Enriched toric $[\vec{D}]$-partitions

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Overview

1. Background
   - Permutations and permutation statistics: Linear vs Cyclic
   - (Cyclic) Quasi-symmetric functions: QSym vs cQSym
2. P-partitions
3. Enriched P-partitions
4. Toric $[\vec{D}]$-partitions
   - DAGs
   - Toric DAGs
   - Toric $[\vec{D}]$-partitions
5. Enriched toric $[\vec{D}]$-partitions
   - Enriched toric $[\vec{D}]$-partitions
   - Cyclic peak functions and algebra of cyclic peaks

Outline

Enriched toric $[\vec{D}]$-partitions
Toric poset, enriched partition, cQSym, cPk
(Liang)

Enriched P-partitions
Ordinary poset, enriched partition, QSym, Pk
(John Stembridge)

Toric $[\vec{D}]$-partitions
Toric poset, ordineary partition, cQSym, cDes
(Adin, Gessel, Reiner, and Roichman)

P-partitions
Ordinary poset, ordinary partition, QSym, Des
(Richard Stanley)

Cyclic Permutations

- Let $\mathfrak{S}_n$ denote the symmetric group of linear permutations $\pi = \pi_1\pi_2\ldots\pi_n$ on the set $\{1, 2, \ldots, n\}$. The corresponding cyclic permutation is an equivalence class of linear permutations under rotation.

$$[\pi] = \{\pi_1\pi_2\ldots\pi_n,\ \ \pi_2\ldots\pi_n\pi_1,\ \ \ldots,\ \ \pi_n\pi_1\ldots\pi_{n-1}\}.$$

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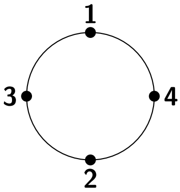
  - **Ex:** For example, $[1423] = \{1423,\ 4231,\ 2314,\ 3142\} = [3142]$.



Linear permutation statistics: Des and Pk

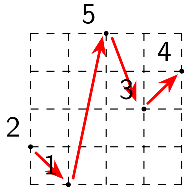
Define the descent set of $\pi \in \mathfrak{S}_n$

$$\operatorname{Des} \pi = \{i \mid \pi_i > \pi_{i+1}\} \subseteq [n-1],$$

and the descent number $\operatorname{des} \pi = |\operatorname{Des} \pi|$. The peak set Pk is defined by

$$\operatorname{Pk} \pi = \{i \mid \pi_{i-1} < \pi_i > \pi_{i+1}\} \subseteq [2, n-1].$$

The peak number is $\operatorname{pk} \pi := |\operatorname{Pk} \pi|$.



$\pi = 21534$

- $\operatorname{Des} \pi = \{1, 3\}$
- $\operatorname{Pk} \pi = \{3\}$

Two more linear permutation statistics: cDes and cPk

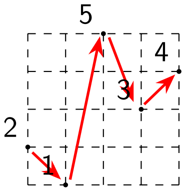
The cyclic descent set cDes of a linear permutation π is defined by

$$\operatorname{cDes}\pi = \{i \mid \pi_i > \pi_{i+1} \text{ where the subscripts are taken modulo } n\} \subseteq [n],$$

and the cyclic descent number $\operatorname{cdes}\pi = |\operatorname{cDes}\pi|$. The cyclic peak set cPk is defined by

$$\operatorname{cPk}\pi = \{i \mid \pi_{i-1} < \pi_i > \pi_{i+1} \text{ where the subscripts are taken modulo } n\} \subseteq [n],$$

The cyclic peak number is $\operatorname{cpk}\pi := |\operatorname{cPk}\pi|$.



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- $\operatorname{cDes}\pi = \{1, 3, 5\}$, $\operatorname{cdes}\pi = 3$
- $\operatorname{cPk}\pi = \{3, 5\}$, $\operatorname{cpk}\pi = 2$

Cyclic permutation statistics: cDes and cPk

Define the cyclic descent set of a cyclic permutation

$$\mathrm{cDes}[\pi] = \{\{\mathrm{cDes}\,\sigma \mid \sigma \in [\pi]\}\},$$

where the double curly brackets denote a multiset. And the cyclic descent number $\mathrm{cdes}[\pi] = \mathrm{cdes}\,\sigma$ for any $\sigma \in [\pi]$. The cyclic peak set cPk is defined by

$$\mathrm{cPk}[\pi] = \{\{\mathrm{cPk}\,\sigma \mid \sigma \in [\pi]\}\}.$$

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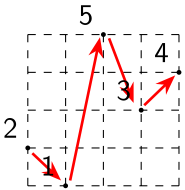

