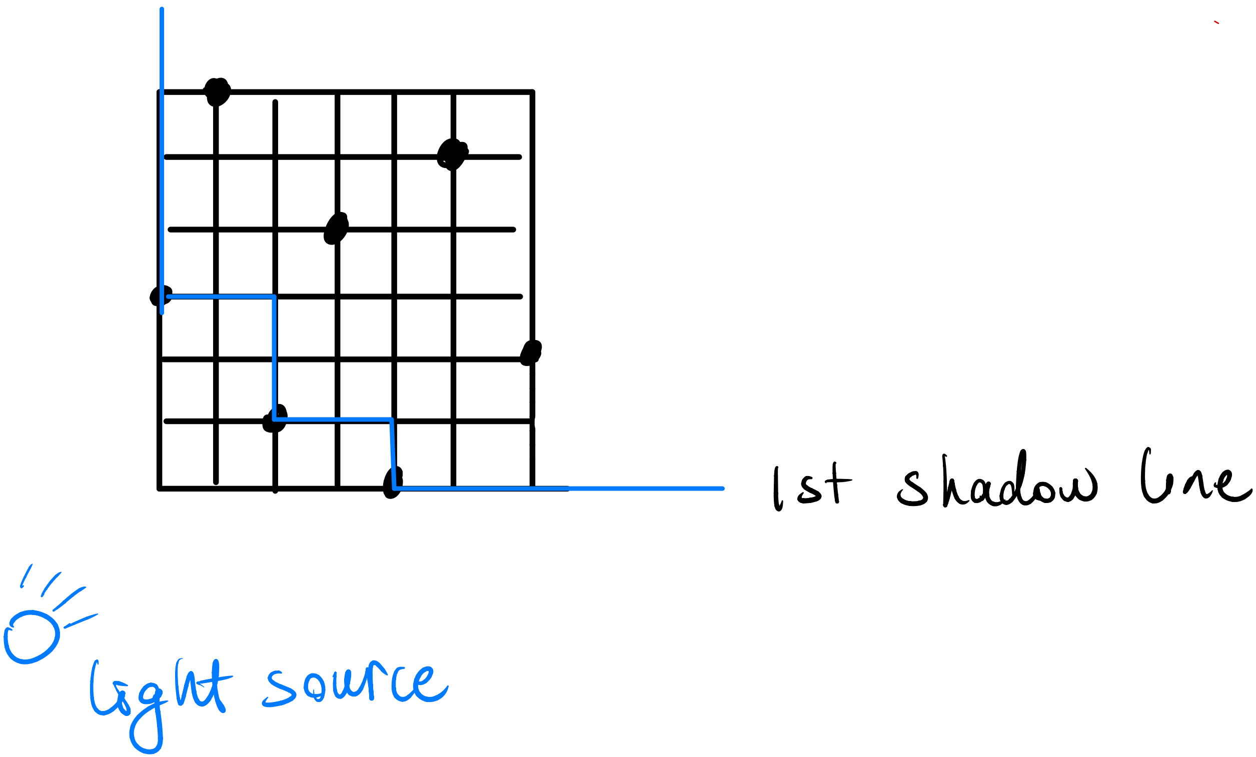

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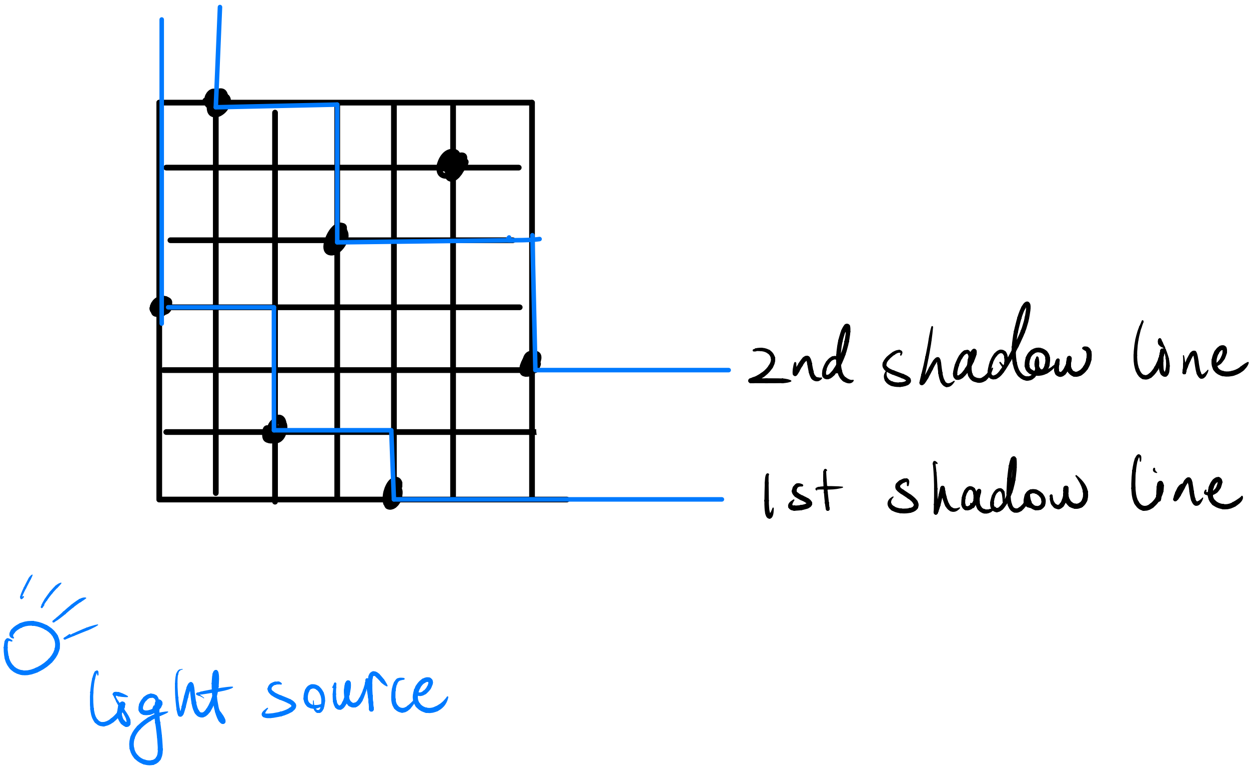

