Maximal Chain Descent Orders

Stephen Lacina

University of Oregon

- Based on my preprint "Maximal Chain Descent Orders," available on Arxiv.
- Thanks to my PhD advisor Patricia Hersh for insightful discussions and guidance throughout this project.

Outline

§1. Background on shelling and lexicographic shelling of posets.

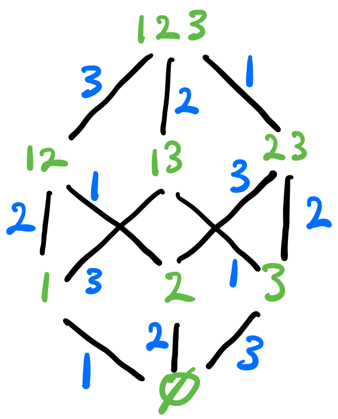
§2. Define maximal chain descent orders on the max'l chains of any poset with an EL-labeling λ and show how they encode the shellings "derived from" λ.

§3. Properties of max'l chain descent orders including subtlety in their cover relations and some structure theorems.

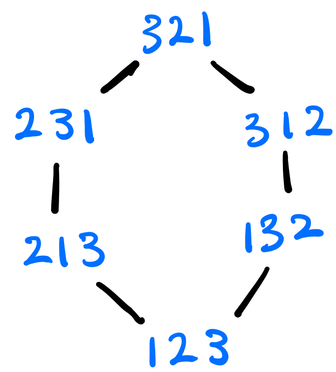
§4. The case of intervals in Young's lattice and standard Young tableaux.

Examples of Maximal Chain Descent Orders

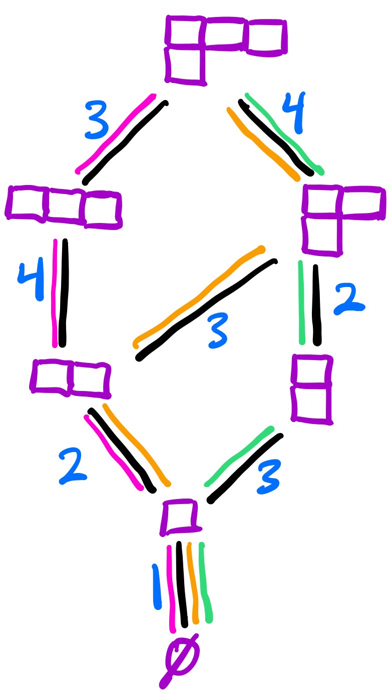
① $B_3 =$



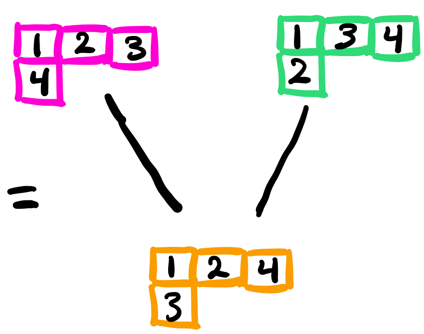
$B_{3\,\lambda}(2) =$



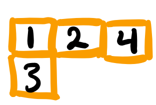
② $Y(\alpha) =$



$Y(\alpha)_\lambda(2) =$



λ EL-labeling induced by SYT



§1 Background: Shelling Simplicial Complexes

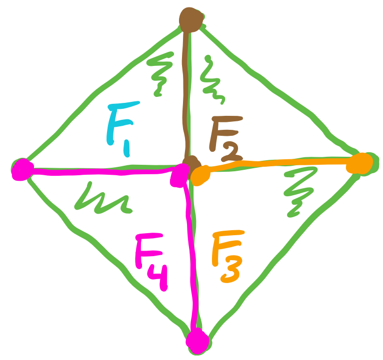
- Δ a finite abstract simplicial complex.
- A **shelling** of Δ is a total order

$$F_1, F_2, \dots, F_t$$

on the facets of Δ such that $\forall\ 1 < j \leq t$

$$\overline{F_j} \cap \left(\bigcup_{i<j} \overline{F_i} \right)$$

is a pure codimension one subcomplex of $\overline{F_j}$.