$\pi = 21534$

- $[21534] = \{21534, 42153, 34215, 53421, 15342\}$
- $\mathrm{cDes}[\pi] = \{\{\,\{1,3,5\}, \{1,2,4\}, \{2,3,5\}, \{1,3,4\}, \{2,4,5\}\,\}\}$
- $\mathrm{cdes}[\pi] = 3$

QSym: Quasi-symmetric Functions

For a formal power series $f \in \mathbb{Q}[[x_1, x_2, \ldots]]$, we use $[x_{i_1}^{a_1} x_{i_2}^{a_2} \ldots x_{i_s}^{a_s}]\, f$ to denote the coefficient of monomial $x_{i_1}^{a_1} x_{i_2}^{a_2} \ldots x_{i_s}^{a_s}$ in the expression of f. And the degree of a monomial $x_{i_1}^{a_1} x_{i_2}^{a_2} \ldots x_{i_s}^{a_s}$ is defined by $a_1 + a_2 + \cdots + a_s$.

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A *quasi-symmetric function* is a formal power series $f \in \mathbb{Q}[[x_1, x_2, \ldots]]$ such that for any sequence of positive integers $a = (a_1, a_2, \ldots, a_s)$, and two increasing sequences $i_1 < i_2 < \cdots < i_s$ and $j_1 < j_2 < \cdots < j_s$ of positive integers,

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Let QSym_n be the set of all quasi-symmetric functions which are homogeneous of degree n, and $\mathrm{QSym} = \oplus_{n\geq 0} \mathrm{QSym}_n$.

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- **Ex:** $5x_1^7 x_2 + 5x_1^7 x_3 + 5x_2^7 x_3 + \cdots$
- Note that a symmetric function must be quasi-symmetric, but the converse is **false**, i.e.,

$$\mathrm{Sym} \subsetneq \mathrm{QSym}\,.$$

Monomial and Fundamental Quasi-symmetric Functions

Definition

Given a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s) \vDash n$, the associated monomial quasi-symmetric function indexed by α is

$$M_\alpha = \sum_{i_1 < i_2 < \cdots < i_s} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \ldots x_{i_s}^{\alpha_s}.$$

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- $\{M_\alpha\}_{\alpha \vDash n}$ form a basis of QSym_n
- There is a natural bijection between subsets of $[n-1]$ and compositions of n. The map $\Phi : 2^{[n-1]} \to \mathrm{Comp}_n$ is defined by

$$\Phi(E) := (e_1 - e_0, e_2 - e_1, \ldots, e_k - e_{k-1}, e_{k+1} - e_k)$$

for any given $E = \{e_1 < e_2 < \cdots < e_k\} \subseteq [n-1]$ with $e_0 = 0$ and $e_{k+1} = n$.

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- $\{M_\alpha\}_{\alpha \vDash n}$ form a basis of QSym_n
- There is a natural bijection between subsets of $[n-1]$ and compositions of n. We can also index monomials by subsets of $[n-1]$.

Another important basis of QSym: fundamental quasi-symmetric functions. The relation between monomials and fundamentals is simple:

$$F_{n,E} = \sum_{E \subseteq L \subseteq [n-1]} M_{n,L}.$$

cQSym: Cyclic Quasi-symmetric Functions

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- Note that a cyclic quasi-symmetric function must be quasi-symmetric, but the converse is **false**, i.e.,

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Given a composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_s) \vDash n$, the associated monomial cyclic quasisymmetric function indexed by α is

$$M_\alpha^{\mathrm{cyc}} = \sum_{i=1}^{s} M_{(\alpha_i, \alpha_{i+1}, \dots, \alpha_{i-1})},$$

where the indices are interpreted modulo s, meaning $\alpha_{j+s} = \alpha_j$.

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If α and $\tilde{\alpha}$ differ by cyclic shifts, then $M_\alpha^{\mathrm{cyc}} = M_{\tilde{\alpha}}^{\mathrm{cyc}}$ — indexed by cyclic compositions, or correspondingly, equivalence classes of subsets under cyclic shifts.

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A cyclic shift of a subset $E \subseteq [n]$ in $[n]$ is a set of the form

$$i + E = \{i + e \pmod{n} \mid e \in E\}.$$

Note that sometimes we will use $E + i$ as well for the same concept.

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If α and $\tilde{\alpha}$ differ by cyclic shifts, then $M_\alpha^{\text{cyc}} = M_{\tilde{\alpha}}^{\text{cyc}}$ — indexed by cyclic compositions, or correspondingly, equivalence classes of subsets under cyclic shifts.

