Boolean growth polytopes and their triangulations

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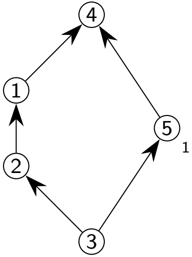
Combinatorial polytopes

- permutahedron, Birkhoff polytope, associahedron, ASM polytope...
- order polytope, flow polytope, matching polytope...
- Cayley polytope: $1 \leq x_1 \leq 2$, $1 \leq x_i \leq 2x_{i-1}$ for $i = 2, \ldots, n$, normalized volume is the number of connected graphs on $n + 1$ vertices [K-Pak 2013]

Boolean growth polytope

$\vec{G}$ directed acyclic graph, $S \subseteq V$, $\alpha \colon S \to \mathbb{Z}$



$\vec{G}, S, \alpha$

$$1 \le x_5 \le 2$$
$$x_3 \le x_2 \le x_3 + 1$$
$$x_2 \le x_1 \le x_2 + 1$$
$$x_1 \le x_4 \le x_1 + 1$$
$$x_3 \le x_5 \le x_3 + 1$$
$$x_5 \le x_4 \le x_5 + 1$$

Outline

- Alternating sign matrices and the ASM polytope
- Order polytope
- Boolean growth polytope
- Volume of the Boolean growth polytope
- Applications to classes of alternating sign matrices

Alternating sign matrices and the ASM polytope

Alternating sign matrices

An *alternating sign matrix (ASM)* is a square matrix with entries in $\{0, 1, -1\}$ such that in each row and each column the non-zero entries alternate and sum up to 1.

Every permutation matrix is an ASM, and there are many other examples.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Robbins numbers

The counting sequence is $1, 1, 2, 7, 42, 429, 7436, 218348, \ldots$

Theorem

The number of $n \times n$ alternating sign matrices is

$$|\,\mathrm{ASM}_n\,| = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}.$$

ASMs were introduced by Robbins and Rumsey in the 1980s when studying the *λ-determinant*, and they formed the conjecture with Mills.

The conjecture was proved by Zeilberger in the early 1990's. Other significant proofs are due to Kuperberg and Fischer.

Corner sum matrices

A *corner sum matrix (CSM)* is a matrix $B = [b_{ij}]_{i,j=0}^{n}$ satisfying the following:

- $b_{i0} = b_{0i} = 0$ for $i = 0, \ldots, n$;
- $b_{in} = b_{ni} = i$ for $i = 0, \ldots, n$;
- $b_{i+1,j} - b_{ij} \in \{0, 1\}$ for $i = 0, \ldots, n-1$, $j = 0, \ldots, n$;
- $b_{i,j+1} - b_{ij} \in \{0, 1\}$ for $i = 0, \ldots, n$, $j = 0, \ldots, n-1$.

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 1 & 2 & 3 & 3 & 4 & 5 \\ 0 & 1 & 1 & 2 & 3 & 3 & 4 \\ 0 & 0 & 1 & 2 & 2 & 3 & 3 \\ 0 & 0 & 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Bijection between CSMs and with ASMs

For an ASM A, $b_{ij} = \sum_{i' \leq i, j' \leq j} a_{i'j'}$ gives a CSM B.

For a CSM B, $a_{ij} = b_{ij} - b_{i,j-1} - b_{i-1,j} + b_{i-1,j-1}$ gives an ASM A.

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \longleftrightarrow \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 1 & 2 & 3 & 3 & 4 & 5 \\ 0 & 1 & 1 & 2 & 3 & 3 & 4 \\ 0 & 0 & 1 & 2 & 2 & 3 & 3 \\ 0 & 0 & 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Symmetries of ASMs

All symmetries of a square can act on an ASM.

That gives us:

- vertically symmetric ASMs (VSASM);
- vertically and horizontally symmetric ASMs (VHSASM);
- diagonally symmetric ASMs (DSASM);
- diagonally and antidiagonally symmetric ASMs (DADSASM);
- half-turn symmetric ASMs (HTSASM);
- quarter-turn symmetric ASMs (QTSASM);
- totally symmetric ASMs (TSASM).

ASM polytope

Every $n \times n$ ASM represents a point in the n^2-dimensional space.