<u>Rmk:</u> The intuitive key is codimension one intersections.

- At step j there is a unique minimal face $R(F_j)$ of $\overline{F_j}$ not contained in any $\overline{F_i}$ for $i<j$,

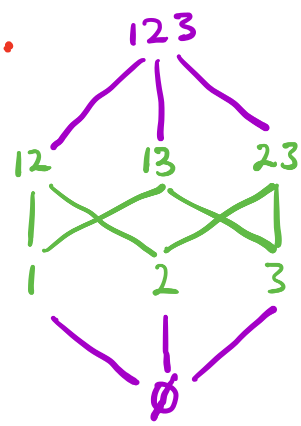
$$R(F_j) = \{ v \in F_j \mid F_j \setminus \{v\} \subset F_i \text{ for some } i<j \}$$

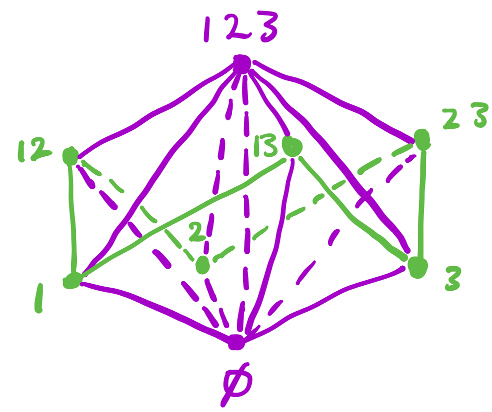
R is called the **restriction map** of the shelling and $R(F_j)$ the **restriction face** of F_j.

Background: Lexicographic Shellings of Posets

Def: The order complex of a poset P is the abstract simplicial complex $\Delta(P)$ whose i-faces are the i-chains $v_0 < v_1 < \dots < v_i$ in P.

e.g.

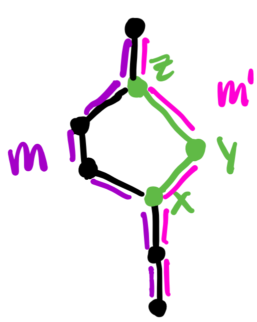
$P =$ 

$\Delta(P) =$ 

- The facets of $\Delta(P)$ are the maximal chains of P, denoted $\mathcal{M}(P)$. Max'l chains must have each $v_i < v_{i+1}$ a cover relation, denoted $v_i \lessdot v_{i+1}$.

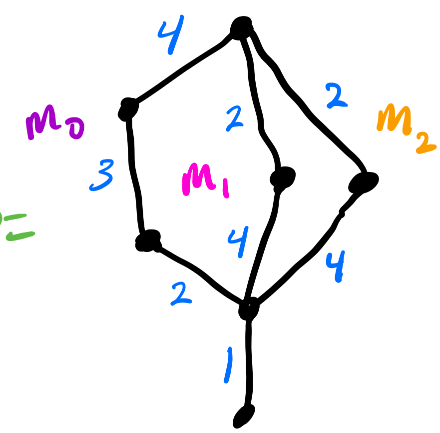
- Max'l chains m and m' have $m \cap m'$ codim' one in m' when $x \lessdot y \lessdot z$ contained in m' and m and m' agree everywhere except at y.

e.g.



- Björner (then Björner + Wachs) proved that labeling the edges of the Hasse diagram of P by integers (subject to some conditions) and ordering the max'l chains by lexicographic (dictionary) order on their label sequences gives a shelling of $\Delta(P)$.

- If two max'l chains have the same label sequence, Björner (Björner + Wachs) showed that the tie can be broken arbitrarily to give a shelling order.