Another important basis of cQSym, the fundamental cyclic quasi-symmetric function indexed by $E \subseteq [n]$, is defined by

$$F_{n,E}^{\text{cyc}} := \sum_{E \subseteq L \subseteq [n]} M_{n,L}^{\text{cyc}}.$$

P-partitions

Definition

Given a poset P on $[n]$, a P-partition is a function $f : [n] \to \mathbb{P}$ such that

(a) $i \leq_P j$ implies $f(i) \leq f(j)$,

(b) $i \leq_P j$ and $i > j$ implies $f(i) < f(j)$.

Denote by $\mathcal{A}(P)$ the set of all P-partitions.

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Associate any $f : [n] \to \mathbb{P}$ with a monomial

$$\mathbf{x}^f = x_{f(1)} x_{f(2)} \cdots x_{f(n)}.$$

One can define the generating function

$$\Gamma_P = \sum_{f \in \mathcal{A}(P)} \mathbf{x}^f.$$

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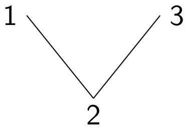
Ex: P is defined on the vertex set $V = \{1, 2, 3\}$ by the Hasse diagram



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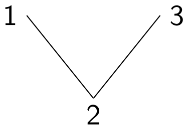

(a) $f(2) \leq f(1)$, $f(2) \leq f(3)$;

(b) $f(2) < f(1)$.

$$\begin{cases} f(2) = f(3) < f(1) \\ f(2) < f(3) < f(1) \\ f(2) < f(3) = f(1) \\ f(2) < f(1) < f(3) \end{cases} \implies \begin{cases} M_{(2,1)} \\ M_{(1,1,1)} \\ M_{(1,2)} \\ M_{(1,1,1)} \end{cases}$$

$$\Gamma_P = M_{(2,1)} + M_{(1,2)} + 2M_{(1,1,1)}.$$

Linear extensions and Fundamental Lemma

For a given P on $V = [n]$, define a linear extension of P to be a total ordering $\pi = \pi_1\pi_2\cdots\pi_n$ of V such that $\pi_i <_P \pi_j$ implies $i < j$. Let $\mathcal{L}(P)$ be the set of linear extensions of P.

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- In particular, π is also a poset on $[n]$. The structure of π-partitions is simple to describe:

$$\mathcal{A}(\pi) = \{f : [n] \to \mathbb{P} \mid f(\pi_1) \leq \cdots \leq f(\pi_n), i \in \operatorname{Des}\pi \text{ implies } f(\pi_i) < f(\pi_{i+1})\}.$$

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Fundamental Lemma

For any poset P on $[n]$, $\mathcal{A}(P)$ can be decomposed as the disjoint union:

$$\mathcal{A}(P) = \bigsqcup_{\pi \in \mathcal{L}(P)} \mathcal{A}(\pi).$$

One can regard total orderings, or equivalently, linear permutations, as the building blocks!

Enriched P-partitions

Stembridge defines $\mathbb{P}'$ to be the set of nonzero integers, totally ordered as

$$-1 \prec 1 \prec -2 \prec 2 \prec -3 \prec 3 \prec \cdots .$$

Definition

An enriched P-partition is a function $f : [n] \to \mathbb{P}'$ such that for all $i \leq_P j$ in P,

(a) $f(i) \preceq f(j)$,

(b) $f(i) = f(j) > 0$ implies $i < j$,

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Denote by $\mathcal{E}(P)$ the set of all enriched P-partitions f.

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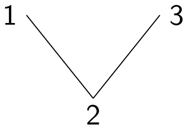

(a) $f(2) \preceq f(1)$, $f(2) \preceq f(3)$;

(b) $f(2) = f(1) = L$ implies $L < 0$;

(c) $f(2) = f(3) = L$ implies $L > 0$.

Weight enumerator and peak set

Define the weight enumerator for enriched P-partitions by the formal series

$$\Delta_P := \sum_{f \in \mathcal{E}(P)} \prod_{i \in [n]} x_{|f(i)|}.$$

By the Fundamental Lemma of enriched P-partitions:

$$\Delta_P = \sum_{\pi \in \mathcal{L}(P)} \Delta_\pi.$$

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Proposition (Stembridge, 1997)

As a quasi-symmetric function, Δ_π has the following expansion of monomial quasi-symmetric functions

$$\Delta_\pi = \sum_{E \subseteq [n-1]:\ \mathrm{Pk}\,\pi \subseteq E \cup (E+1)} 2^{|E|+1} M_{n,E}.$$

Weight enumerator and peak set

Define the weight enumerator for enriched P-partitions by the formal series

$$\Delta_P := \sum_{f \in \mathcal{E}(P)} \prod_{i \in [n]} x_{|f(i)|}.$$

By the Fundamental Lemma of enriched P-partitions:

$$\Delta_P = \sum_{\pi \in \mathcal{L}(P)} \Delta_\pi.$$

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Takeaway: Δ_π depends on $\operatorname{Pk}\pi$.

