Homomesy on permutations

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A ICERM Research Community in Algebraic Combinatorics project

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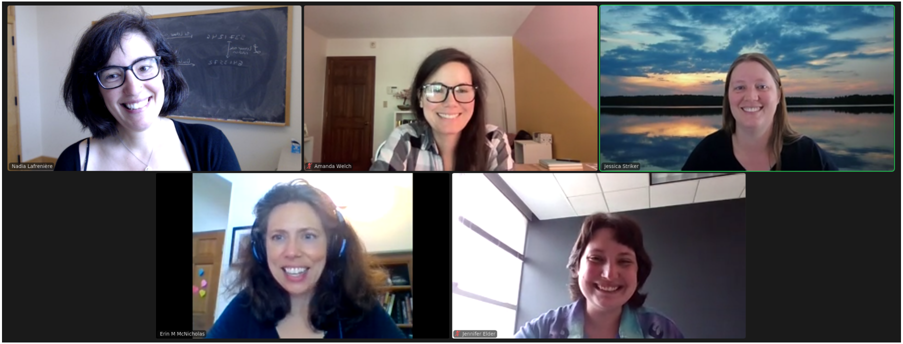
Collaborators



We met through the ICERM Research Community in Algebraic Combinatorics

Objects and actions

Mathematical inquiry often begins with the study of *objects* and the question:

"What are the objects like?"

It then moves to the study of *actions* and the question,

"How do the objects behave?"

For objects and actions arising from algebraic combinatorics, we call this study

Dynamical Algebraic Combinatorics.

What objects? What actions? A systematic study.

- One object: permutations
- Multiple actions!

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- Goals:
  - characterize homomesy
  - understand the maps that exhibit a lot of homomesy.

    $\Longrightarrow$ Systematic study.

What objects? What actions? A systematic study.

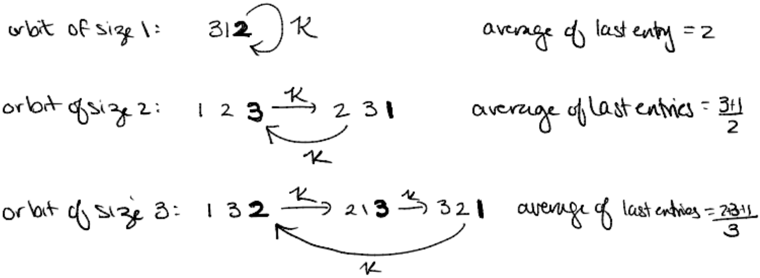
- One object: permutations
- Multiple actions!
- Goals:
  - characterize homomesy
  - understand the maps that exhibit a lot of homomesy.

    $\Longrightarrow$ Systematic study.
- Outcomes:
  - 120+ proven occurrences of homomesy
  - Counter-examples for all other pairs of a map and a statistic from FindStat (7,000 +)

Homomesy

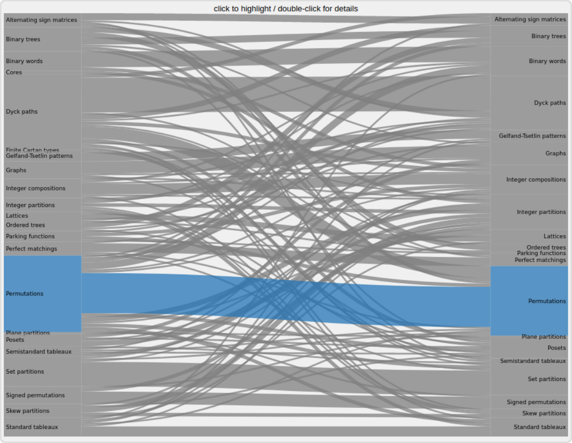
Definition (J. Propp and T. Roby 2015)

A statistic on a set exhibits *homomesy* with respect to an action when the orbit-average of the statistic equals the global average of that statistic.



Homomesy on Permutations using Findstat

Findstat contains 19 bijective maps, and 387 statistics, on permutations.



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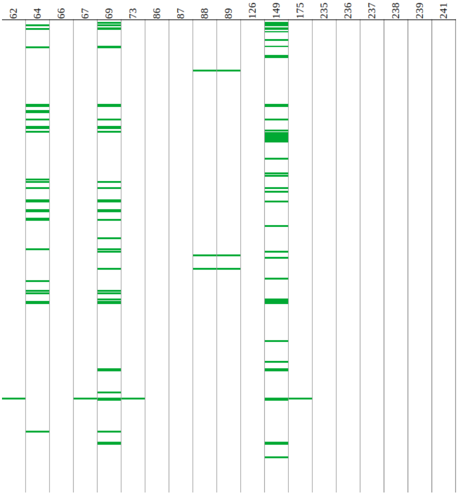