Define the ASM polytope $\mathcal{A}_n$ as the convex hull of all $n \times n$ ASMs.

It was defined by Behrend–Knight (2007) and Striker (2009). It has dimension $(n-1)^2$, its vertices and integer points are precisely the ASMs, and we can describe the entire face structure.

ASM polytope

Define the *normalized volume* $\operatorname{vol} A$ of a set $A \subseteq \mathbb{R}^n$ as $n!$ times the usual volume of A.

If A is the simplex with vertices $v_0, \ldots, v_n$, then its (normalized) volume is

$$\operatorname{vol} A = |\det(v_1 - v_0, \ldots, v_n - v_0)|.$$

The normalized volume (in the $(n-1)^2$-dimensional space) of $\mathcal{A}_n$ for $n = 1, \ldots, 4$ is $1, 1, 4, 1376$.

Volume of the ASM polytope

Theorem [Izanloo (2019) and Behrend–Izanloo (2026+)]

We have

$$\operatorname{vol} \mathcal{A}_n = \sum_{A \in \mathrm{ASM}_{n-2}} |L(\vec{G}_A)|.$$

Here $\vec{G}_A$ is a certain directed graph corresponding to $A \in \mathrm{ASM}_{n-2}$, and $L(\vec{G})$ is the set of *linear extensions* (topological orderings) of a directed acyclic graph $\vec{G} = (V, \vec{E})$, i.e. the ways to order the elements of $\vec{E}$ linearly so that the edge orientation is respected.

Using this formula, they found $\operatorname{vol} \mathcal{A}_n$ for $n = 1, \ldots, 7$:

$$1, 1, 4, 1376, 201675688, 37350087969236232,$$

$$20423967603561169141089171040$$

Comparison with the Birkhoff polytope

The *Birkhoff polytope* $\mathcal{B}_n$ is the convex hull of all $n \times n$ permutation matrices.

Theorem [De Loera–Liu–Yoshida, 2009]

The normalized volume of $\mathcal{B}_n$ equals

$$\sum_{(\sigma_1,\ldots,\sigma_{n-1}) \in (S_n)^{n-1}} \det\big(M(\sigma_1,\ldots,\sigma_{n-1})\big)^2.$$

The volume has been computed up to $n = 10$, where the result is 5091038988117504946842559205930853037841762820367901333706255223000.

Order polytope

Order polytope

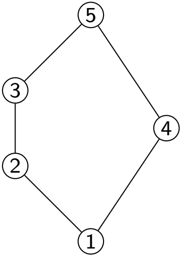
Given a finite poset P, define the *order polytope* $\mathcal{O}(P)$ as the set of points $x = (x_p)_{p \in P}$ satisfying:

- $0 \leq x_p \leq 1$ for $p \in P$;
- if $p \leq q$, then $x_p \leq x_q$.

It is enough to assume $x_p \geq 0$ for all minimal elements of P, $x_p \leq 1$ for all maximal elements of P, and $x_p \leq x_q$ if $p \lessdot q$.

The order polytope was defined by Stanley in 1986.

Example

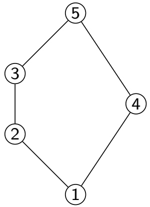


The order polytope $\mathcal{O}(P)$ is defined by inequalities

$$x_1 \geq 0, \quad x_5 \leq 1, \quad x_1 \leq x_2, \quad x_2 \leq x_3,$$

$$x_3 \leq x_5, \quad x_1 \leq x_4, \quad x_4 \leq x_5.$$

The vertices of the order polytope



The vertices are precisely the integer points, so for our example the vertices are $(0,0,0,0,0)$, $(0,0,0,0,1)$, $(0,0,1,0,1)$, $(0,1,1,0,1)$, $(0,0,0,1,1)$, $(0,0,1,1,1)$, $(0,1,1,1,1)$, $(1,1,1,1,1)$.

The integer points are in a bijection with antichains of P.

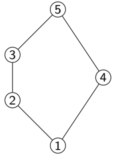
The volume of the order polytope $\mathcal{O}(P)$ is equal to the number of linear extensions of $\vec{P}$, the Hasse diagram of P.

For our example, the volume is 3.

Proof via a triangulation

Assign a simplex of volume 1 to each $\pi \in L(\vec{P})$: the first vertex is $(0, \ldots, 0)$, and then add 1's in the order determined by π.

