Proofs of Borwein Conjectures

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Prologue

The “birth” of the Borwein Conjecture

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September 1993: Workshop on *“Symbolic Computation in Combinatorics”*, Cornell University, USA (organised by Peter Paule and Volker Strehl)

George Andrews gave a two-part lecture on *“AXIOM and the Borwein Conjecture”*.

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Consider the product

$$(1-q)(1-q^2)(1-q^4)(1-q^5)\cdots(1-q^{3n-2})(1-q^{3n-1}).$$

Then the sign pattern of the coefficients in the expansion of this polynomial is $+--+--+--\cdots$.

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Example. $n=3$:

$$\begin{aligned}(1-q)(1-q^2)(1-q^4)(1-q^5)(1-q^7)(1-q^8)\\ = 1-q-q^2+q^3-q^4+2q^6-q^7-q^8\\ +3q^9-q^{10}-q^{11}+2q^{12}-2q^{13}-2q^{14}+2q^{15}-q^{16}-q^{17}\\ +3q^{18}-q^{19}-q^{20}+2q^{21}-q^{23}+q^{24}-q^{25}-q^{26}\\ +q^{27}\end{aligned}$$

The "birth" of the Borwein Conjecture

More formally:

Let

$$(a;q)_m := \prod_{i=0}^{m-1}(1-aq^i).$$

Conjecture (PETER BORWEIN)

Let the polynomials $A_n(q)$, $B_n(q)$ and $C_n(q)$ be defined by the relationship

$$\frac{(q;q)_{3n}}{(q^3;q^3)_n} = A_n(q^3) - qB_n(q^3) - q^2C_n(q^3).$$

Then these polynomials have non-negative coefficients.

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There is a nice trick which allows one to use the q-binomial theorem in order to find elegant formulae for $A_n(q)$, $B_n(q)$, $C_n(q)$:

$$\begin{aligned}
&(1-q)(1-q^2)(1-q^4)(1-q^5)\cdots(1-q^{3n-2})(1-q^{3n-1})\\
&\quad=(1-q)(1-q^4)\cdots(1-q^{3n-2})\cdot(1-q^2)(1-q^5)\cdots(1-q^{3n-1})\\
&\quad=(-1)^n q^{(3n+1)n/2}(1-q^{-3n+1})\cdots(1-q^{-5})(1-q^{-2})\\
&\qquad\qquad\qquad\cdot(1-q)(1-q^4)\cdots(1-q^{3n-2})\\
&\quad=(-1)^n q^{(3n+1)n/2}\,(q^{-3n+1};q^3)_{2n}.
\end{aligned}$$

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$$(1-q)(1-q^2)(1-q^4)(1-q^5)\cdots(1-q^{3n-2})(1-q^{3n-1}) = (-1)^n q^{(3n+1)n/2}\,(q^{-3n+1};q^3)_{2n}.$$

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Here we need the q-binomial theorem:

$$(z;q)_N = (1-z)(1-qz)\cdots(1-q^{N-1}z) \\ = \sum_{k=0} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} N \\ k \end{bmatrix}_q z^k,$$

where the q-binomial coefficient is defined by

$$\begin{bmatrix} N \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}}.$$

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Thus, we obtain

$$(1-q)(1-q^2)(1-q^4)(1-q^5)\cdots(1-q^{3n-2})(1-q^{3n-1})\\ = \sum_{j=-n}^{n} (-1)^j q^{(3j+1)j/2} \begin{bmatrix} 2n \\ n+j \end{bmatrix}_{q^3}.$$

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Since the q-binomial coefficient is on base q^3, it is easy to separate the terms with exponent $\equiv s$ modulo 3, $s = 0, 1, 2$:

$$A_n(q) = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(9j+1)/2} \begin{bmatrix} 2n \\ n+3j \end{bmatrix}_q,$$

$$B_n(q) = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(9j-5)/2} \begin{bmatrix} 2n \\ n+3j-1 \end{bmatrix}_q,$$

$$C_n(q) = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(9j+7)/2} \begin{bmatrix} 2n \\ n+3j+1 \end{bmatrix}_q.$$

What did we know?

