

Asymptotic Face Distributions in Random Reduced $SL(3)$ Webs

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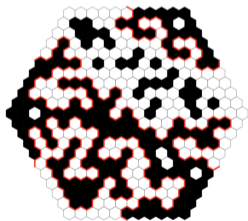
Yale University

March 19, 2026

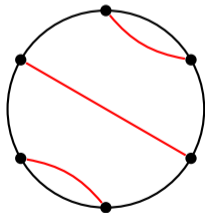
Michigan State Combinatorics Seminar

Why study random $SL(3)$ webs?

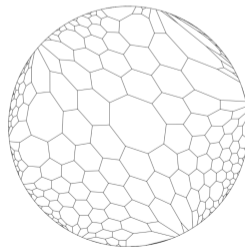
- $SL(2)$ webs \leftrightarrow non-crossing matchings
 - Appear as interfaces under certain boundary conditions in statistical mechanics models.
 - Well-understood probabilistically
- $SL(3)$ webs: a higher-rank analogue of non-crossing matchings
 - Appear in higher-rank dimer models
 - Study their typical large-scale geometry (face sizes)



Percolation interface



Corresponding $SL(2)$
non-crossing matching

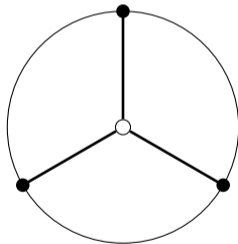


Random $SL(3)$ web

What is an $SL(3)$ web?

Definition

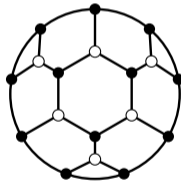
An $SL(3)$ web is a planar bipartite trivalent graph embedded in a disk with $3n$ black boundary vertices of degree 1.



Example: the smallest $SL(3)$ web ($n = 1$)

Definition

An $SL(3)$ web is **reduced** (or **non-elliptic**) if every interior face has at least 6 sides.



Theorem ([Kup96])

Let $V = \mathbb{C}^3$. Reduced $SL(3)$ webs with $3n$ black boundary vertices represent a basis of

$$\text{Inv}_{SL(3,\mathbb{C})}(V^{\otimes 3n}).$$

Therefore

$$\#\{\text{reduced } SL(3) \text{ webs}\} = \frac{2(3n)!}{n!(n+1)!(n+2)!} = \#\{3 \times n \text{ SYT}\}$$

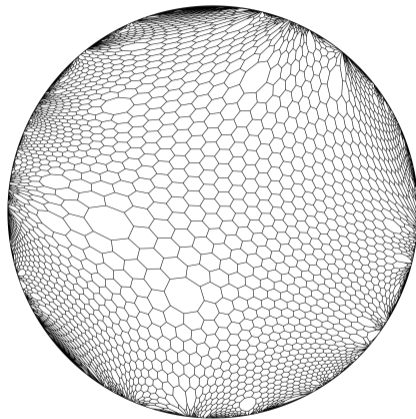


Figure: Uniform random reduced web with $n = 800$ (2400 boundary vertices)

- 1 Use Tymoczko's bijection [Tym07]:

$$\{3 \times n \text{ SYT}\} \longleftrightarrow \{m\text{-diagrams}\} \longleftrightarrow \{\text{reduced } SL(3) \text{ webs}\}$$

- 2 Encode tableaux as lattice paths in the quadrant, which become a random walk under the uniform model.
- 3 Translate m -diagram crossings into hitting events of the random walk.
- 4 Compute hitting probabilities (discrete harmonic functions) to compute face probabilities.

Theorem

In a large uniformly random reduced $SL(3)$ web, faces far from the boundary are mostly hexagonal.