For example, for the linear extension $\pi = 12435$ of the poset



we get vertices $(0,0,0,0,0), (0,0,0,0,1), (0,0,1,0,1)$, $(0,0,1,1,1), (0,1,1,1,1), (1,1,1,1,1)$.

Given a generic point $x \in \mathbb{R}^n$, the order of the coordinates gives a linear extension, x is in the simplex corresponding to that linear extension.

Boolean growth polytope

Goals...

- Find a direct explanation for the formula for the volume of the ASM polytope, i.e. find an explicit triangulation of the ASM polytope.
- Explain the similarities between the formulas for the volume of the order polytope and the ASM polytope, i.e. find a common generalization.

First step

Instead of the ASM polytope, look at the *CSM polytope*, the convex hull of all corner sum matrices.

There is a volume-preserving map between the two polytopes, so this is equivalent.

The upside is that a point in the CSM polytope satisfies similar inequalities as a point in the order polytope: we have $x_{i,j} \leq x_{i+1,j}$ and $x_{i,j} \leq x_{i,j+1}$.

First step

However, we also have $x_{i+1,j} \leq x_{i,j} + 1$ and $x_{i,j+1} \leq x_{i,j} + 1$.

Furthermore, the values are not all between 0 and 1, and some values are fixed: $x_{i0} = x_{0i} = 0$, $x_{in} = x_{ni} = i$.

We can remove the fixed coordinates. The coordinates of adjacent points must lie on an interval of length 1; for example, $x_{i1} \in [0, 1]$, $x_{i,n-1} \in [i-1, i]$.

Second step

Look at rectangular CSMs of size $k \times n$. For $k = 2$ and $n = 2, 3, \ldots$, we get polytopes with $2, 7, 16, 30, 50 \ldots$ vertices and volume $1, 4, 38, 294, 2172 \ldots$ in dimensions $1, 4, 6, 8, 10 \ldots$

This is more manageable than polytopes with $1, 2, 7, 42, 429 \ldots$ vertices and volume $1, 1, 4, 1376, 201675688 \ldots$ in dimensions $0, 1, 4, 9, 16 \ldots$!

(Moral: when you are studying ASMs, look at rectangular ASMs as well.)

Marked graphs...

For a *directed acyclic graph* $\vec{G} = (V, \vec{E})$, write $G = (V, E)$ for the underlying undirected graph. Call a step from u to v in G *forward* if $uv \in \vec{E}$, and *backward* if $vu \in \vec{E}$.

Definition

For a directed acyclic graph $\vec{G}$ and a subset $S \subseteq V$, we say that the map $\alpha \colon S \to \mathbb{Z}$ is *valid* for G and S if:

- every connected component of G contains an element of S;
- for a walk from $u \in S$ to $v \in S$ in G with k forward steps, $\alpha(v) \leq \alpha(u) + k$, and if the walk contains an element of $V \setminus S$, then $\alpha(v) < \alpha(u) + k$.

We call the triple $(\vec{G}, S, \alpha)$ with α valid for $\vec{G}$ and S a *marked graph*. We write $n = |V|$.

… and the Boolean growth polytope

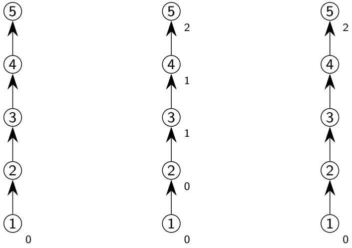
Definition

Given a marked graph $(\vec{G}, S, \alpha)$, define the *Boolean growth polytope* $\mathcal{BGP}(\vec{G}, S, \alpha)$ as the set of points $x = (x_u)_{u \in V}$ satisfying:

- $\alpha(u) \leq x_u \leq \alpha(u) + 1$ for $u \in S$;
- $x_u \leq x_v \leq x_u + 1$ if $uv \in \vec{E}$.

If we take $\vec{G} = \vec{P}$, $S = P$ and $\alpha(u) = 0$ for all u, we get the order polytope.