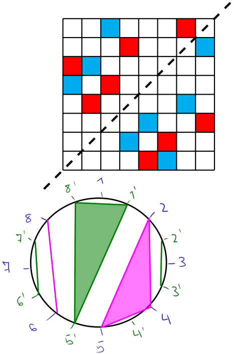
Legend:

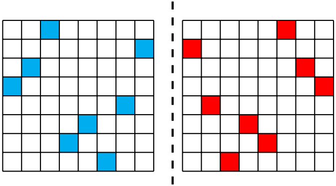
- 64 Reverse
- 69 Complement
- 88 Kreweras complement
- 89 Inverse Kreweras comp.
- 149 Lehmer code rotation

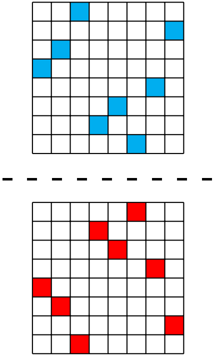
Geometric actions ≈ Homomesic maps

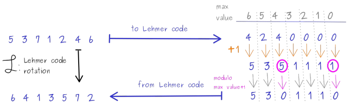
Observation

Only a few maps exhibit homomesy!









Other maps in Findstat: First Fundamental transform, Cactus evacuation, Inverse, . . .

A warm-up: reverse and complement

The reverse of a permutation $\sigma_1 \ldots \sigma_n$ is $\sigma_n \ldots \sigma_1$.
The complement of a permutation $\sigma_1 \ldots \sigma_n$ is $(n+1-\sigma_1)(n+1-\sigma_2)\ldots(n+1-\sigma_n)$.

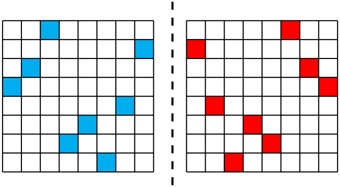


Figure: Reverse map

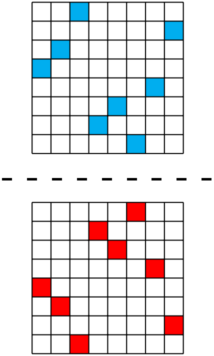


Figure: Complement map

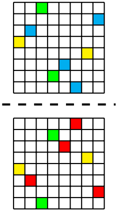
Homomesic statistics: reverse and complement

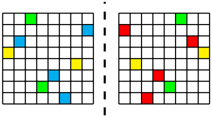
- Several statistics relative to descents:

$$i \text{ is descent} \implies \begin{cases} i \text{ is ascent for comp.} \\ n+1-i \text{ is ascent for reverse} \end{cases}$$

- Several statistics relative to inversions:

$$(a,b) \text{ is inv.} \implies \begin{cases} (a,b) \text{ is noninversion for comp.} \\ (n+1-b, n+1-a) \text{ is noninv.} \\ \qquad\qquad\qquad\qquad \text{for reverse} \end{cases}$$





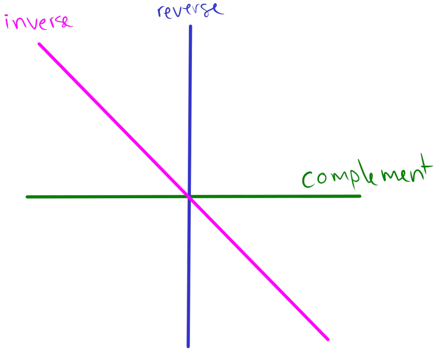
Homomesic statistics: reverse and complement

- Major index is homomesic for the reverse, but not for the complement.
- The disorder of a permutation is homomesic for the complement, but not for the reverse.

One proof technique is to compute the average over an orbit; a more interesting one is through connections between the maps.

Reverse, inverse and complement

- $\mathcal{R}(\sigma^{-1}) = \mathcal{C}(\sigma)^{-1}$
- $(\mathcal{R} \circ \mathcal{C})^2 = e$
- $(\mathcal{R} \circ \mathcal{I})^4 = e$, where $\mathcal{I}(\sigma) = \sigma^{-1}$



Remark

No statistic is homomesic under the inverse map.

Sample homomesy example: Disorder

The *disorder of a permutation* corresponds to the complexity of some sorting algorithm.

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The *disorder of a permutation* corresponds to the complexity of some sorting algorithm.

Observation

The disorder of a permutation is given by

$$\sum_{i=1}^{n-1} (n-i)\ \delta_{(i,i+1) \text{ is an inversion pair}}.$$

Theorem

The disorder of a permutation is homomesic for the reverse, but not for the complement.

Sample homomesy example: Major index

The *major index* is the sum of the descents.