Compare with:

Theorem (Andrews, Baxter, Bressoud, Burge, Forrester, Viennot)

Let K be a positive integer, and m, n, α, β be non-negative integers, satisfying $\alpha + \beta < 2K$ and $\beta - K \le n - m \le K - \alpha$. Then the polynomial

$$\sum_{j \in \mathbb{Z}} (-1)^j q^{jK \frac{j(\alpha+\beta)+\alpha-\beta}{2}} \begin{bmatrix} m+n \\ n-Kj \end{bmatrix}_q$$

is the generating function for partitions inside an $m \times n$ rectangle that satisfy some so-called "hook difference conditions" specified by α, β and K.

What did we know?

In order to apply this theorem to the Borwein Conjecture, we have to choose $m = n$, $\alpha = 5/3$, $\beta = 4/3$ and $K = 3$.

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In order to apply this theorem to the Borwein Conjecture, we have to choose $m = n$, $\alpha = 5/3$, $\beta = 4/3$ and $K = 3$.

Alas, α and β are not integers!

Many people have tried to adapt the (combinatorial) arguments of Andrews et al. in order to cope with this situation, to no avail.

What did we know?

David Bressoud extended the mystery by making the following much more general conjecture.

Conjecture (David Bressoud)

Let m and n be positive integers, α and β be positive rational numbers, and K be a positive integer such that αK and βK are integers. If $1 \le \alpha + \beta \le 2K + 1$ (with strict inequalities if $K = 2$) and $\beta - K \le n - m \le K - \alpha$, then the polynomial

$$\sum_{j=-\infty}^{\infty} (-1)^j q^{j(K(\alpha+\beta)j + K(\alpha-\beta))/2} \begin{bmatrix} m+n \\ m-Kj \end{bmatrix}$$

has non-negative coefficients.

What did we know?

Moderate progress on this generalised conjecture has been made. Alexander Berkovich and Ole Warnaar proved Bressoud's conjecture for several infinite families in several papers in the period 2000–2020.
However, literally no progress at all has been made on the original Borwein Conjecture, for lack of an idea how to approach it.

What did we know?

A partial result is:

Proposition (Andrews)

The power series $A_\infty(q)$, $B_\infty(q)$, $C_\infty(q)$ have non-negative coefficients. More precisely, we have

$$A_\infty(q) = \frac{(q^4, q^5, q^9; q^9)_\infty}{(q;q)_\infty},$$

$$B_\infty(q) = \frac{(q^2, q^7, q^9; q^9)_\infty}{(q;q)_\infty},$$

$$C_\infty(q) = \frac{(q^1, q^8, q^9; q^9)_\infty}{(q;q)_\infty},$$

where we use the short notation

$$(a_1, a_2, \ldots, a_k; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \cdots (a_k;q)_\infty.$$

What did we know?

The proof uses Jacobi's triple product identity

$$\sum_{k=-\infty}^{\infty} (-1)^k q^{\binom{k}{2}} z^k = (q;q)_\infty \, (z;q)_\infty \, (q/z;q)_\infty ,$$

a special case of which is Euler's pentagonal number theorem

$$(q;q)_\infty = \sum_{k=-\infty}^{\infty} (-1)^k q^{k(3k-1)/2}.$$

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$$(q;q)_\infty = \sum_{k=-\infty}^{\infty} (-1)^k q^{k(3k-1)/2}.$$

Namely, we have

$$\frac{(q;q)_\infty}{(q^3;q^3)_\infty} = \frac{\sum_{k=-\infty}^{\infty} (-1)^k q^{k(3k-1)/2}}{(q^3;q^3)_\infty}.$$

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where we use the short notation

$$(a_1, a_2, \dots, a_k; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \cdots (a_k; q)_\infty.$$

What did we know?