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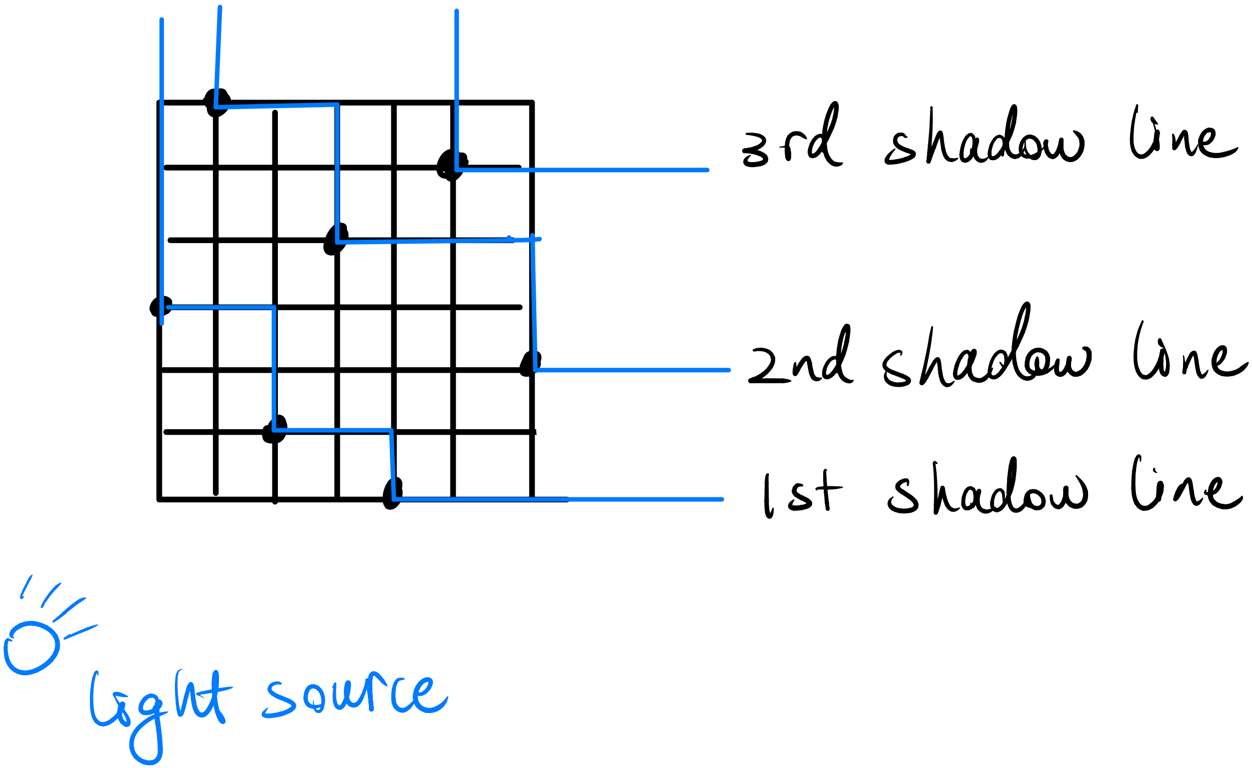

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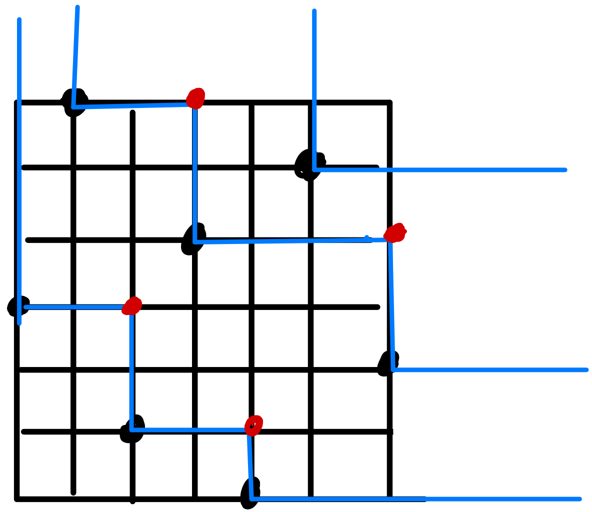
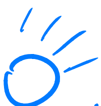


• shadow points

$S(w) = \{ \bullet \}$

$= \{(3,4), (4,7), (5,2), (7,5)\}$



light source

Viennot Shadow

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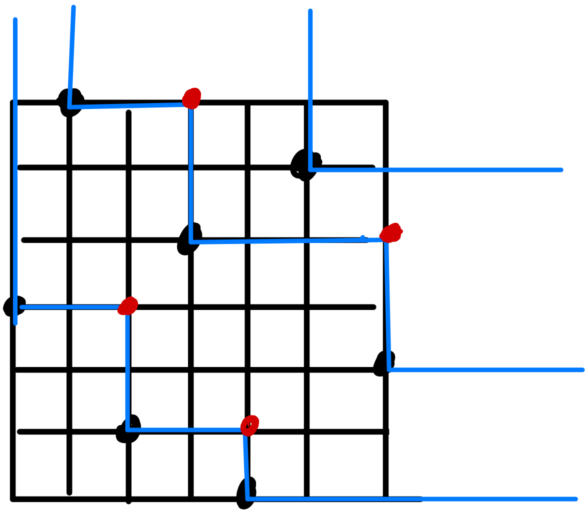

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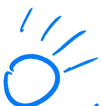