Basic examples: three paths



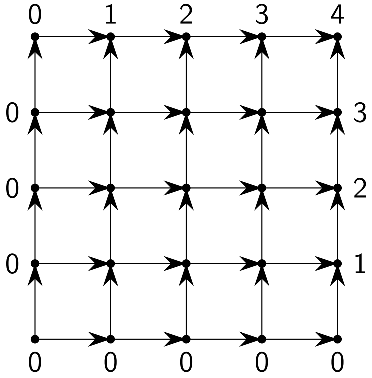
# of integer vertices: 32, 13, 20, volume: 120, 16, 66.

More generally, take an n-path ($\alpha(n) = k$ for the last case).

# of integer vertices: 2^n, F_{n+1}, $\binom{n+1}{k+1}$, volume: $n!$, E_n, $A(n, k)$.

Basic example: ASM polytope

The ASM polytope $\mathcal{A}_6$ is the Boolean growth polytope for the following marked graph:



Alcoved polytopes

Lam and Postnikov (2007) defined *alcoved polytopes* as polytopes of the form

$$\{x \in \mathbb{R}^d : \alpha_i \leq x_i \leq \beta_i \text{ for all } i, \gamma_{ij} \leq x_j - x_i \leq \delta_{ij} \text{ for all } i < j\}$$

for some $\alpha_i, \beta_i, \gamma_{ij}, \delta_{ij} \in \mathbb{Z}$.

Boolean growth polytopes are clearly a subclass of alcoved polytopes.

Why does a map have to be valid?

- every connected component of G contains an element of S;

This guarantees that the inequalities define a bounded polyhedron, i.e., a polytope.

Why does a map have to be valid?

- for a path from $u \in S$ to $v \in S$ in G with k forward steps, $\alpha(v) \leq \alpha(u) + k$

This guarantees that the polytope is non-empty and full-dimensional.

Why does a map have to be valid?

- ... and if the path contains an element of $V \setminus S$, then $\alpha(v) < \alpha(u) + k$.

Otherwise we could add that vertex to S.

Vertices of the BG polytope

Theorem [Behrend–K, 2026+]

The vertices of $\mathcal{BGP}(\vec{G}, S, \alpha)$ are precisely its integer points.

Facets of the BG polytope

Theorem [Behrend–K, 2026+]

The facets of $\mathcal{BGP}(\vec{G}, S, \alpha)$ are given by:

- hyperplanes $x_v = x_u$, where uv is not direct and $u \notin S$ or $v \notin S$ or $u, v \in S$, $\alpha(v) = \alpha(u)$;
- hyperplanes $x_v = x_u + 1$, where uv is not scenic and $u \notin S$ or $v \notin S$ or $u, v \in S$, $\alpha(v) = \alpha(u) + 1$;
- hyperplanes $x_u = \alpha(u)$, $u \in S$, where there is no edge vu with $v \in S$, $\alpha(v) = \alpha(u)$, and no edge uv with $v \in S$, $\alpha(v) = \alpha(u) + 1$;
- hyperplanes $x_u = \alpha(u) + 1$, $u \in S$, where there is no edge vu with $v \in S$, $\alpha(v) = \alpha(u) - 1$ and no edge uv with $v \in S$, $\alpha(v) = \alpha(u)$.

Volume of the Boolean growth polytope

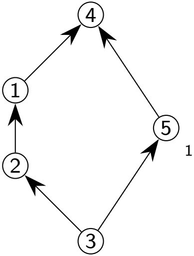
$A(\vec{G}, S, \alpha)$ and $\vec{G}_x$

Define $A(\vec{G}, S, \alpha)$ as the set of integer points $x \in \mathcal{BGP}(\vec{G}, S, \alpha)$ satisfying $x_u = \alpha(u)$ for $u \in S$.

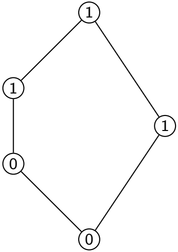
Given $x \in A(\vec{G}, S, \alpha)$, define the directed graph $\vec{G}_x$ on V so that we have an edge uv if $uv \in \vec{E}$, $x_v = x_u + 1$, and we have an edge vu if $uv \in \vec{E}$, $x_v = x_u$.

In other words, *keep* the direction of the edge of the graph if the coordinates increase along it, and *reverse* it otherwise.