P=


$\overset{1234}{m_0}, \overset{142}{m_1}, \overset{142}{m_2}$

$\overset{1234}{m_0}, \overset{142}{m_2}, \overset{142}{m_1}$

lex. shellings of $\Delta(P)$

Rmk: One view of maximal chain descent orders is that they are what we see by looking deeply at the idea that edge labelings give multiple shellings of $\Delta(P)$.

EL-labelings

Def: (Björner; Björner + Wachs)

Edge labeling λ of finite, bounded poset P is called an EL-labeling if for all $x < y$ in P,

(i) $\exists!$ saturated chain

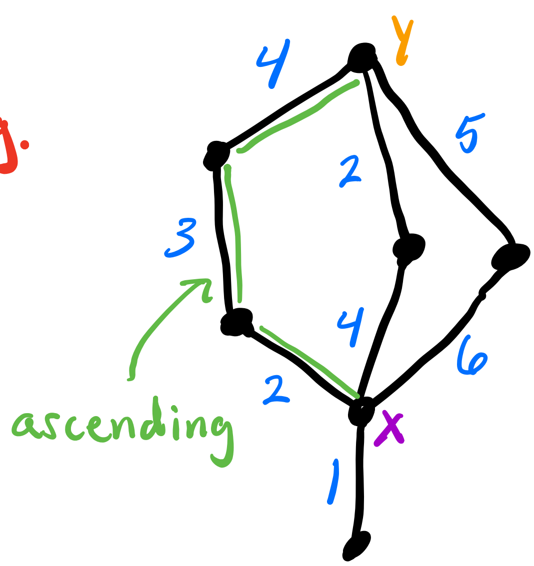
$$m: x \lessdot x_0 \lessdot x_1 \lessdot \cdots \lessdot x_{k-1} \lessdot y$$ such that

$$\lambda(x, x_0) \leq \lambda(x_0, x_1) \leq \cdots \leq \lambda(x_{k-1}, y).$$

Call m an ascending chain.

(ii) m lex. precedes all other max'l chains of $[x, y]$.

e.g.



Thm: (Björner; Björner + Wachs)

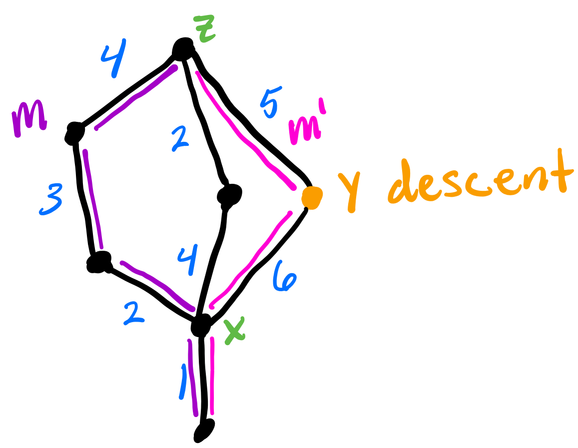
Lex. order (with ties broken arbitrarily) on the max'l chains of finite, bounded poset P induced by EL-labeling λ is a shelling order of $\Delta(P)$.

A key observation:

If max'l chain m' contains $x \lessdot y \lessdot z$ such that $\lambda(x,y) \nleq \lambda(y,z)$, then there is a unique max'l chain m which agrees with m' except on (x,z) and $m \cap [x,z]$ ascends w.r.t. λ.

- Call y a descent of m' w.r.t. λ.

e.g.



Rmk: Descents encode the restriction map of any lex. shelling from λ.

$$R(m) = \{x \in m \mid x \text{ descent of } m \text{ w.r.t. } \lambda\}$$

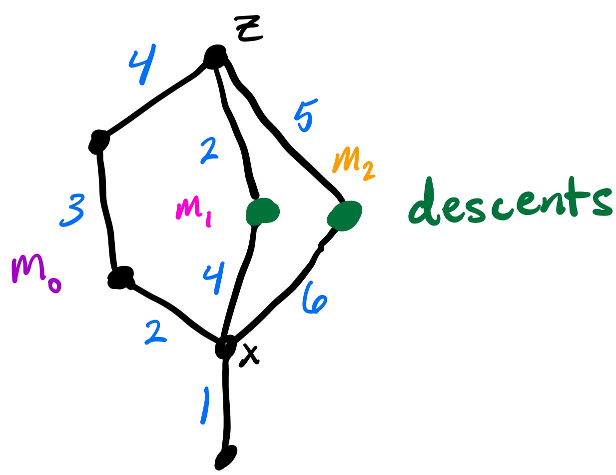
§2 Definition of a Maximal Chain Descent Order

- λ an EL-labeling of finite, bounded poset P.