Even more generally:

Theorem (Andrews, P. Borwein and Garvan)

For any prime number p, if

$$\frac{(q;q)_\infty}{(q^p;q^p)_\infty} = \sum_{j=0}^{\infty} c_p(j) q^j,$$

then $c_p(j)$ and $c_p(j+p)$ have the same sign for all j.

Preliminaries

November 2017:

November 2017: Chen Wang tells me that he wants to prove the Borwein Conjecture.

His starting point is another set of formulae of Andrews:

Theorem (Andrews)

Let, as before,

$$\frac{(q;q)_{3n}}{(q^3;q^3)_n} = A_n(q^3) - qB_n(q^3) - q^2C_n(q^3).$$

Then

$$A_n(q) = \sum_{j=0}^{n/3} \frac{q^{3j^2}(1-q^{2n})(q^3;q^3)_{n-j-1}(q;q)_{3j}}{(q;q)_{n-3j}(q^3;q^3)_{2j}(q^3;q^3)_j},$$

$$B_n(q) = \sum_{j=0}^{(n-1)/3} \frac{q^{3j^2+3j}(1-q^{3j+2}+q^{n+1}-q^{n+3j+2})(q^3;q^3)_{n-j-1}(q;q)_{3j}}{(q;q)_{n-3j-1}(q^3;q^3)_{2j+1}(q^3;q^3)_j},$$

$$C_n(q) = \sum_{j=0}^{(n-1)/3} \frac{q^{3j^2+3j}(1-q^{3j+1}+q^{n}-q^{n+3j+2})(q^3;q^3)_{n-j-1}(q;q)_{3j}}{(q;q)_{n-3j-1}(q^3;q^3)_{2j+1}(q^3;q^3)_j}.$$

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Wang had experimentally observed that, in this sum, the term for $j=0$ gives the main contribution to the coefficients in the polynomial, while the other terms contribute much less.

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His idea hence was to estimate the contributions of the terms and show — at least for large n — that indeed the first term dominated the other terms.

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One and half years later, by using saddle point approximations for large n and a computer check for small n, he succeeded to fully prove the Borwein Conjecture.

Theorem (Chen Wang)

Let the polynomials $A_n(q)$, $B_n(q)$ and $C_n(q)$ be defined by the relationship

$$\frac{(q;q)_{3n}}{(q^3;q^3)_n} = A_n(q^3) - qB_n(q^3) - q^2C_n(q^3).$$

Then these polynomials have non-negative coefficients.

C. Wang, *An analytic proof of the Borwein Conjecture*, Adv. Math. **394** (2022), Paper No. 108028, 54 pp.

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Conjecture (Second Borwein Conjecture)

Let the polynomials $\alpha_n(q)$, $\beta_n(q)$ and $\gamma_n(q)$ be defined by the relationship

$$\frac{(q;q)_{3n}^2}{(q^3;q^3)_n^2} = \alpha_n(q^3) - q\beta_n(q^3) - q^2\gamma_n(q^3).$$

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Conjecture (Third Borwein Conjecture)

Let the polynomials $\nu_n(q)$, $\phi_n(q)$, $\chi_n(q)$, $\psi_n(q)$ and $\omega_n(q)$ be defined by the relationship

$$\frac{(q;q)_{5n}}{(q^5;q^5)_n} = \nu_n(q^5) - q\phi_n(q^5) - q^2\chi_n(q^5) - q^3\psi_n(q^5) - q^4\omega_n(q^5),$$

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Conjecture (Chen Wang: the "Cubic Borwein Conjecture")

Let the polynomials $\widetilde{\alpha}_n(q)$, $\widetilde{\beta}_n(q)$ and $\widetilde{\gamma}_n(q)$ be defined by the relationship

$$\frac{(q;q)_{3n}^3}{(q^3;q^3)_n^3} = \widetilde{\alpha}_n(q^3) - q\widetilde{\beta}_n(q^3) - q^2\widetilde{\gamma}_n(q^3).$$

Then these polynomials have non-negative coefficients.

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Is Wang's proof just an isolated instance, or can similar ideas also lead to proofs of the other conjectures?