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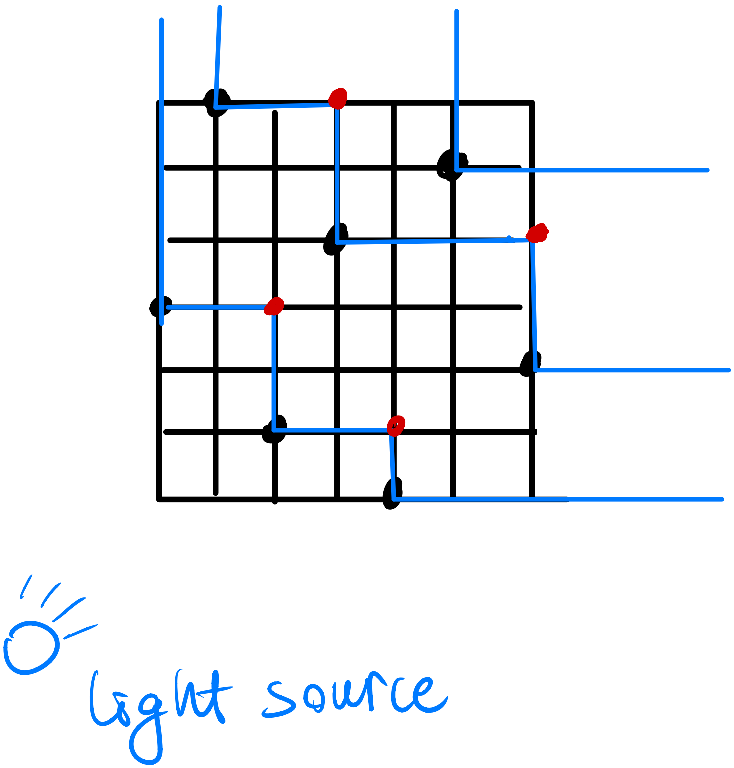
Fact: $|S(w)|=n-\mathrm{lis}(w)$

Viennot Shadow

$w = [4,7,2,5,1,6,3] \in \mathfrak{S}_7$



$s(w) = x_{3,4}\, x_{4,7}\, x_{5,2}\, x_{7,5}$

Theorem (Rhoades)

$\{s(w) : w \in \mathfrak{S}_n\}$ descends to a basis of $\mathbb{C}[X_{n\times n}]/I_n$. This is the standard monomial basis w.r.t the Toeplitz order.

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Cor $\mathrm{Hilb}(\mathbb{C}[X_{n\times n}]/I_n ; q) = a_{n,n} + a_{n,1} q + \cdots + a_{n,1} q^{n-1}$

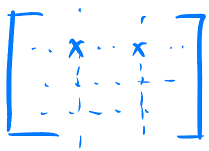
$$= \sum_{w \in \mathfrak{S}_n} q^{n - \mathrm{lis}(w)}$$

Graded Module Structure

Recall that I_n is generated by

$$x_{i,j}^2$$

$$x_{i,j}\cdot x_{i,j'}$$



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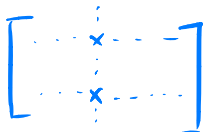
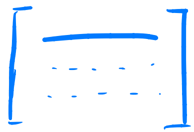


$$x_{i,1}+x_{i,2}+\cdots+x_{i,n}$$



$$x_{1,j}+x_{2,j}+\cdots+x_{n,j}$$

Graded Module Structure

Recall that I_n is generated by

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$$x_{i,j} \cdot x_{i',j}$$

$$x_{i,1} + x_{i,2} + \cdots + x_{i,n}$$

$$x_{1,j} + x_{2,j} + \cdots + x_{n,j}$$

Q: what is the (graded) $\mathfrak{S}_n \times \mathfrak{S}_n$ representation structure of $\mathbb{C}[X_{n\times n}]/I_n$?

Theorem (Rhoades)

$$\left(\mathbb{C}[x_{n\times n}]/I_n\right)_d \cong \bigoplus_{\substack{\lambda \vdash n \\ \lambda_1 = n-d}} \mathrm{End}(V^\lambda)$$

$$\cong \bigoplus_{\substack{\lambda \vdash n \\ \lambda_1 = n-d}} V^\lambda \otimes V^\lambda$$

Where does this In come from?

Where does this I_n come from?

Orbit Harmonics!

Orbit Harmonics

$\mathbf{x}_N = (x_1, x_2, \ldots, x_N)$ variables

$\mathbb{C}[\mathbf{x}_N]$ polynomial ring

$Z \subseteq \mathbb{C}^N$ finite locus of points

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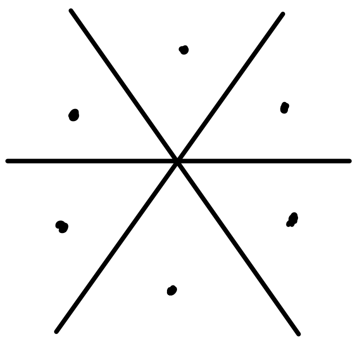