More precisely, if a face is at depth at least d , then

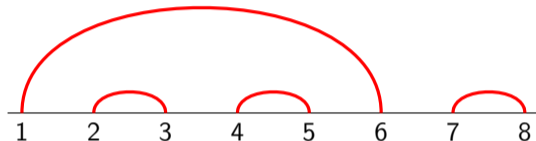
$$\mathbb{P}(\text{face has size } 6 + 2k) = O(d^{-2k}).$$

Catalan bijection: tableaux, Dyck paths, and noncrossing matchings

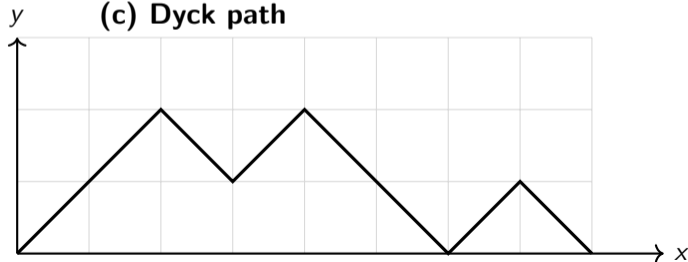
(a) Tableaux

3	5	6	8
1	2	4	7

(b) arc diagram



(c) Dyck path



Tymoczko's bijection: tableau \rightarrow m -diagram [Tym07]

Start from a standard Young tableau T of shape $3 \times n$ with entries $1, \dots, 3n$.

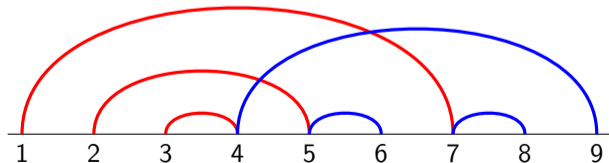
- Draw boundary points $1, \dots, 3n$ on a line.
- **Red arcs**: connect row 1 entries to row 2 entries (Catalan matching).
- **Blue arcs**: connect row 2 entries to row 3 entries.
- Each triple (i, j, k) forms an m -shape (middle row used twice).

(a) Tableaux

6	8	9
4	5	7
1	2	3

(b) m -diagram

- first arcs (row 1 \rightarrow row 2)
- second arcs (row 2 \rightarrow row 3)

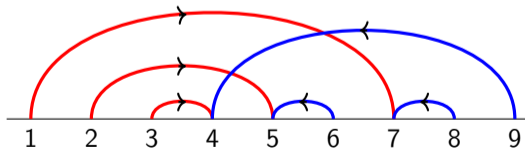


m -diagram \rightarrow reduced web

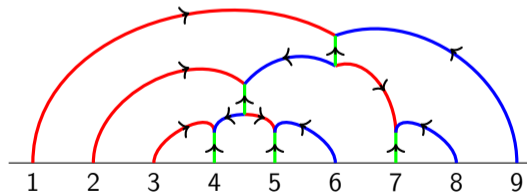
From the m -diagram:

- Replace each middle point j of an $m = (i, j, k)$ by a trivalent Y vertex.
- Orient all edges toward the Y vertex.
- Resolve each red/blue crossing into two trivalent vertices.

(a) m -diagram



(b) reduced web



Lattice path model

Reading entries $1, 2, \dots, 3n$ of T in increasing order gives a walk

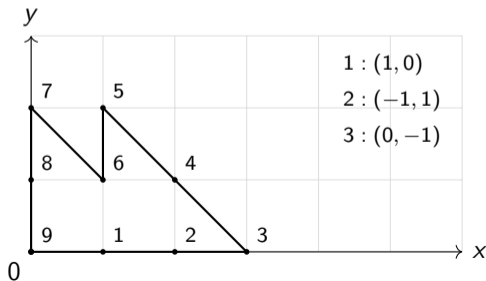
$$S_t = (A_t, B_t) \in \mathbb{Z}_{\geq 0}^2, \quad t = 0, 1, \dots, 3n,$$

with steps

$$s_1 = (1, 0), \quad s_2 = (-1, 1), \quad s_3 = (0, -1),$$

depending on whether t lies in row 1, 2, 3 respectively. SYT $\Rightarrow S_t$ stays in the quadrant and returns to $(0, 0)$ at time $3n$.