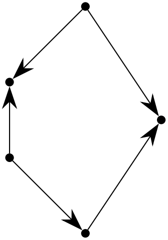
Example



$\vec{G}, S, \alpha$



$x \in A(\vec{G}, S, \alpha)$



$\vec{G}_x$

Volume of the BG polytope

The graph $\vec{G}_x$ is again acyclic, so it has linear extensions.

Theorem [Behrend–K, 2026+]

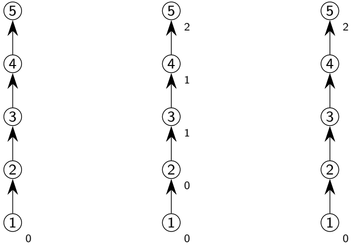
The normalized volume of $\mathcal{BGP}(\vec{G}, S, \alpha)$ is

$$\sum_{x \in A(\vec{G}, S, \alpha)} |L(\vec{G}_x)|.$$

For our example, there are 10 elements of $A(\vec{G}, S, \alpha)$ and $3 + 8 + 3 + 8 + 8 + 3 + 3 + 8 + 8 + 3 = 55$ linear extensions, so the volume is 55.

Examples

For the order polytope, $A(\vec{P}, P, 0)$ contains only one element x, $x_p = 0$ for all $p \in P$, and $|L(\vec{G}_x)| = |L(\vec{P})|$, so this recovers Stanley's result.



$\#$ of integer vertices: 2^n, F_{n+1}, $\binom{n+1}{k+1}$, volume: $n!$, E_n, $A(n,k)$.

Examples

If $(\vec{G}, S, \alpha)$ is the marked graph corresponding to $\mathcal{A}_n$, then the elements of $A(\vec{G}, S, \alpha)$ are in bijection with the integer points of the BG polytope corresponding to $\mathcal{A}_{n-2}$, which explains the Behrend–Izanloo formula. We were able to compute

$$\operatorname{vol} \mathcal{A}_8 = 6611671548765659977879930617132471066227536 8$$

Volume of alcoved polytopes

Lam and Postnikov proved that the volume of an alcoved polytope in $\mathbb{R}^d$ is a sum of numbers of certain integer points over all permutations in S_d.

This is *much* less efficient than our formula.

Triangulation of the BG polytope: idea of proof

Given $x \in A(\vec{G}, S, \alpha)$ and $\pi \in L(\vec{G}_x)$, define a simplex with volume 1 as follows: the first vertex is x, and then we add 1s to all values in the order determined by π.

Given a generic point $x \in \mathcal{BGP}(\vec{G}, S, \alpha)$, then $\lfloor x \rfloor \in A(\vec{G}, S, \alpha)$. The linear extension π is determined by the order of the fractional parts $x - \lfloor x \rfloor$. Then x is in the simplex determined by $\lfloor x \rfloor$ and π.

Ehrhart polynomial

If $\mathcal{P}$ is a polytope, and $t\mathcal{P}$ is the polytope formed by expanding $\mathcal{P}$ by a factor of t, write $E_{\mathcal{P}}(t)$ for the number of integer points in $t\mathcal{P}$.

If all vertices of the polytope have integer coordinates, then $E_{\mathcal{P}}(t)$ is a polynomial in t, we call it the *Ehrhart polynomial* of the polytope $\mathcal{P}$.

The leading coefficient is equal to the (non-normalized) volume of $\mathcal{P}$.

Ehrhart polynomial

Choose an appropriate ordering of the vertices of $\vec{G}$.

Theorem [Behrend–K, 2026+]

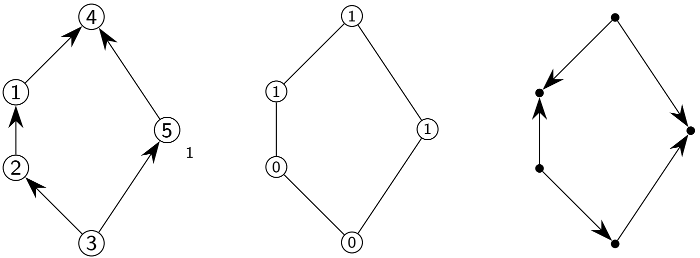
The Ehrhart polynomial of $\mathcal{BGP}(\vec{G}, S, \alpha)$ is

$$E_{\mathcal{BGP}(\vec{G},S,\alpha)}(t) = \sum_{x \in A(\vec{G},S,\alpha)} \sum_{\pi \in L(\vec{G}_x)} \binom{t + n - \operatorname{des} \pi}{n}.$$