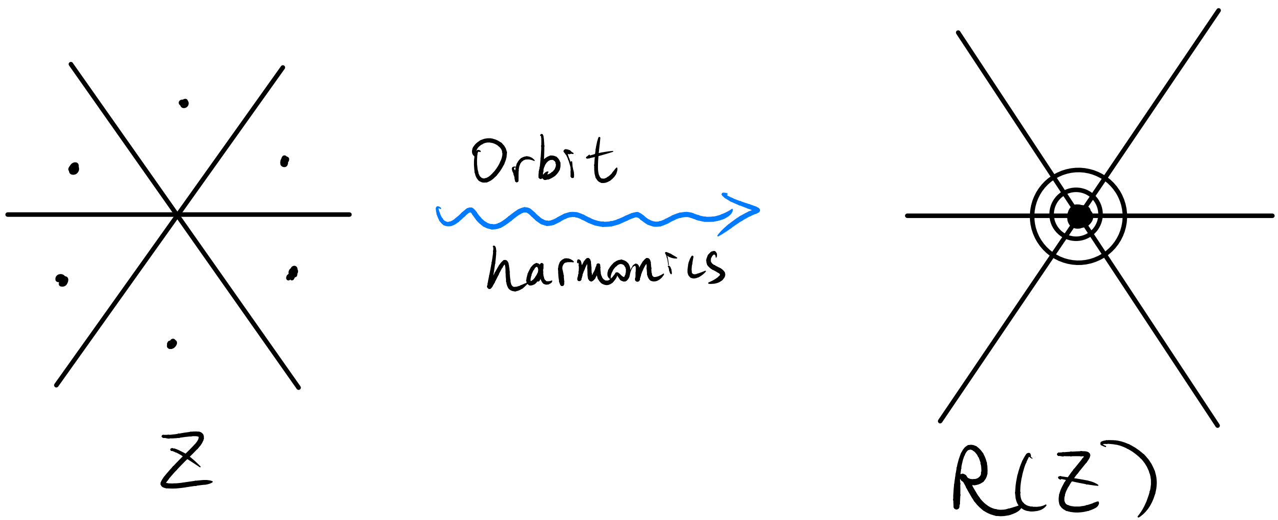
Z

Orbit Harmonics

$x_N = (x_1, x_2, \cdots, x_N)$ — variables

$\mathbb{C}[x_N]$ — polynomial ring

$Z \subseteq \mathbb{C}^N$ — finite locus of points



Orbit Harmonics

$Z = \{(0,0), (1,1)\} \subseteq \mathbb{C}^2$

finite locus of points

$I(Z) = \langle x(x-1), x(y-1), y(x-1), y(y-1) \rangle$

vanishing ideal

Orbit Harmonics

$Z = \{(0,0), (1,1)\} \subseteq \mathbb{C}^2$ finite locus of points

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vanishing ideal

$\mathbb{C}[Z] \cong \mathbb{C}[x_N] / I(Z)$ as v.s.

Orbit Harmonics

$$f(x,y) = x^3+y^3+2x^2y+3x^2+y^2+xy+2$$

$$T(f) = x^3+y^3+2x^2y$$

Orbit Harmonics

$$f(x,y) = x^3+y^3+2x^2y+3x^2+y^2+xy+2$$

$$\tau(f) = x^3+y^3+2x^2y$$

$I \subseteq \mathbb{C}[\mathbf{x}_N]$ ideal

$$\operatorname{gr} I = \langle \tau(f) : f \in I \rangle$$

associated graded ideal

Orbit Harmonics

$$* \quad \mathbb{C}[Z] \cong \mathbb{C}[x_N]/I(Z) \cong \mathbb{C}[x_N]/\operatorname{gr} I(Z)$$

orbit harmonics

$$R(Z) := \mathbb{C}[x_N]/\operatorname{gr} I(Z)$$

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– why is this useful?

Example

$n \in \mathbb{N}$, $I = \langle e_d(x_1, \ldots, x_n) : 1 \leq d \leq n \rangle \subseteq \mathbb{C}[x_1, \ldots, x_n]$

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Chevalley: $\mathbb{C}[x_1, \dots, x_n]/I \cong \mathbb{C}[\mathfrak{S}_n]$

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$Z = \{(w(1), w(2), \ldots, w(n)) : w \in \mathfrak{S}_n\} \subseteq \mathbb{C}^n$

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$\forall\, 1 \leq d \leq n$, $\quad e_d(x_1, \cdots x_n) - e_d(1, 2, \cdots, n) \in I(Z)$

So $I \subseteq \operatorname{gr} I(Z)$

Example

$$I \subseteq \operatorname{gr} I(Z),$$

$$n! = \dim R(Z) \leq \dim \mathbb{C}[x_1, \ldots, x_n]/I = n!$$

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$$n! = \dim R(Z) \leq \dim \mathbb{C}[x_1, \dots, x_n]/I = n!$$

$$\Rightarrow I = \operatorname{gr} I(Z).$$

we recover Chevally's result that

$$\mathbb{C}[x_1, \dots, x_n]/I \cong \mathbb{C}[\mathfrak{S}_n]$$

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point locus Z	$R(Z)$
regular $\mathfrak{S}_n$-orbit	coinvariant ring (Kostant)

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λ-tableau	Garsia-Haiman module V_λ (Haiman)

$\mathfrak{S}_n$ as matrices

$$w = [4, 1, 2, 5, 3] \in \mathfrak{S}_5$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \in \mathrm{Mat}_{n \times n}(\mathbb{C})$$

Theorem (Rhoades)

$$I_n = \operatorname{gr} I(\mathfrak{S}_n)$$

Note that $I(\mathfrak{S}_n)$ contains

$$x_{i,j}^2 - x_{i,j}$$

$$x_{i,j} \cdot x_{i,j'}$$

$$x_{i,j} \cdot x_{i',j}$$

$$x_{i,1} + x_{i,2} + \cdots + x_{i,n} - 1$$

$$x_{1,j} + x_{2,j} + \cdots + x_{n,j} - 1$$

What other matrix loci can

we consider?

Colored Permutations

$$\mathfrak{S}_{n,r} := \mathbb{Z}_r \wr \mathfrak{S}_n$$

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One-line notation $\quad w = [4^2\ 2^0\ 5^1\ 3^0\ 1^2] \in \mathfrak{S}_{5,3}$

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Matrix form

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \zeta^2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \zeta^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \zeta & 0 & 0 \end{bmatrix}$$

$$\zeta = e^{2\pi i/3}$$

Colored Permutations

$\underline{\Lambda} = (\lambda^0, \lambda^1, \ldots, \lambda^{r-1})$ is an **r-partition** if each λ^i is a partition and

$$\sum_{i=0}^{r-1} |\lambda^i| = n.$$

Colored Permutations

$\underline{\lambda} = (\lambda^0, \lambda^1, \cdots, \lambda^{r-1})$ is an r-partition if each λ^i is a partition and

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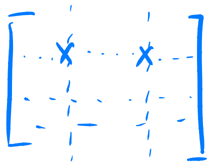
Conjugacy classes of $\mathfrak{S}_{n,r}$ are labelled by r-partitions of n.