6	8	9
4	5	7
1	2	3

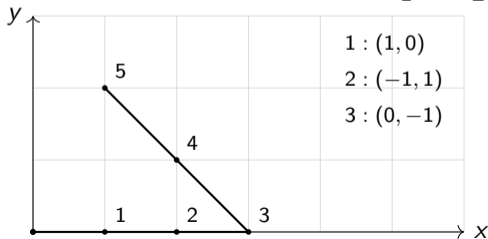
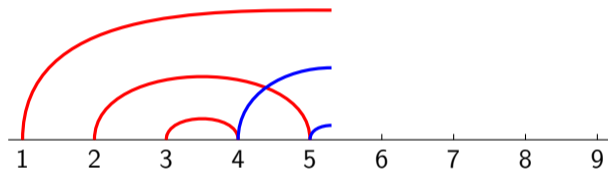


Bijection for a partially filled $3 \times n$ tableau

We build the m -diagram from the tableau step by step. The lattice path position $S_t = (R_t, B_t)$ records the number of open arcs: $R_t = \#$ red arcs open. $B_t = \#$ blue arcs open.

Row 1 \rightarrow open red, Row 2 \rightarrow close red and open blue, Row 3 \rightarrow close blue.

4	5	
1	2	3



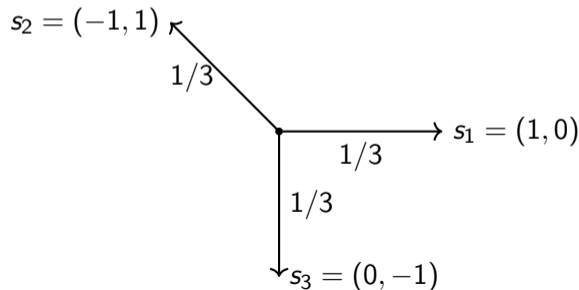
Local i.i.d. behavior

After scaling, the lattice path converges to a Brownian excursion in the quadrant.

Away from the endpoints, the steps behave approximately like independent moves.

$$\mathbb{P}(s_1) = \mathbb{P}(s_2) = \mathbb{P}(s_3) = \frac{1}{3}.$$

Entries are equally likely to appear in any of the three rows of the Young tableau. We compute local crossing probabilities in the m -diagram using this i.i.d. model.



What can happen to a red arc?

We open a red arc at position r_1 .

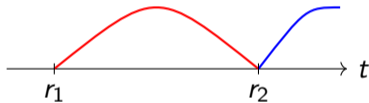
There are only three possible behaviors:

- **(1) No crossings:** The arc closes without interacting with any blue arcs
- **(2) Left crossing:** A previously opened blue arc crosses it from the left
- **(3) Right crossings:** When the red arc closes, it crosses $k \geq 1$ blue arcs that were opened after r_1

Three crossing events for an open red arc

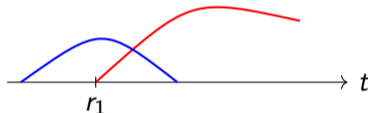
We label the events as follows:

E_1^R : red closes



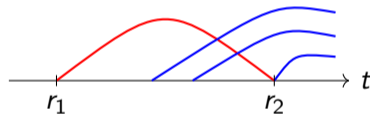
red closes, blue opens

E_2^R : blue crosses from left



blue arc starts left of r_1

$E_{3,k}^R$: k blue crosses from right

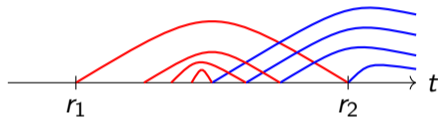
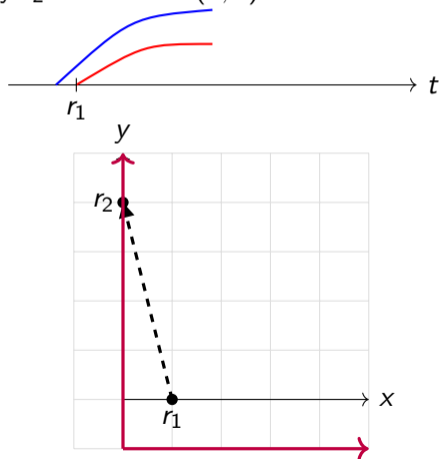


blue arcs start inside and exit right ($k = 2$)

We will compute $\mathbb{P}(E_1^{R,B})$, $\mathbb{P}(E_2^{R,B})$, and $\mathbb{P}(E_{3,k}^{R,B})$ using the lattice path model.