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PROBLEM: There are no reasonable explicit formulae for the polynomials $\alpha_n(q)$, $\beta_n(q)$, etc. in these conjectures. In particular, there is no analogue of Andrews'

$$B_n(q) = \sum_{j=0}^{(n-1)/3} \frac{q^{3j^2+3j}(1-q^{3j+2}+q^{n+1}-q^{n+3j+2})(q^3;q^3)_{n-j-1}(q;q)_{3j}}{(q;q)_{n-3j-1}(q^3;q^3)_{2j+1}(q^3;q^3)_j},$$

and it is unlikely that a formula of this kind exists for $\alpha_n(q)$, $\beta_n(q)$, etc.
Thus, it seems that we cannot even get started.

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Preliminary calculations suggested that the quantities that would have to be estimated here are very similar to those that were at stake in Wang's proof.

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OBSTACLES:
(1) Now the (dominant) saddle points will not be real but genuinely complex numbers.
(2) We have to work with approximate saddle points.

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OBSTACLES:
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POTENTIAL BENEFITS:
Will lead to uniform proofs of sign pattern assertions of this kind.

Summary of Results

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C. Wang, C.K., *An asymptotic approach to Borwein-type sign pattern theorems*, `ar`χ`iv:2201.12415`.

Contains a uniform proof of:

- the First Borwein Conjecture,
- the Second Borwein Conjecture,
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- the First Borwein Conjecture,
- the Second Borwein Conjecture,
- "two thirds" of Wang's Cubic Borwein Conjecture.

Further work will lead to a proof of (at least) "three fifth" of the Third Borwein Conjecture.

Outline of Approach

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1. show that the conjectures hold for the "first few" and the "last few" coefficients;
2. represent the coefficients by a contour integral;
3. divide the contour into two parts, the "peak part" (the part close to the dominant saddle points of the integrand) and the remaining part, the "tail part";
4. for "large" n, bound the error made by approximating the "peak part" by a Gaußian integral (the "peak error");
5. for "large" n, bound the error contributed by the "tail part" (the "tail error");
6. verify the conjectures for "small" n;
7. put everything together to complete the proofs.

The "Borwein polynomial"

Let

$$P_n(q) := \frac{(q;q)_{3n}}{(q^3;q^3)_n}$$
$$= (1-q)(1-q^2)(1-q^4)(1-q^5)\cdots(1-q^{3n-2})(1-q^{3n-1}).$$

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- The First Borwein Conjecture is about $P_n(q)$.
- The Second Borwein Conjecture is about $P_n^2(q)$.
- Wang's Cubic Borwein Conjecture is about $P_n^3(q)$.

Step 1: the "first few" coefficients

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For the FIRST BORWEIN CONJECTURE:

- Recall that it is a theorem that the infinite product

$$P_\infty(q) = \frac{(q;q)_\infty}{(q^3;q^3)_\infty}$$

has the sign pattern $+--+--\cdots$.

- What is the difference between this and $P_n(q)$?

$$\frac{(q;q)_{3n}}{(q^3;q^3)_n} = \frac{(q;q)_\infty}{(q^3;q^3)_\infty} \cdot \frac{(q^{3n+3};q^3)_\infty}{(q^{3n+1};q)_\infty} = \frac{(q;q)_\infty}{(q^3;q^3)_\infty} \cdot \left(1+O(q^{3n+1})\right).$$

Consequently, the first $3n+1$ coefficients (and hence also the last $3n+1$ coefficients) of $P_n(q)$ and $P_\infty(q)$ agree!

Similarly for the SECOND BORWEIN CONJECTURE:

Step 1: the "first few" coefficients

Similarly for the Second Borwein Conjecture:

- A result of Kane (2004) shows the sign pattern $+ - - + - - \cdots$ for the coefficients of
$$\frac{(q;q)_\infty^2}{(q^3;q^3)_\infty}$$
(with one exception).
- Multiplication by $1/(q^3;q^3)_\infty$ then implies the sign pattern $+ - - + - - \cdots$ for the coefficients of
$$P_\infty^2(q) = \frac{(q;q)_\infty^2}{(q^3;q^3)_\infty^2}.$$
- By the earlier difference argument, the first $3n$ coefficients (and hence also the $3n$ last coefficients) of $P_n^2(q)$ and $P_\infty^2(q)$ agree!