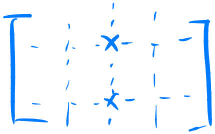
$I_{n,r} \subseteq \mathbb{C}[X_{n\times n}]$ is the ideal generated by

$$x_{i,j}^{r+1}$$

$$x_{i,j} \cdot x_{i,j'}$$



$$x_{i,j} \cdot x_{i',j}$$



$$x_{i,1}^{r} + x_{i,2}^{r} + \cdots + x_{i,n}^{r}$$

$$x_{1,j}^{r} + x_{2,j}^{r} + \cdots + x_{n,j}^{r}$$

Theorem (L.) $I_{n,r} = \operatorname{gr} I(\mathfrak{S}_{n,r})$

Note that $I(\mathfrak{S}_{n,r})$ contains

$$x_{i,j}^{r+1} - x_{i,j}$$

$$x_{i,j} \cdot x_{i,j'}$$

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$$x_{i,1}^r + x_{i,2}^r + \cdots + x_{i,n}^r - 1$$

$$x_{1,j}^r + x_{2,j}^r + \cdots + x_{n,j}^r - 1$$

$$w = [7^0\ 3^1\ 6^2\ 1^0\ 8^0\ 4^1\ 2^0\ 5^2] \in \mathfrak{S}_{8,3}$$

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$$C_0(w) = \{(1,7), (4,1), (5,8), (7,2)\}$$

$$C_1(w) = \{(2,3), (6,4)\}$$

$$C_2(w) = \{(3,6), (8,5)\}$$

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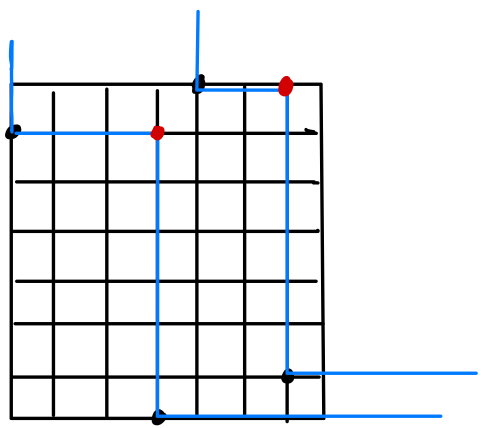

$$S(C_0(w)) = \{(4,7), (7,8)\}$$

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$$C_2(w) = \{(3,6), (8,5)\}$$

$$S(C_0(w)) = \{(4,7), (7,8)\}$$

$$S(w) = x_{2,3} \cdot x_{6,4} \cdot x_{3,6}^2 \cdot x_{8,5}^2 \cdot x_{4,7}^3 \cdot x_{7,8}^3$$

Theorem (L.) $I_{n,r} = \mathrm{gr}\, I(\mathfrak{S}_{n,r})$, and $\{s(w) : w \in \mathfrak{S}_{n,r}\}$ descends to a basis of $R(\mathfrak{S}_{n,r})$. This is again the SUB w.r.t. the Toeplitz order.

Cor

$$\mathrm{Hilb}(R(\mathfrak{S}_{n,r});q) = \sum_{w \in \mathfrak{S}_{n,r}} q^{r \cdot n - r \cdot \mathrm{lis}(C_0(w)) - \sum_{i=1}^{r-1} (r-i) \cdot |C_i(w)|}$$

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$$= \sum c_{n,r,k} \, q^{rn-k}$$

where

$$c_{n,r,k} = \#\left\{ w \in \mathfrak{S}_{n,r} : r \cdot \mathrm{lis}(C_0(w)) + \sum_{i=1}^{r-1} (r-i) \, |C_i(w)| = k \right\}$$

Unimodality

For $\{c_{n,r,k}\}$, log-concavity fails in general.

e.g. for $r=2$, fails for $n \geq 9$.

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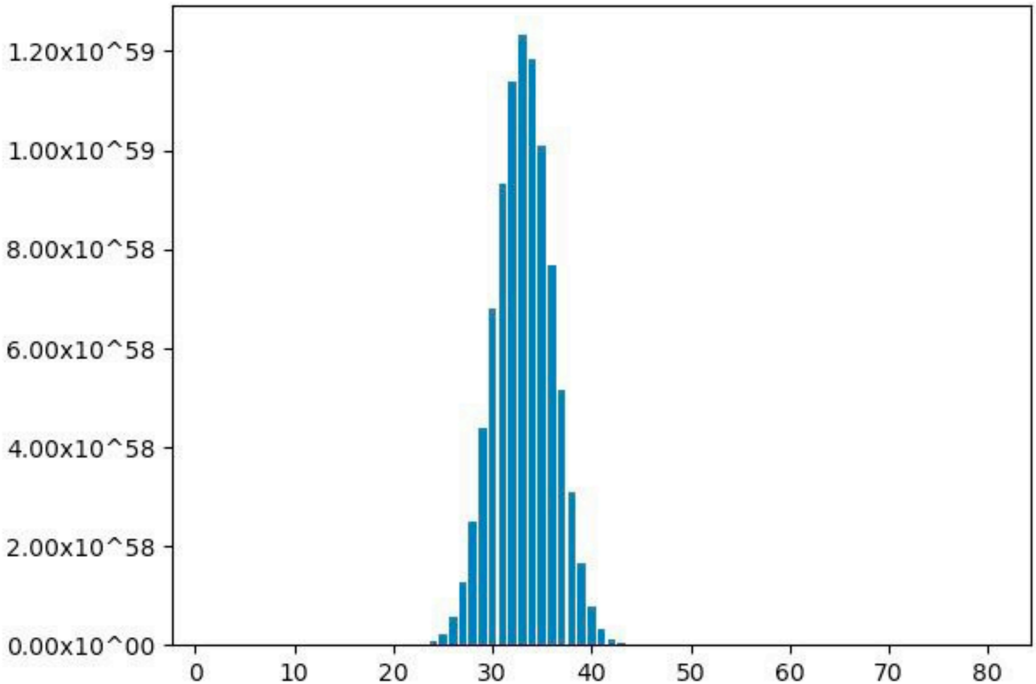
But we do conjecture that $\{c_{n,r,k}\}$ is unimodal.

Distribution



$\{c_{n,r,k}\}$ when $r=2$ and $n=40$.