DAGs

Every DAG, abbreviation for directed acyclic graph, has a unique transitive closure; transitive $\vec{D}$ induces a poset on the vertex set V by if $i \to j$ in $\vec{D}$ then $i <_{\vec{D}} j$.

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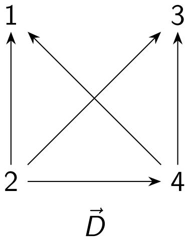
Ex: $\vec{D}$ has vertex set $V = \{1, 2, 3, 4\}$.



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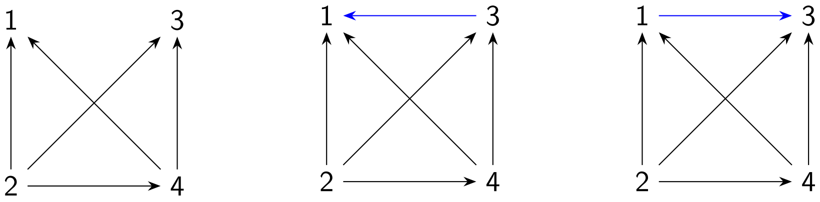

$\vec{D}$ $\quad$ $\vec{D}_1 = 2431$ $\quad$ $\vec{D}_2 = 2413$

Observation: Both $\vec{D}_1$ and $\vec{D}_2$ extend $\vec{D}$, so 2431 and 2413 linearly extend $\vec{D}$.

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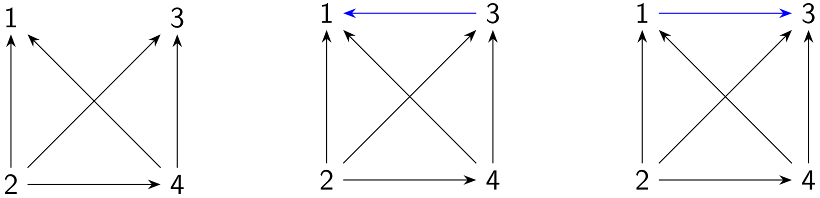

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Observation: Both $\vec{D}_1$ and $\vec{D}_2$ extend $\vec{D}$, so 2431 and 2413 linearly extend $\vec{D}$.

Equivalently, one can define a linear extension of $\vec{D}$ as being a DAG $\vec{D}'$ on the same vertex set whose arc set contains that of $\vec{D}$.

Toric DAGs

Definition (Flip)

Suppose i is a source (all arrows out) or a sink (all arrows in) in $\vec{D}$, we say $\vec{D}'$ is obtained from $\vec{D}$ by a flip at i if $\vec{D}'$ is obtained by reversing all arrows containing i.

Define the equivalence relation $\equiv$ on DAGs as follows: $\vec{D}' \equiv \vec{D}$ if and only if $\vec{D}'$ is obtained from $\vec{D}$ by a sequence of flips. A toric DAG $[\vec{D}]$ is the equivalence class of a DAG $\vec{D}$.

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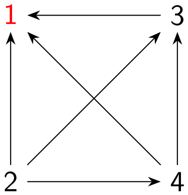

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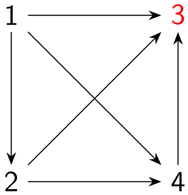
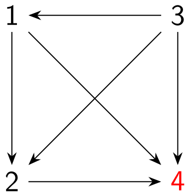
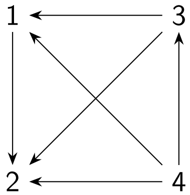


$\vec{D}_2 = 1243$



$\vec{D}_3 = 3124$



$\vec{D}_4 = 4312$

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Question: If $\vec{D} = \pi$ is a total linear order, what's the corresponding $[\vec{D}]$?

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Proposition (Develin-Macauley-Reiner, 2016)

If $\vec{D} = \pi$ is a total linear order with $\pi = \pi_1 \ldots \pi_n$, then there is a bijection between toric DAG $[\vec{D}]$ and cyclic permutation $[\pi]$.