Step 1: the "first few" coefficients

Similarly for the CUBIC BORWEIN CONJECTURE:

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- Borwein, Borwein and Garvan (1994) showed that

$$\frac{(q;q)_\infty^3}{(q^3;q^3)_\infty} = \sum_{m,n\in\mathbb{Z}} q^{3(m^2+mn+n^2)} - q \sum_{m,n\in\mathbb{Z}} q^{3(m^2+mn+n^2+m+n)}.$$

- Multiplication by $1/(q^3;q^3)_\infty^2$ then implies the sign pattern $+--+--\cdots$ for the coefficients of

$$P_\infty^3(q) = \frac{(q;q)_\infty^3}{(q^3;q^3)_\infty^3}.$$

- By the earlier difference argument, the first $3n$ coefficients (and hence also the $3n$ last coefficients) of $P_n^3(q)$ and $P_\infty^3(q)$ agree!

Step 1: the "first few" coefficients

Summary: With $\delta = 1, 2, 3$, it "suffices" to prove that

$$\langle q^m \rangle \, P_n^{\delta}(q)$$

has the sign pattern $+ - - + - - \cdots$ for $3n \le m \le \deg P_n^{\delta}(q) - 3n$.

Step 2: the contour integral representation

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By Cauchy's formula, we have

$$\langle q^m \rangle P_n^{\delta}(q) = \frac{1}{2\pi i} \int_{\Gamma} P_n^{\delta}(q) \frac{dq}{q^{m+1}},$$

where $\delta = 1, 2, 3$.

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where $\delta = 1, 2, 3$.
We choose as contour Γ a circle of radius r, where r has to be chosen appropriately. After substitution $q = re^{i\theta}$, we obtain

$$\langle q^m \rangle \, P_n^\delta(q) = \frac{r^{-m}}{2\pi} \int_{-\pi}^{\pi} P_n^\delta(re^{i\theta}) e^{-mi\theta} \, d\theta.$$

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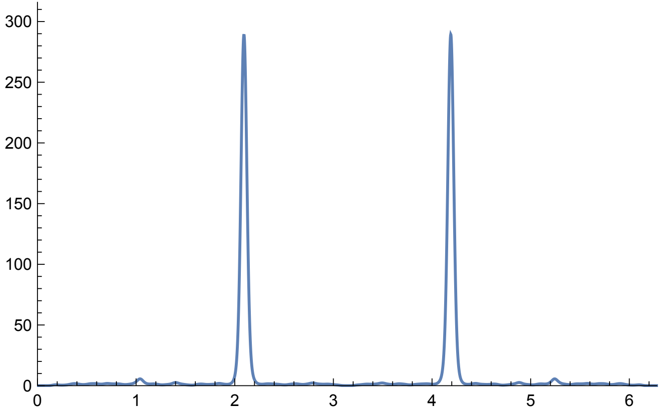

$|P_{10}(q)|$ at $q = .95e^{i\theta}$ at logarithmic scale

Step 3: "peak part" and "tail part"

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We need to cut the integration domain into two pieces: to this end, we choose an (appropriate) cut-off θ_0.

- The peak part is $I_{\text{peak}} := [-2\pi/3 - \theta_0, -2\pi/3 + \theta_0] \cup [2\pi/3 - \theta_0, 2\pi/3 + \theta_0]$.
- The tail part is $I_{\text{tail}} := [-\pi, \pi] \setminus I_{\text{peak}}$.