Graded Structure

$\mathfrak{S}_{n,r} \times \mathfrak{S}_{n,r}$ acts on

$$\begin{bmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n} \end{bmatrix}$$

by left & right multiplication

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by left & right multiplication

$$\mathbb{C}[\mathfrak{S}_{n,r}] \cong \mathbb{C}[X_{n \times n}] / I(\mathfrak{S}_{n,r}) \cong \underset{\substack{\uparrow \\ \text{graded}}}{R(\mathfrak{S}_{n,r})}$$

as $\mathfrak{S}_{n,r} \times \mathfrak{S}_{n,r}$ modules

Graded Structure

Theorem (L.)

$$(R(G_{n,r}))_d \cong \bigoplus_{\Delta \vdash_r n} \mathrm{End}(V^{\Delta})$$

$$r \cdot \lambda_1^0 + \sum_{i=1}^{r-1} i \cdot |\lambda^i| = rn - d$$

Graded Structure

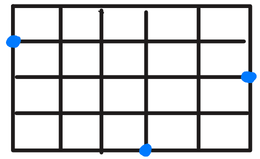
Theorem (L.)

$$(R(G_{n,r}))_d \cong \bigoplus_{\substack{\Delta \vdash_r n \\ r\cdot\lambda^0_1 + \sum_{i=1}^{r-1} i\cdot|\lambda^i| = rn-d}} \mathrm{End}(V^{\Delta})$$

(*) $\mathrm{End}(V^{\Delta}) \cong V^{\Delta} \otimes V^{\Delta'}$

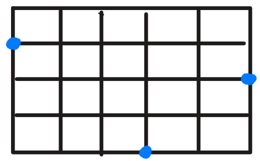
where $\Delta' = (\lambda^0, \lambda^{r-1}, \lambda^{r-2}, \ldots, \lambda^1)$

Rook Placements

$R =$  $\in Z_{6,5,3}$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Rook Placements

$R =$  $\in Z_{6,5,3}$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$UZ_{n,m,r} = \bigsqcup_{r' \geq r} Z_{n,m,r'}$$

"Upper rook placements"

Rook Placements

$Z_{n,m,r}$:
- harder to analyze
- no "clean" formula
- no basis for $R(Z_{n,m,r})$.

(Zhu, 2025 studied the properties)

Rook Placements

$Z_{n,m,r}$: • harder to analyze

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• no basis for $R(Z_{n,m,r})$.

(Zhu, 2025 studied the properties)

We'll focus on $UZ_{n,m,r}$.

* when $n=m=r$, $UZ_{n,m,r} = \mathfrak{S}_n$

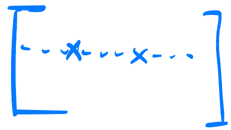
when $r=0$, $\mathcal{U}\mathcal{Z}_{n,m,0} = \{\text{all rook placements}\}$

when $r=0$, $UZ_{n,m,0} = \{\text{all rook placements}\}$

Thm (L., Zhu, 2025)

$\mathrm{gr}\, I(UZ_{n,m,0})$ is generated by:

$x_{i,j} \cdot x_{i,j'}$



$x_{i,j} \cdot x_{i',j}$

Furthermore, $\{m(R) : R \text{ is a rook-placement}\}$ descends to the SMB of $R(UZ_{n,m,0})$ w.r.t. the Toeplitz order.

Thm (L., Zhu, 2025)

For general n, m, r, $\mathrm{gr}\, I(UZ_{n,m,r})$ is generated by

$$x_{ij} \cdot x_{i,j'}$$

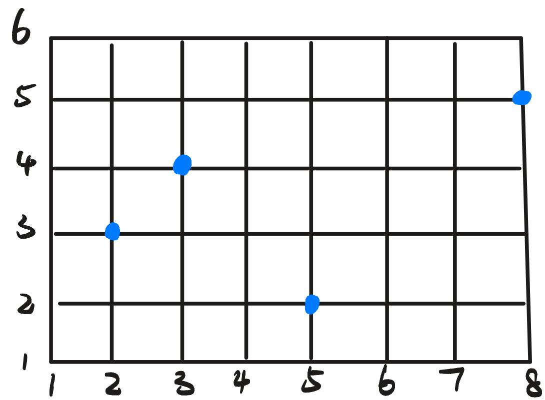
$$x_{ij} \cdot x_{i',j}$$

$$\prod_{k=1}^{n-r+1} \left(\sum_{j=1}^{m} x_{i_k, j} \right)$$ product of $(n-r+1)$ column sums

$$\prod_{k=1}^{m-r+1} \left(\sum_{i=1}^{n} x_{i, j_k} \right)$$ product of $(m-r+1)$ row sums