Observation

The disorder of a permutation is given by

$$\sum_{i=1}^{n-1}(n-i)\ \delta_{(i,i+1)\text{ is an inversion pair}} = \sum_{i=1}^{n-1}(n-i)\ \delta_{i\text{ is a descent of }\sigma^{-1}}.$$

The major index is given by

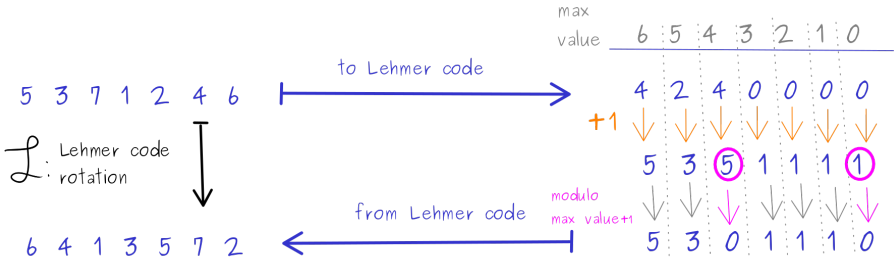
$$\sum_{i=1}^{n-1} i\ \delta_{i\text{ is a descent}}.$$

Theorem

The major index is homomesic for the complement, but not for the reverse.

A more interesting map: the Lehmer code rotation

The *Lehmer code* of a permutation encodes the number of inversions starting at a given position.



Sample result: Lehmer code rotation

1234	(0, 0, 0, 0)	2134	(1, 0, 0, 0)
2341	(1, 1, 1, 0)	3241	(2, 1, 1, 0)
3412	(2, 2, 0, 0)	4312	(3, 2, 0, 0)
4132	(3, 0, 1, 0)	1243	(0, 0, 1, 0)
1324	(0, 1, 0, 0)	2314	(1, 1, 0, 0)
2431	(1, 2, 1, 0)	3421	(2, 2, 1, 0)
3124	(2, 0, 0, 0)	4123	(3, 0, 0, 0)
4231	(3, 1, 1, 0)	1342	(0, 1, 1, 0)
1423	(0, 2, 0, 0)	2413	(1, 2, 0, 0)
2143	(1, 0, 1, 0)	3142	(2, 0, 1, 0)
3214	(2, 1, 0, 0)	4213	(3, 1, 0, 0)
4321	(3, 2, 1, 0)	1432	(0, 2, 1, 0)

All orbits have size $\mathrm{lcm}(1, 2, 3, \ldots, n)$.

Sample result: Lehmer code rotation

Theorem (Elder-L.-McNicholas-Striker-Welch)

45 statistics are homomesic under the Lehmer code rotation, including:

- *right-to-left minima, not left-to-right minima*
- *descents*
- *inversions*
- *various permutation patterns*
- 1^{st} *entry, not last*
- *rank*

Rank is homomesic for reverse and Lehmer code rotation

The *rank* of a permutation is its rank among the permutations of n ordered lexicographically.

σ	123	132	213	231	312	321
$L(\sigma)$	000	010	100	110	200	210

Observation

Sorting lexicographically following the Lehmer code is equivalent to sorting lexicographically following the permutations.

Rank is homomesic for reverse and Lehmer code rotation

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Sorting lexicographically following the Lehmer code is equivalent to sorting lexicographically following the permutations.

Proposition

The rank is given by $rank(\sigma) = 1 + \sum_{i=1}^{n-1} L(\sigma)_i (n-i)!$, *where* $L(\sigma)_i$ *is the i-th entry of the Lehmer code of* σ.

Theorem

The rank is homomesic for the Lehmer code rotation and the reverse.

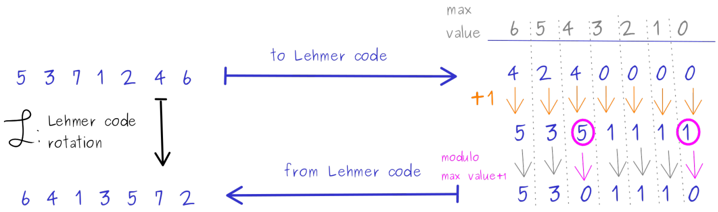
What is Lehmer code rotation?

Remark

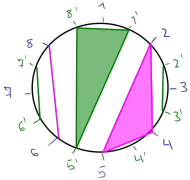
We tested several maps dictated by the Lehmer code, and none of them exhibited homomesy!

Question

What is the geometric meaning of the Lehmer code rotation?



Kreweras complement on permutations



Properties

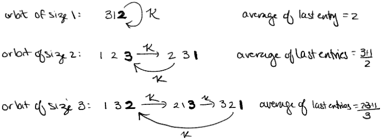
- The Kreweras complement on permutations is an extension of the well-known Kreweras complement on non-crossing partitions.
- On non-crossing partitions, $\mathcal{K}^2$ is equivalent to a rotation.