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The cut-off is chosen as

$$\theta_0 := \frac{10}{81} \cdot \frac{1 - r^3}{1 - r^{3n}},$$

where r is chosen so as to minimise $r^{-m} \left| P_n^\delta(re^{2\pi i/3}) \right|$; it is the unique solution to the approximate saddle point equation

$$r \,\mathrm{Re} \left(\frac{d}{dr} \log P_n(re^{2\pi i/3}) \right) = \frac{m}{\delta}.$$

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Lemma

For all integers $n \geq 1$ and $m \in (0, \delta \deg P_n)$, with $\delta \in \{1, 2, 3\}$, the approximate saddle point equation

$$r \operatorname{Re}\left(\frac{d}{dr} \log P_n(re^{2\pi i/3})\right) = \frac{m}{\delta}.$$

has a unique solution $r = r_{m,n} \in \mathbb{R}^+$. Moreover, if $3n \leq m \leq (\delta \deg P_n)/2$, then we have $r_0 < r \leq 1$, where

$$r_0 = e^{-\sqrt{4\delta/27n}}.$$

Furthermore, as a function in m, the solution $r = r_{m,n}$ to the approximate saddle point equation is increasing.

Steps 4 and 5: bounding the approximation errors

$$\langle q^m \rangle \, P_n^\delta(q) = \frac{r^{-m}}{2\pi} \int_{-\pi}^{\pi} P_n^\delta(re^{i\theta}) e^{-mi\theta} \, d\theta,$$

Steps 4 and 5: bounding the approximation errors

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- The *peak part* is estimated by a Gaußian integral. A relative error of $\varepsilon_{0,P_n^\delta}(m,r)$ occurs.
- The *tail part* is bounded above by a fraction of this Gaußian integral. A relative error of $\varepsilon_{1,P_n^\delta}(r)$ occurs.

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$$\langle q^m \rangle \, P_n^\delta(q) = \frac{r^{-m}}{2\pi} \int_{-\pi}^{\pi} P_n^\delta(re^{i\theta}) e^{-mi\theta} \, d\theta,$$

The fundamental inequality that results from these considerations is:

$$\left| \frac{r^m \sqrt{2\pi g_{Q_n}(r)}}{\operatorname{erf}\left(\theta_0 \sqrt{g_{Q_n}(r)/2}\right)} \frac{1}{|Q_n(re^{2\pi i/3})|} [q^m] Q_n(q) \right.$$
$$\left. - 2\cos\left(\arg Q_n(re^{2\pi i/3}) - 2m\pi/3\right) \right|$$
$$\leq \epsilon_{0,Q_n}(m,r) + \epsilon_{1,Q_n}(r),$$

where $Q_n(q) = P_n^\delta(q)$ and

$$g_{Q_n}(r) = -\operatorname{Re} \frac{\partial^2}{\partial\theta^2} \log Q_n(re^{i\theta}) \Big|_{\theta=2\pi/3}.$$

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Hence: there are two tasks:

1. Bound the argument $\arg P_n(re^{2\pi i/3})$.
2. Make sure that $\epsilon_{0,Q_n}(m,r) + \epsilon_{1,Q_n}(r)$ is smaller than $2\cos\left(\arg Q_n(re^{2\pi i/3}) - 2m\pi/3\right)$, where $Q_n(q) = P_n^\delta(q)$.

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Lemma

For $n \in \mathbb{Z}^+$, $\arg P_n(re^{2\pi i/3})$ is increasing with respect to r. Moreover, for $r \in (0,1]$ and $n \in \mathbb{Z}^+$, we have $\arg P_n(re^{2\pi i/3}) \in (-\pi/18, 0]$.

Steps 4 and 5: bounding the approximation errors

Together with precise bounds on the peak and tail errors $\epsilon_{0,Q_n}(m,r)$ and $\epsilon_{1,Q_n}(r)$, this leads to proofs of the sign pattern $+--+--\cdots$ for the coefficients for the following cases:

- $P_n(q)$ for $n \geq 5300$;
- $P_n^2(q)$ for $n \geq 7000$;
- $\langle q^m \rangle P_n^3(q)$ for $n \geq 3150$ and $m \equiv 0, 1 \pmod 3$.