Equivalently,

$$\sum_{t=0}^{\infty} E_{\mathcal{BGP}(\vec{G},S,\alpha)}(t) z^t = \frac{1}{(1-z)^{n+1}} \sum_{\substack{x \in A(\vec{G},S,\alpha) \\ \pi \in L(\vec{G}_x)}} z^{\operatorname{des} \pi}.$$

Ehrhart polynomial



$\vec{G}, S, \alpha$ $\quad x \in A(\vec{G}, S, \alpha)$ $\quad \vec{G}_x$

The linear extensions are

$$23415, 23451, 24135, 24315, 24351, 42135, 42315, 42351$$

and their contribution is

$$z + z + z + z^2 + z^2 + z^2 + z^2 + z^2.$$

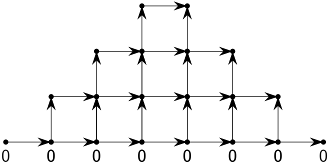
Applications to classes of alternating sign matrices

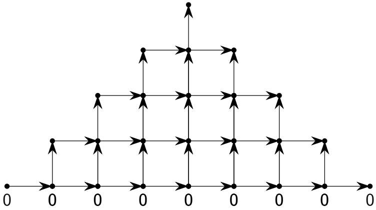
Classes of ASM matrices

We know how to realize the following classes of ASMs as the vertices of Boolean growth polytopes: $\mathcal{A}_n$, $\mathcal{A}_{2n+1}^{VS}$, $\mathcal{A}_{2n+1}^{VHS}$, $\mathcal{A}_n^{DS}$, $\mathcal{A}_n^{DADS}$, $\mathcal{A}_n^{HTS}$, $\mathcal{A}_{2n+1}^{TS}$ (but *not* $\mathcal{A}_n^{QTS}$).

We can also realize the *partial alternating sign matrix polytope* (Heuer–Striker), the *ASM-CRY family of polytopes* (Mészáros–Morales–Striker), and the *quilt polytope* (Billey–K).

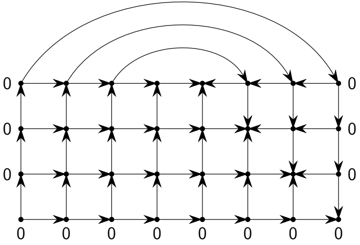
DADSASMs:

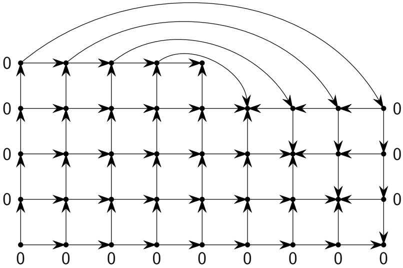




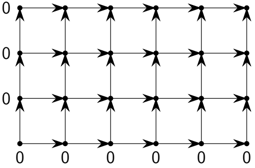
Classes of ASM matrices

HTSASMs:





PASMs:



Volumes

Volume for DADSASMs:

$$1, 1, 1, 4, 19, 1008, 62540, 84104000, 131649104624$$

$$7337064856162400, 47340853208144247 1088,$$

$$1615683104763109990538920704,$$

$$6355214714622694282695412854143984,$$

$$1838740394716134595142131492746079990278272$$

Volume for HTSASMs:

$$1, 1, 1, 8, 158, 345580, 1951876112, 5980439562366860,$$

$$471906885263693516302 92, 626262134884447124616748965534548,$$

$$2121341915605803800215174012766810278882 1468$$

Volumes

Volume for PASMs:

k \ n	1	2	3	4	5	6
1	1	1	1	1	1	1
2		6	43	308	2214	16071
3			5036	696658	96467610	13287071526
4				3106156252	16003351841356	82679058946411074
5					4419401234483684374	1386985375288955960127443
6						34983921571100943860765356655132

Questions

Can you describe the entire face structure?

Can you say anything about the h^* polynomial?

Can you compute the volume of the polytope of quarter-turn symmetric ASMs?

Can other polytopes in the literature be realized as Boolean growth polytopes?

What is the connection with the flow polytope?

THANK YOU