Definition

On permutations, the Kreweras complement is defined as

$$\mathcal{K}(\sigma) = c \circ \sigma^{-1}, \text{ with } c = (1234 \ldots n)$$

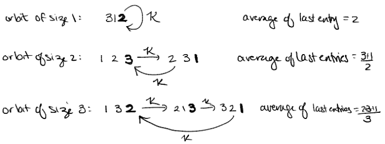
Kreweras complement: orbit sizes

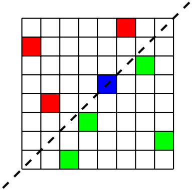


Theorem (Elder-L.-McNicholas-Striker-Welch)

- *The order of $\mathcal{K}$ is $2n$.*
- *If n is even, there is no orbit of odd size.*
- *We know the full distribution for orbit lengths.*

Kreweras complement: homomesic statistics





Theorem (Elder-L.-McNicholas-Striker-Welch)

The following statistics are homomesic under the Kreweras complement:

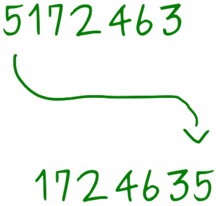
- *The number of exceedances*
- *The number of weak deficiencies*
- *The last entry*
- *When n is even, the lower middle entry*

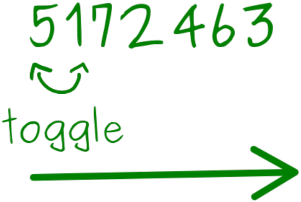
A few more questions

Question

- Are maps that have a natural geometric meaning more likely to exhibit homomesy? Why?
- Why is the inverse “poor” in homomesic statistics? Does it have too many fixed points?
- What is the meaning of the Lehmer code rotation?
- Are there any maps missing in FindStat that would exhibit homomesy?

Rotation action on permutations





Rowmotion means toggling in a well-defined order.

$$\text{Rotation} \overset{?}{=} \text{Rowmotion on permutations}$$

Rotation $\stackrel{?}{=}$ Rowmotion

Theorem (Dowling, 2022+)

There are between 29 and 34 FindStat statistics homomesic for rotation, including:

- *Exceedances*
- *Small and big exceedances*
- *Each entry*
- *The inversion sum (but not the inversion number)*

Dynamical Algebraic Combinatorics on Permutations

1. Cyclic Sieving Phenomenon

The Cyclic Sieving Phenomenon

- Introduced in 2004 by Reiner, Stanton and White to describe a surprising phenomenon occurring in representation theory.
- For a polynomial p and a group action X of order d to exhibit the CSP, we need the evaluation of p at all powers of the d-th root of unity to be a non-negative integer.
- We further need that $p(\zeta^k)$ is the number of fixed points of X^k, with $\zeta = e^{\frac{i\pi}{d}}$.
- For the CSP to be interesting, the polynomial must be *meaningful*.

The Cyclic Sieving Phenomenon

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Definition

- The *statistic-generating function* of a statistic $\mathrm{st} : S \to \mathbb{N}$ is $f^{\mathrm{st}(q)} = \sum_{s \in S} q^{\mathrm{st}(s)}$.
- A statistic on a set exhibits the *cyclic sieving phenomenon* with respect to an action X if f^{st} exhibits it for X.

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- A statistic on a set exhibits the *cyclic sieving phenomenon* with respect to an action X if f^{st} exhibits it for X.

Example

Let the statistic be the number of descents on permutations of 3. Then, $f^{\mathrm{des}}(q) = 1 + 4q + q^2$.

Known occurrence of the CSP

Theorem (Sagan, Shareshian, Wachs, 2009)

Let $st = maj - exc$, the major index minus the number of exceedences. The Cyclic Sieving Phenomenon occurs for this statistics with the action of the conjugation by the long cycle $(1, 2, \ldots, n)$.

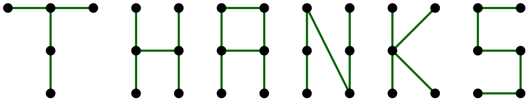
Are there other occurrences of the CSP on permutations?

Other occurrences of the CSP

Conjecture

The CSP occurs for the following actions with the statistics in FindStat:

- *Reverse and complement (41 statistics, including 19 Mahonian ones)*
- *Lehmer code rotation (1 statistic: rank)*
- *Corteel and Invert Laguerre heap (15 statistics)*
- *Alexandersson-Kebede map (2 statistics)*
- *Rotation (24 statistics: Mahonian ones, rank, specific entries)*
- *Conjugation by long cycle (7 statistics, including maj−exc)*


THANKS