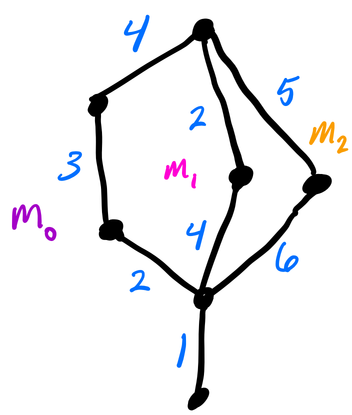
Def: For max'l chains $m, m' \in \mathcal{M}(P)$, say m increases by a polygon move to m', denoted $m \to m'$, if m' has a descent at y w.r.t. λ and m is the unique chain from the key observation above.

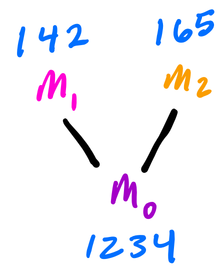
Rmk: $m \to m'$ implies $\lambda(m) <_{lex} \lambda(m')$

e.g. $m_0 \to m_1$
$m_0 \to m_2$
$m_1 \not\to m_2$

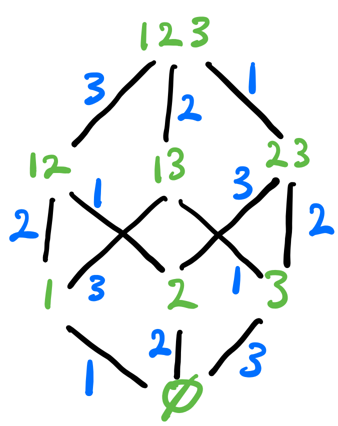


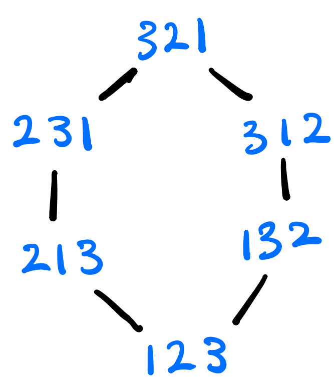
Def: (Hersh & L.)
The max'l chain descent order induced by λ is the reflexive & transitive closure of $m \to m'$, denoted $\preceq_\lambda$. Denote $(\mathcal{M}(P), \preceq_\lambda)$ by $P_\lambda(2)$.

① $P =$ 

$P_\lambda(2) =$ 

Rmk: $P_\lambda(2)$ is not lex. order from λ.

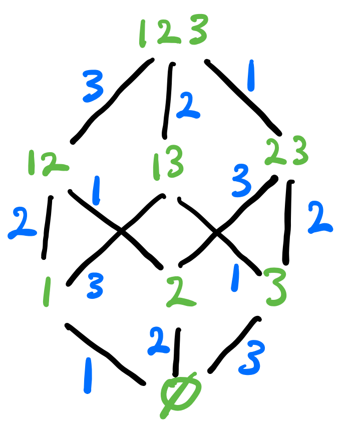
② $B_3 =$ 

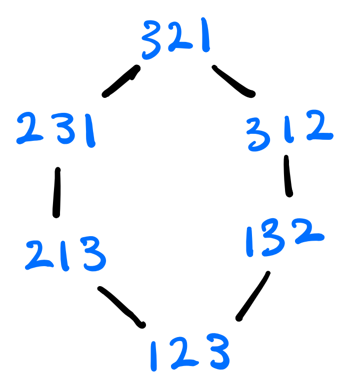
$B_{3\,\lambda}(2) =$ 

Boolean Lattice and the Weak Order of Type A

- The Boolean lattice B_n, poset of subsets of $[n]$, has a natural EL-labeling $\lambda(B, B\cup\{i\}) = i$.