Example of crossing event as hitting event

A crossing event can be encoded by a subpath of the lattice path. In this example, we illustrate the event $E_{3,3}^R$. We start the subpath begins at entry r_1 at location $(1, 0)$ and ends at entry r_2 at location $(0, 4)$. The first time the path reaches purple axes is a crossing event.



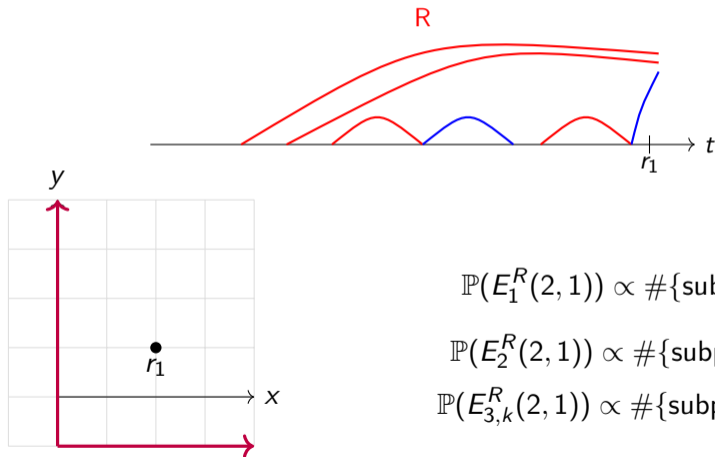
$$\mathbb{P}(E_1^R) \propto \#\{\text{subpaths } (1, 0) \rightarrow (0, 1)\}$$

$$\mathbb{P}(E_2^R) \propto \#\{\text{subpaths } (1, 0) \rightarrow (a, -1)\}$$

$$\mathbb{P}(E_{3,k}^R) \propto \#\{\text{subpaths } (1, 0) \rightarrow (0, k + 1)\}$$

Starting at other locations

When we have multiple arcs open, we can calculate the crossing event probabilities by starting the subpath at a different location.



$$\mathbb{P}(E_1^R(2, 1)) \propto \#\{\text{subpaths } (2, 1) \rightarrow (0, 1)\}$$

$$\mathbb{P}(E_2^R(2, 1)) \propto \#\{\text{subpaths } (2, 1) \rightarrow (a, -1)\}$$

$$\mathbb{P}(E_{3,k}^R(2, 1)) \propto \#\{\text{subpaths } (2, 1) \rightarrow (0, k+1)\}$$

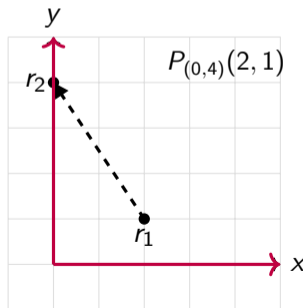
Hitting probabilities are discrete harmonic

Let $P_a(x, y)$ be the probability that the walk starting at (x, y) first hits the boundary at a . Let $g(x, y)$ be the probability that the random walk hits the x -axis before the y -axis