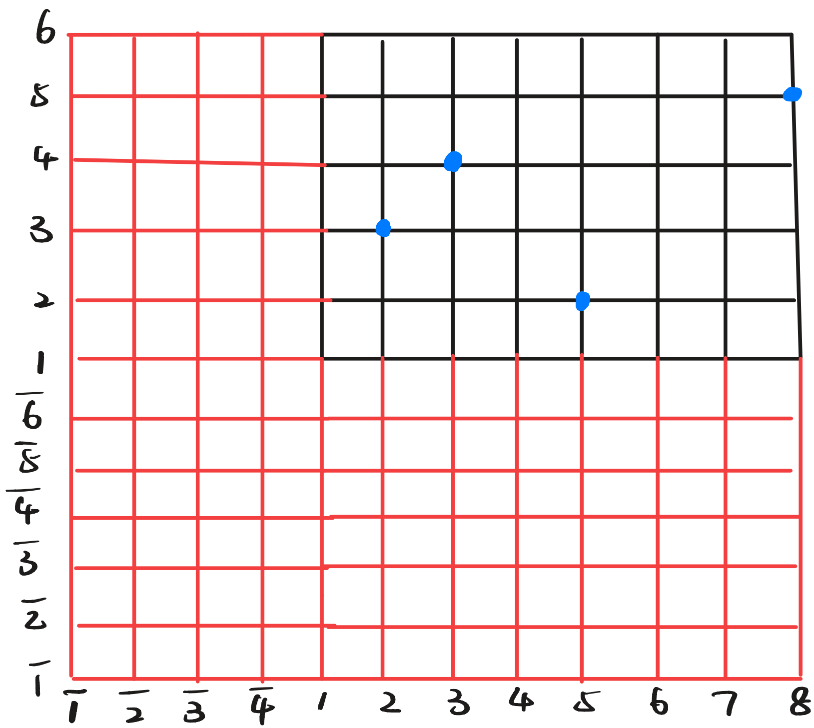
Generalized Shadow

$$R = \{(2,3), (3,4), (5,2), (8,5)\} \in \mathcal{UZ}_{8,6,2}$$



Generalized Shadow

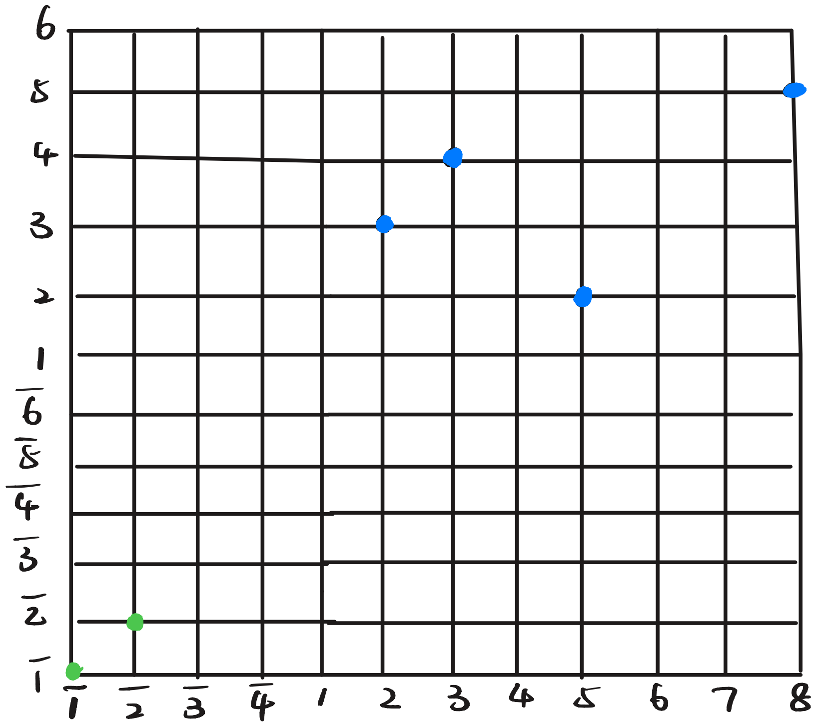
$$R = \{(2,3), (3,4), (5,2), (8,5)\} \in UZ_{8,6,2}$$



add m-r columns & n-r rows

Generalized Shadow

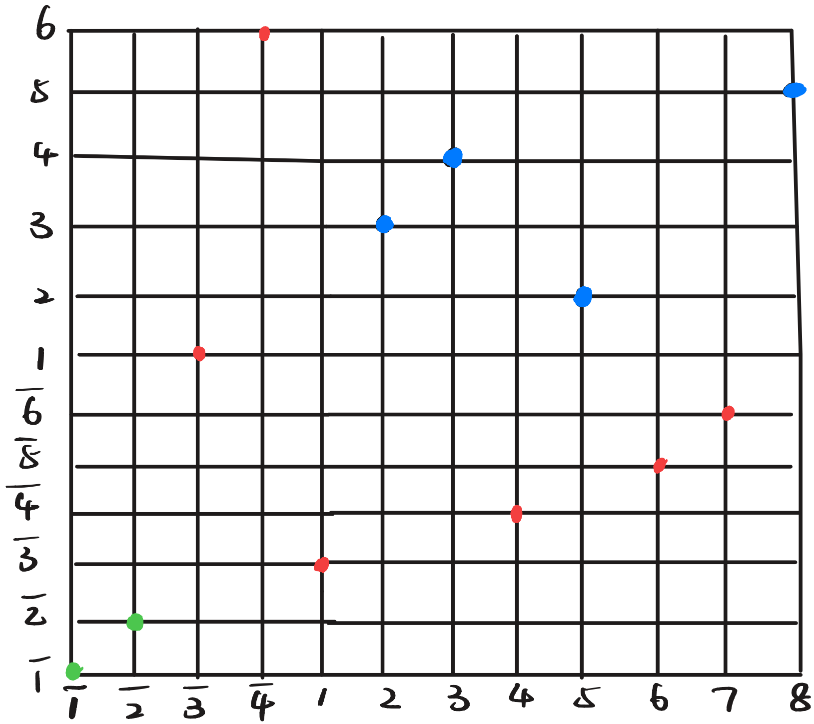
$$R = \{(2,3), (3,4), (5,2), (8,5)\} \in \mathcal{UZ}_{8,6,2}$$



for $i \le |R| - r$:
add a point at $(\bar{i}, \bar{i})$

Generalized Shadow

$$R = \{(2,3),(3,4),(5,2),(8,5)\} \in \mathcal{UZ}_{8,6,2}$$



for $i \le |R| - r$:
add a point at $(\bar{i}, \bar{i})$

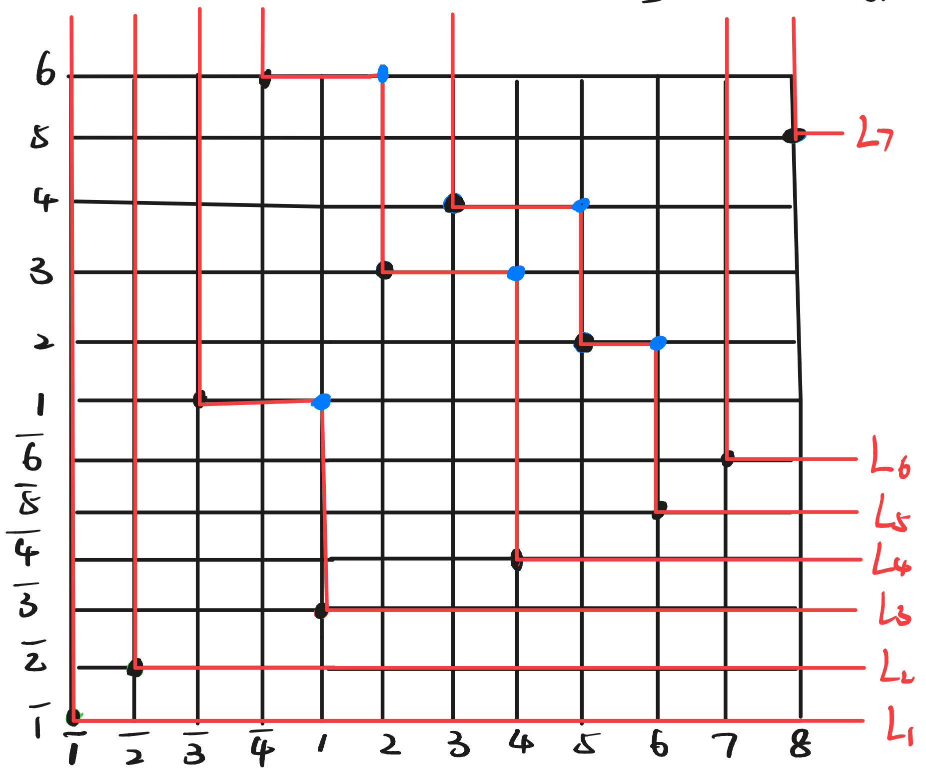
columns $\bar{1} \to \overline{m-r}$ form increasing pattern