Thm: (L.)
The max'l chain descent order $B_{n_\lambda}(2)$ is isomorphic to the weak order on S_n (type A) via the map $m \mapsto \lambda(m)$.

$B_3 =$ 

$B_{3_\lambda}(2) =$ 

$P_\lambda(2)$ and Shellings Derived from λ

Thm: (L.)
Let P be a finite, bounded poset with EL-labeling λ. For any total order $\Omega: m_1, m_2, \ldots, m_t$ on the max'l chains of, TFAE:

(i) Ω is a linear extension of $P_\lambda(2)$.

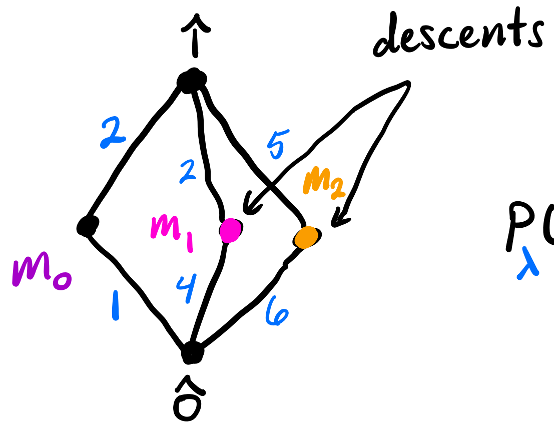
(ii) Ω is a shelling order of $\Delta(P)$ whose restriction map is given by
$R(m_i) = \{x \in m_i \mid x \text{ descent of } m_i \text{ w.r.t. } \lambda\}$

for each $1 \leq i \leq t$.

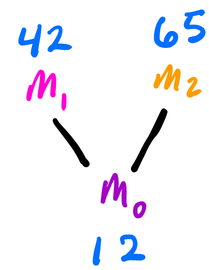
Rmk: We can think of $P_\lambda(2)$ as the structure of the set of shellings "derived from λ."

Rmk: For the Boolean lattice, the lin. exts. of $B_{n_\lambda}(2)$ recover Björner's shellings of the type A Coxeter complex.

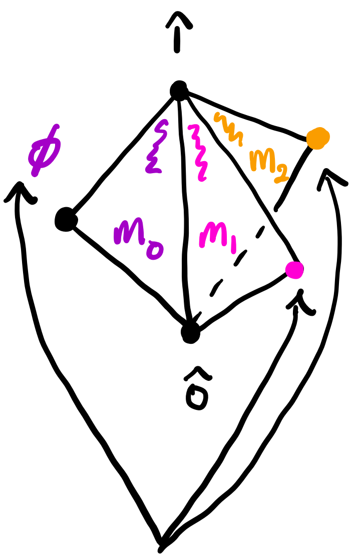
$P =$



$P_\lambda(2) =$



$\Delta(P) =$



Restriction faces of the shellings

$\underbrace{m_0, m_1, m_2}_{\text{lex.}}$ + $\underbrace{m_0, m_2, m_1}_{\text{not lex.}}$

Obs: No total order on max'l chains which begins with m_1 or m_2 has the same restriction map despite being shellings.

Rmk: Lex. shellings from λ are among the lin. exts. of $P_\lambda(2)$, but some lin. exts. are not lex. shellings.