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Question: If $\vec{D} = \pi$ is a total linear order, what's the corresponding $[\vec{D}]$?

The cyclic permutation $[\pi]$!

For a general toric DAG $[\vec{D}]$, define

$$\mathcal{L}^{\text{tor}}([\vec{D}]) := \{\, [\pi] : [\pi] \text{ torically extends } [\vec{D}] \,\}.$$

Toric $[\vec{D}]$-partitions

Definition

A toric $[\vec{D}]$-partition is a function $f : [n] \to \mathbb{P}$ which is a $\vec{D}'$-partition for at least one DAG $\vec{D}' \in [\vec{D}]$. Denote by $\mathcal{A}^{\text{tor}}([\vec{D}])$ the set of all toric $[\vec{D}]$-partitions f.

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- Similarly define the generating function

$$\Gamma^{\mathrm{cyc}}_{[\vec{D}]} = \sum_{f \in \mathcal{A}^{\mathrm{tor}}([\vec{D}])} \mathbf{x}^f .$$

It turns out that

$$\Gamma^{\mathrm{cyc}}_{[\pi]} = F^{\mathrm{cyc}}_{n, \mathrm{cDes}\,\pi} \in \mathrm{cQSym} .$$

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$$\mathcal{E}^{\text{tor}}([\vec{D}]) = \bigcup_{\vec{D}' \in [\vec{D}]} \mathcal{E}(\vec{D}'), \qquad \left(\mathcal{E}^{\text{tor}}([\pi]) = \bigcup_{\pi' \in [\pi]} \mathcal{E}(\pi') \right).$$

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Lemma (Fundamental Lemma of enriched toric $[\vec{D}]$-partitions, L, 2022)

For a DAG $\vec{D}$, the set of all enriched toric $[\vec{D}]$-partitions is a disjoint union of the set of enriched toric $[\pi]$-partitions of all toric extensions $[\pi]$ of $[\vec{D}]$:

$$\mathcal{E}^{\text{tor}}([\vec{D}]) = \bigsqcup_{[\pi] \in \mathcal{L}^{\text{tor}}([\vec{D}])} \mathcal{E}^{\text{tor}}([\pi]).$$

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- Fundamental Lemma ✓
- Denote the weight enumerator for enriched toric $[\vec{D}]$-partitions by $\Delta^{\text{cyc}}_{[\vec{D}]}$. The Fundamental Lemma implies,

$$\Delta^{\text{cyc}}_{[\vec{D}]} = \sum_{[\pi] \in \mathcal{L}^{\text{tor}}([\vec{D}])} \Delta^{\text{cyc}}_{[\pi]}$$

Cyclic Peak Functions

Proposition (L, 2022)

1. For any given cyclic permutation $[\pi]$ of length n, we have

$$\Delta_{[\pi]}^{\text{cyc}} = \sum_{E \subseteq [n] :\, \text{cPk}(\pi) \subseteq E \cup (E+1)} 2^{|E|} M_{n,E}^{\text{cyc}}.$$

The sum is independent of the choice of representative π of $[\pi]$.

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Takeaway: $\Delta^{\mathrm{cyc}}_{[\pi]}$ depends on $\mathrm{cPk}\,\pi$, or equivalently, $\mathrm{cPk}[\pi]$; one can associate to every cyclic peak set S a cyclic quasi-symmetric function $K^{\mathrm{cyc}}_S := \Delta^{\mathrm{cyc}}_{[\pi]}$ for any permutation π with peak $\mathrm{cPk}\,\pi = S$. That is our cyclic peak function.

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2. For any cyclic peak set S in $[n]$, we have
$$2^{-|S|} K^{\text{cyc}}_S = \sum_{E \subseteq [n] :\, S \subseteq E \triangle (E+1)} F^{\text{cyc}}_{n,E},$$
where $\triangle$ denotes symmetric difference.

Algebra of cyclic peaks

Let Λ_n be the space of cyclic quasi-symmetric functions spanned by cyclic peak function K_S^{cyc}, where S ranges over cyclic peak sets in $[n]$. Also let

$$\Lambda := \oplus_{n \geq 0} \Lambda_n.$$

Proposition (L, 2022)

Λ is a graded subring of cQSym, with K_S^{cyc} form a basis of Λ. We call Λ the algebra of cyclic peaks.

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More interesting stuff:

- Order polynomial
- Cyclic shuffle-compatibility (a joint work with Bruce Sagan and Yan Zhuang in preparation)

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Thank You!

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