rows $\bar{1} \to \overline{n-r}$ form increasing pattern

$$\mathrm{Ex}(R) \in \mathfrak{S}_{n+m-r}$$

Generalized Shadow

$$R = \{(2,3),(3,4),(5,2),(8,5)\} \in \mathcal{UZ}_{8,6,2}$$



$$S(EX(R)) = \{(1,1),(2,6),(4,3),(5,4),(6,2)\}$$

$$es(R) = x_{1,1} \cdot x_{2,6} \cdot x_{4,3} \cdot x_{5,4} \cdot x_{6,2}$$

Thm (L., Zhu 2025)

$\{ es(R) : R \in UZ_{n,m,r} \}$ descends to a basis of $R(UZ_{n,m,r})$.

This is the SMB w.r.t the Toeplitz order.

Thm (L., Zhu 2025)

$\{es(R) : R \in UZ_{n,m,r}\}$ descends to a basis of $R(UZ_{n,m,r})$.

This is the SMB w.r.t the Toeplitz order.

Cor

$$\mathrm{Hilb}(R(UZ_{n,m,r}); q) = \sum_{R \in UZ_{n,m,r}} q^{n+m-r-\mathrm{lis}(EX(R))}$$

Graded Structure

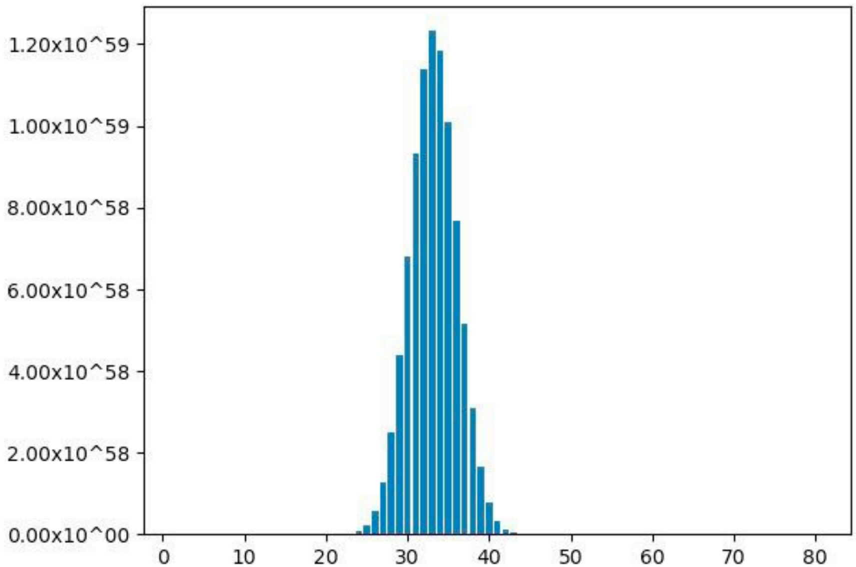
$\mathbb{C}[UZ_{n,m,r}] \cong R(UZ_{n,m,r})$ as ungraded $\mathfrak{S}_n \times \mathfrak{S}_m$ - modules

Thm (L., Zhu 2025)

$$R(UZ_{n,m,r})_d \cong \sum_{\mu \vdash d} \{s_\mu \cdot h_{n-d} \otimes s_\mu \cdot h_{m-d}\}_{\lambda_1 \leq n+m-d-r}$$

Future Directions

Question To what distribution does $\mathrm{Hilb}(R(\mathfrak{S}_{n,r}); q)$ converge to as $n \to \infty$? What if we also let $r \to \infty$?



$\{c_{n,r,k}\}$ when $r=2$ and $n=40$.

Equivariant log-concavity

$V = \bigoplus_d V_d$ graded G-representation.

G-equivariant log-concave:

$\exists \varphi : V_{d-1} \otimes V_{d+1} \hookrightarrow V_d \otimes V_d$

that commutes with the action of G.

Conjecture (Rhoades)

$R(\mathfrak{S}_n)$ is $\mathfrak{S}_n \times \mathfrak{S}_n$-equivariant log-concave.

Conjecture (Rhoades)

$R(\mathfrak{S}_n)$ is $\mathfrak{S}_n \times \mathfrak{S}_n$-equivariant log-concave.

Conjecture (L., Zhu)

$R(UZ_{n,m,r})$ is $\mathfrak{S}_n \times \mathfrak{S}_m$ - equivariant log-concave

Other Matrix Loci

- $G(r,p,n)$ $\left(\mathfrak{S}_{n,r} = G(r,1,n) \right)$
- Weyl groups H_3 and F_4
- Involutions (L., Ma, Rhoades, Zhu)
- Contingency tables (Oh–Rhoades)
- Other cycle types in $\mathfrak{S}_n$

Thanks for Listening!

- Increasing subsequences, matrix loci, and orbit harmonics (Rhoades, 2024)
- Viennot shadows and graded module structure in colored permutation groups (L., 2025)
- An extension of Viennot's shadow to rook placements via orbit harmonics (L., Zhu, 2025)