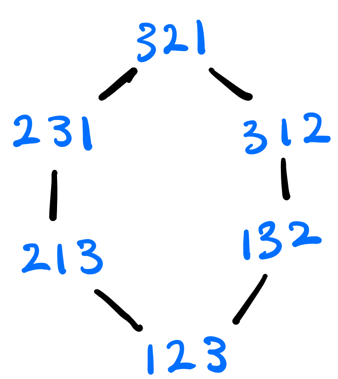
§3 Facts About $P_\lambda(2)$

Lemma: (L.)
Max'l chain descent order $P_\lambda(2)$ has a $\hat{0}$ given by the unique ascending max'l chain of P w.r.t. λ.

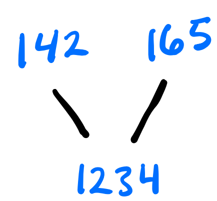
Lemma: (L.)
If max'l chain m of P has descending label sequence w.r.t. λ, then m is a maximal element of $P_\lambda(2)$.

Rmk: Descending max'l chains do <u>not</u> in general give all maximal elements of $P_\lambda(2)$.

$B_{3\lambda}(2)=$



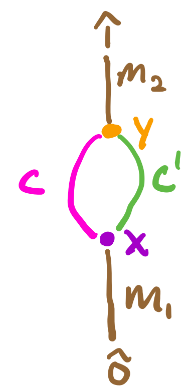
$P_\lambda(2)=$



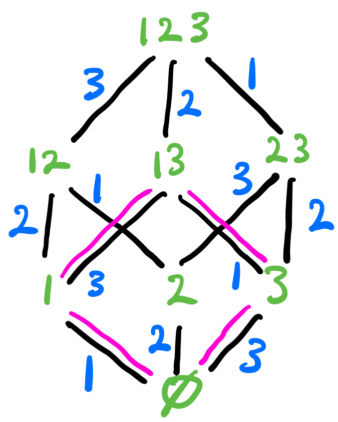
Rmk: Restricting λ to any closed interval $[x,y]$ of P gives an EL-labeling of $[x,y]$.
This gives a lifting property for $P_\lambda(2)$.

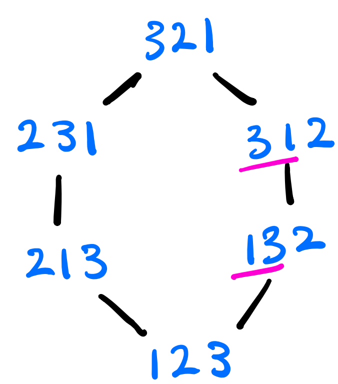
Lemma: (L.)
Let c and c' be max'l chains of $[x,y]$. Let m_1 be a max'l chain of $[\hat{0},x]$ and m_2 be a max'l chain of $[y,\hat{1}]$.
If $c \preceq_\lambda c'$ in $[x,y]_\lambda(2)$, then

$m_1 * c * m_2 \preceq_\lambda m_1 * c' * m_2$ in $P_\lambda(2)$.



e.g.

$B_3 =$ 

$B_{3\lambda}(2) =$ 

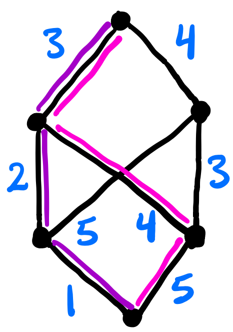
Rmk: $[x,y]_\lambda(2)$ appears as subposets of $P_\lambda(2)$, but not necessarily induced subposets.

Cover Relations of $P_\lambda(2)$

Rmk: Every cover relation of $P_\lambda(2)$ comes from a polygon move.

Perhaps surprisingly, the converse is <u>not</u> true.

e.g.

$P =$ 

$$P_\lambda(2) = \begin{array}{c} 543 \\ | \\ 534 \\ | \\ 154 \\ | \\ 123 \end{array}$$

polygon move, but not a cover relation

Def: (L.)

Let λ be an EL-labeling of finite, bounded poset P. Say λ is polygon complete if $m \to m'$ implies $m \lessdot_\lambda m'$.

- Many well known families of EL-labelings are polygon complete.

Thm: (L.)