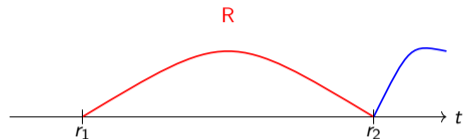
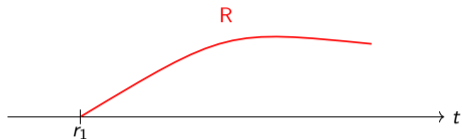
$$(g(x, y) = \sum_{i=1}^{\infty} P_{(i,0)}(x, y))$$

$$\mathbb{P}(E_1^R(x, y)) = P_{(0,2)}(x, y + 1), \mathbb{P}(E_2^R(x, y)) = g(x, y + 1), \mathbb{P}(E_{3,k}^R(x, y)) = P_{(0,k+2)}(x, y + 1)$$

$$P_a(x, y) = \frac{1}{3} \left(P_a(x + 1, y) + P_a(x - 1, y + 1) + P_a(x, y - 1) \right)$$



Explicit crossing probability



Example: probability a new red arc closes without further crossings (E_1^R) is

$$P_{(0,2)}(1, 1) = \frac{243\sqrt{3}}{40\pi} - 3 \approx 0.3493.$$

This comes from solving the discrete Dirichlet problem for the $P_a(x, y)$

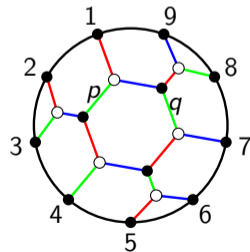
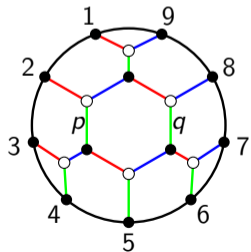
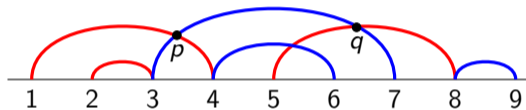
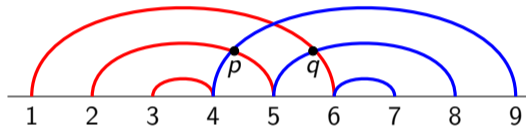
Lemma

The hitting probabilities $P_a(x, y)$ can be computed exactly, giving constants in $\mathbb{Q}(\sqrt{3}, \pi)$.

Faces in reduced web

Lemma

If a face of the reduced web has size $2k$, then the corresponding m -diagram contains $2k - 2$ arc segments bounding the face.



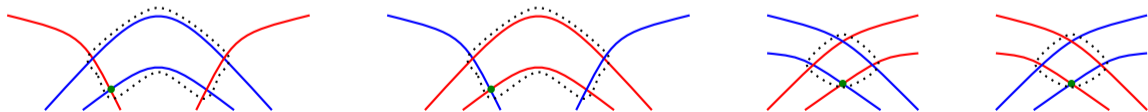
Face Diagrams of size 6

Definition

A **face diagram** is an abstract local configuration in an m -diagram consisting of a single interior face together with the surrounding arcs that form its immediate boundary.

Lemma

Let F be an interior face of an m -diagram whose boundary is formed by segments of exactly four arcs. There are exactly four possible local face diagrams around F , which are shown below. After resolution to a web, each such quadrilateral becomes a face of size 6.

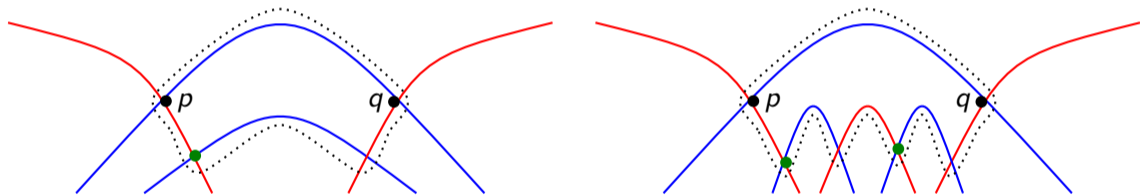


4 arc segments in the m -diagram \implies hexagonal face in the reduced web.

Face Diagram Extensions

Definition

The **face diagram extension operation** is a local transformation that produces a new face diagram corresponding to a face of size $2k + 2$ from one of size $2k$.



Any face diagram F can be obtained by repeating the diagram extension operation to one of the four face diagrams of size 4.