Step 6: computer verification for “small” n

By a straightforward computer programme, one can verify the sign pattern $+--+--\cdots$ for the coefficients for the following cases:

- $P_n(q)$ for $n < 5300$;
- $P_n^2(q)$ for $n < 7000$;
- $\langle q^m \rangle\, P_n^3(q)$ for $n < 3150$ and $m \equiv 0, 1 \pmod 3$.

This proves the First Borwein Conjecture, the Second Borwein Conjecture, and “two thirds” of the Cubic Borwein Conjecture.

Some of the (nasty) details

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Lemma

Suppose that $x_0 > 0$ *and* $f \in C^4([-x_0, x_0]; \mathbb{C})$ *with* $f(0) = 0$. *We define* $f_k := f^{(k)}(0)$ *for* $k = 1, 2$ *as well as*

$$f_3 := 3\int_0^1 (1-t)^2 \sup_{|x|\le tx_0} |f^{(3)}(x)|\,dt$$

$$f_4 := 4\int_0^1 (1-t)^3 \sup_{|x|\le tx_0} \left|f^{(4)}(x)\right|\,dt,$$

and we write $g = -\operatorname{Re} f_2$ *for simplicity. Suppose further that* $f_1 \in \mathbb{R}$, $g > 0$, *that* $\mu_3 := \frac{x_0 f_3}{3g} \in (0,1)$, *and that* $\mu_4 := \frac{x_0\sqrt{f_4}}{\sqrt{8g}} \in (0,1)$. *Then we have*

$$\left|\sqrt{\frac{g}{2\pi}}\int_{-x_0}^{x_0}\left(e^{f(x)} - e^{-gx^2/2}\right)dx\right| \le \operatorname{erf}\left(x_0\sqrt{\frac{g}{2}}\right)\cosh(f_1x_0)$$

$$\times\left(\frac{|\Im f_2| + f_1^2}{2g} + \frac{4f_3\beta_1(\mu_3)}{9\sqrt{\pi}g^3} + \frac{f_4\beta_3(\mu_4)}{3\sqrt{\pi}g^2} + \frac{4f_1f_3\beta_2(\mu_3)}{3\sqrt{\pi}g^2} + \frac{\sqrt{2}f_1f_4\beta_4(\mu_4)}{3\sqrt{\pi}g^{5/2}}\right),$$

where

$$\beta_1(\mu) := \sup_{w>0} \frac{w^{3/2}}{\operatorname{erf}(\mu\sqrt{w})} \int_0^\mu e^{-wy^2} \left(\cosh(wy^3) - 1\right) \, dy,$$

$$\beta_2(\mu) := \sup_{w>0} \frac{w^{3/2}}{\operatorname{erf}(\mu\sqrt{w})} \int_0^\mu y e^{-wy^2} \sinh(wy^3) \, dy,$$

$$\beta_3(\mu) := \sup_{w>0} \frac{w^{3/2}}{\operatorname{erf}(\mu\sqrt{w})} \int_0^\mu e^{-wy^2} \sinh(wy^4) \, dy,$$

$$\beta_4(\mu) := \sup_{w>0} \frac{w^{2}}{\operatorname{erf}(\mu\sqrt{w})} \int_0^\mu y e^{-wy^2} \sinh(wy^4) \, dy.$$

Some of the (nasty) details

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Lemma

Let $\gamma(s,a) := \int_0^a e^{-x}x^{s-1}dx$ *be the lower incomplete gamma function. Suppose that* $c, d, \mu \in \mathbb{R}^+$ *with* $d > c$. *Then we have*

$$\sup_{w\in\mathbb{R}^+} w^{-c}\gamma(d,\mu w) \leq \frac{\mu^c\Gamma(d-c+1)}{c\sqrt{2\pi(d-c)}}.$$

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Lemma

For $r \in (0,1)$, $n \in \mathbb{Z}^+$, *and* $\theta \in [-\pi,\pi]$, *we have*

$$\sum_{k=1}^{n} r^{k-1}\cos k\theta \le \frac{1-r^n}{1-r}\sqrt{\frac{1}{1+4\kappa\tan^2(\theta/2)}},$$

where

$$\kappa = \frac{(1+r)(1-r^n)(1-r^{n/6})}{(1-r)^2}.$$

Epilogue

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What is the problem with the Cubic Borwein Conjecture?