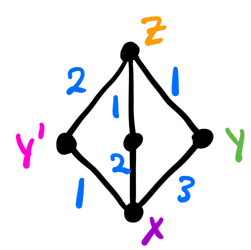
The following families of EL-labelings are polygon complete:

- Stanley's M-chain R-labelings of any finite supersolvable lattice (Björner noticed these are EL-labelings)
- Björner's minimal labelings of any finite geometric lattice
- Dyer's reflection order EL-labelings of closed intervals in Bruhat order of any Coxeter group
- Björner + Wachs' EL-labeling of the Tamari lattice (early non-graded example)
- Kallipoliti + Mühle's EL-labeling of closed intervals of any Cambrian semilattice
- McNamara + Thomas' interpolating EL-labelings

Rmk: The key to the previous theorem is proving each of the EL-labelings satisfies the following concrete condition.

Def: (L.)
Let λ be an EL-labeling of finite, bounded poset P. Say λ is polygon strong if for each descent $x \lessdot y \lessdot z$ w.r.t. λ (i.e. $\lambda(x,y) \not\leq \lambda(y,z)$), we have $\lambda(y,z) < \lambda(y',z)$ where $y' \lessdot z$ is contained in the unique ascending max'l chain of $[x,z]$.

e.g.



Rmk: Polygon strong is a relaxation of Björner's notion of strongly lex. shellable, which he introduced for entirely different reasons.

Thm: (L.)
Polygon strong implies polygon complete.

Characterization of Polygon Complete EL-labelings

- Rather technical; characterize EL-labelings which are not poly. complete.

Thm: (L.) P a finite, bounded poset with EL-labeling λ. λ fails to be polygon complete if and only if P has a max'l chain of length at least three and $\exists\ y, x_1, x_2, \dots, x_n, z_1, z_2, \dots, z_n \in P$ with $n \geq 2$ which, under the convention $x_{n+1} = x_1$, satisfy:

(i) $z_i \lessdot x_{i+1} \lessdot y$ is a descent w.r.t λ and $x_i \lessdot y$ is contained in the unique ascending max'l chain c_i of $[z_i, y]$ w.r.t. λ $\forall\ 1 \leq i \leq n$.

(ii) $\exists$ saturated chains of length at least one of the following form: m and m' from $\hat{0}$ to x_1 such that $m \to m'$ and m_i from $\hat{0}$ to z_i for $1 \leq i \leq n$ satisfying $m \preceq_\lambda m_1 * c_1^{x_1}$ in $[\hat{0}, x_1]_\lambda(2)$, $m_i * z_i * x_{i+1} \prec_\lambda m_{i+1} * c_{i+1}^{x_{i+1}}$ in $[\hat{0}, x_{i+1}]_\lambda(2)$ $\forall\ 1 \leq i \leq n$, and $m_n * x_1 \preceq_\lambda m'$ in $[\hat{0}, x_1]_\lambda(2)$.

Some Structural Results

Thm: (L.)
Let P be a rank n supersolvable lattice and λ be any of Stanley's M-chain R-labelings of P. Then every interval of $P_\lambda(2)$ is isomorphic to some interval of the weak order on S_n.

Rmk: By Birkhoff's Fundamental Theorem of Finite Distributive Lattices every finite distributive lattice L is isomorphic to the poset of order ideals of some finite poset P_L.

- Each linear extension of P_L induces an EL-labeling of L (Stanley's M-chain R-labeling).

Thm: (L.)
Let λ any EL-labeling of finite distributive lattice L from a lin. ext. of P_L. Then $L_\lambda(2)$ is isomorphic to some order ideal of the weak order on $S_{|P_L|}$.