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Recall:
One of our tasks was: make sure that $\epsilon_{0,Q_n}(m,r) + \epsilon_{1,Q_n}(r)$ is smaller than $2\cos\left(\arg P_n^3(re^{2\pi i/3}) - 2m\pi/3\right)$.
To help us, we have:

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For $r \in (0,1]$ and $n \in \mathbb{Z}^+$, we have $\arg P_n(re^{2\pi i/3}) \in (-\pi/18, 0]$.

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To help us, we have:

Lemma

For $r\in(0,1]$ and $n\in\mathbb{Z}^+$, we have $\arg P_n(re^{2\pi i/3})\in(-\pi/18,0]$.

The same problem will be encountered when dealing with the Third Borwein Conjecture.

What else?

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Computer experiments led us to new conjectures.

Conjecture (A modulus 4 "Borwein Conjecture")

Let n be a positive integer and $\delta \in \{1,2,3\}$. Furthermore, consider the expansion of the polynomial

$$\frac{(q;q)_{4n}^{\delta}}{(q^4;q^4)_n^{\delta}} = \sum_{m=0}^{D} c_m^{(\delta)}(n) q^m,$$

which has degree $D = 6\delta n^2$. Then

$$c_{4m}^{(\delta)}(n) \ge 0 \quad \text{and} \quad c_{4m+2}^{(\delta)}(n) \le 0, \qquad \text{for all } m \text{ and } n,$$

while

$$c^{(\delta)}_{4m+1}(n) \le 0, \quad \textit{for} \begin{cases} 0 \le m \le \frac{1}{8}(6\delta n^2 - 8), & \textit{if } n \textit{ is even,} \\ 0 \le m \le \frac{1}{8}(6\delta n^2 - 8 + 2\delta), & \textit{if } n \textit{ is odd,} \end{cases}$$

and

$$c^{(\delta)}_{4m+3}(n) \ge 0, \ \textit{for} \begin{cases} 0 \le m \le \frac{1}{8}(6\delta n^2 - 8), & \textit{if } n \textit{ is even,} \\ 0 \le m \le \frac{1}{8}(6\delta n^2 - 6\delta + 8\chi(\delta = 3)), & \textit{if } n \textit{ is odd,} \end{cases}$$

with the exception of two coefficients: for $\delta = 1$ *and* $n = 5$*, we have* $c^{(1)}_{71}(5) = -1$ *and* $c^{(1)}_{79}(5) = 1$.

Epilogue

Conjecture (A modulus 7 "Borwein Conjecture")

For positive integers n, consider the expansion of the polynomial

$$\frac{(q;q)_{7n}}{(q^7;q^7)_n} = \sum_{m=0}^{21n^2} d_m(n) q^m.$$

Then

$$d_{7m}(n) \ge 0 \quad \text{and} \quad d_{7m+1}(n), d_{7m+3}(n), d_{7m+4}(n), d_{7m+6}(n) \le 0, \qquad \text{for all } m \text{ and } n,$$

while

$$d_{7m+5}(n) \begin{cases} \ge 0, & \text{for } m \le 3\alpha(n)n^2, \\ \le 0, & \text{for } m > 3\alpha(n)n^2, \end{cases}$$

where $\alpha(n)$ seems to stabilise around 0.302.

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One must simply admit that until now “combinatorial” attacks have not led to any progress on the Borwein Conjectures. By contrast, the first proof of the First Borwein Conjecture by Wang has been accomplished using analytic methods, as well as the proofs that I have shown here.

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Gaurav Bhatnagar and Michael Schlosser made several conjectures of "Borwein type" which are also "asymptotic" conjectures.

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I guess the last word in this matter has not yet been spoken . . .