§4 Interesting Case of Max'l Chain Descent Orders: Intervals in Young's Lattice

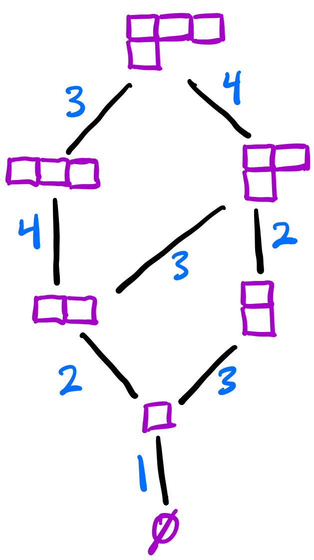
- For Young diagrams μ and α with μ contained in α, the max'l chains of $[\mu, \alpha]$ in Young's Lattice biject with the standard tableaux of skew shape α/μ.

- Each stnd. tab. T of shape α/μ gives an EL-labeling λ_T of $[\mu, \alpha]$ by

$$\lambda_T(\alpha_1, \alpha_2) = \text{entry of } T \text{ in box } \alpha_2/\alpha_1$$

e.g.

$Y(\alpha) =$



λ_T EL-labeling induced by SYT $T = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$

Rmk: The λ_Ts are the lin. ext. EL-labelings of $[\mu, \alpha]$ as a finite distributive lattice (Stanley's M-chain R-labelings).

- For stnd. tab. T of shape α/μ, define a partial order $\leq_T$ on the stnd. tab. of shape α/μ $(ST(\alpha/\mu))$ as the transitive closure of $Q \to R$ if R is Q except with entries i and $i+1$ swapped and the box with entry i in Q has smaller entry in T than the box with entry $i+1$ in Q.

e.g.

$$T = \begin{array}{ccc} 1 & 2 & 4 \\ 3 & 5 & \\ 6 & & \end{array} \quad \text{and} \quad \begin{array}{c} Q \\ \shortparallel \\ \begin{array}{ccc} 1 & 2 & 4 \\ 3 & 6 & \\ 5 & & \end{array} \end{array} \to \begin{array}{c} R \\ \shortparallel \\ \begin{array}{ccc} 1 & 2 & 5 \\ 3 & 6 & \\ 4 & & \end{array} \end{array}$$

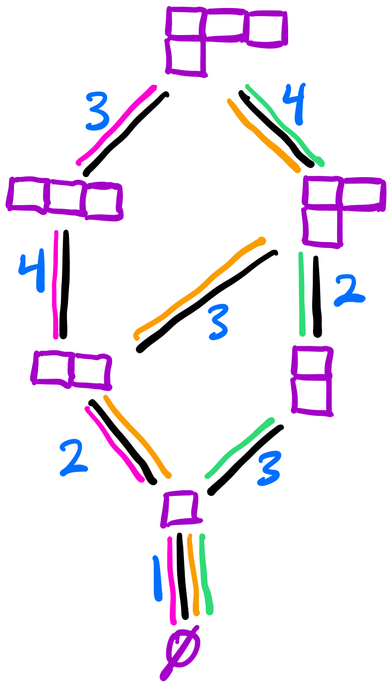
$$123465 \to 123645$$

<u>Thm:</u> (L.)

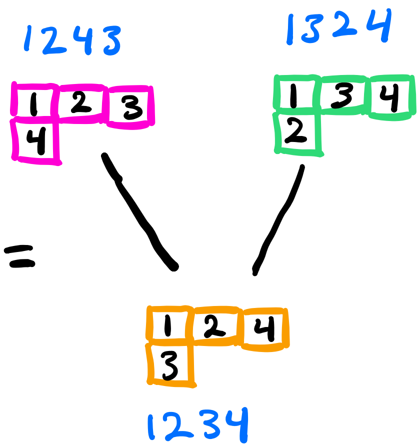
Let T be a standard tableau of shape α/μ. The max'l chain descent order $[\mu,\alpha]_{\lambda_T}^{(2)}$ is isomorphic to $(ST(\alpha/\mu), \leq_T)$.

Moreover, both are isomorphic to some order ideal of weak order on $S_{\#\text{boxes of } T}$.

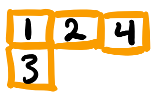
$Y(\alpha) =$



$Y(\alpha)_\lambda(2) =$



λ EL-labeling induced by SYT



Thank you!