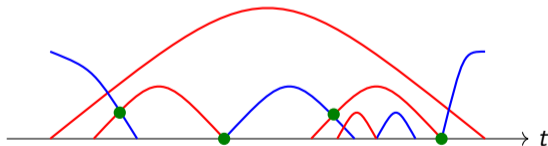
$$2k \text{ arc segments in the } m\text{-diagram} \implies 2k + 2 \text{ edges in the reduced web.}$$

Face types from crossing sequences

Definition

The **face type** of a face records:

- the local face diagram,
- the crossing counts (c_1, \dots, c_k) from the green vertices to the boundary following the right segment.
- the starting color, either R or B .



Example face type: $((1, 1, 2, 1), B) \rightarrow E_{3,2}^B(0, 1), E_1^R(1, 0), E_{3,2}^B(0, 1), E_1^R(2, 0), E_2^B(0, 1)$

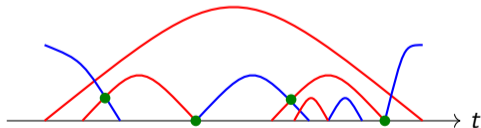
Limiting frequency of a face type

Theorem

For any fixed face type τ , the limiting frequency of faces of type τ is:

$$\lim_{n \rightarrow \infty} \frac{T(\tau, n)}{n |W_n|} = \prod_j P_{a_j}(x_j, y_j) \cdot \begin{cases} g(x_*, y_*), \\ 1 - g(x_*, y_*). \end{cases}$$

- each required crossing contributes one discrete hitting probability $P_a(x, y)$,
- the final closing event contributes $g(x, y)$ or $1 - g(x, y)$.



$$P_{(3,0)}(1, 1) P_{(0,2)}(1, 1) P_{(3,0)}(1, 1) P_{(0,2)}(2, 1) (1 - g(1, 1)).$$

Theorem

Let $d \rightarrow \infty$ with $n \rightarrow \infty$ and $d = o(n)$. For faces at distance at least d from the boundary in a uniformly random reduced $SL(3)$ web, we have:

- 1 The probability that a uniformly chosen such face has size 6 converges to 1 as $d \rightarrow \infty$.
- 2 More generally, for finite d , the probability that such a face has size $6 + 2k$ is $O(d^{-2k})$ for $k \geq 1$, where the constant depends only on k .

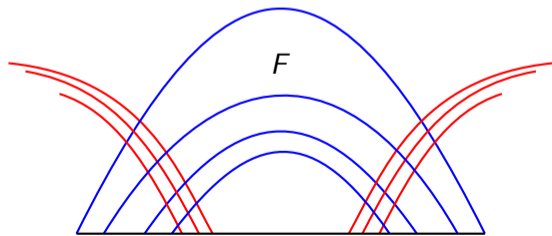
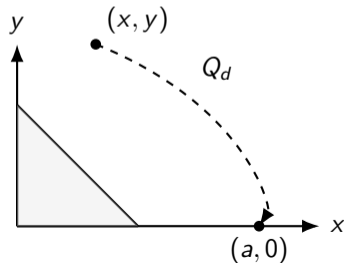
Depth d and truncated domains

Definition

Define the truncated quadrant $Q_d = \{(x, y) \in \mathbb{Z}_{\geq 0}^2 : x + y \geq d\}$. We also define the analogous Dirichlet problem solutions in Q_d by $P_a(\cdot; d)$ and $g(\cdot; d)$.

Lemma

If a face F is distance $\geq d$ from the boundary (in dual graph distance), then the lattice path remains in the wedge Q_d for all steps between the first and last arc in the face diagram of F .



The cost of a face diagram extension operation

- Fix a base diagram and condition on the crossing data (F_a, F_b) , where F_a (resp. F_b) denotes the number of arcs crossing the segment from p to s (resp. from p to e) in the m -diagram.
- We compare the probability of seeing a face diagram F and the extended version \tilde{F} at a distance $\geq d$. Note $F_a, F_b \geq d$.
- Only the area in the grey region differs.
- A face of size $6 + 2k$ is obtained by $2k$ extensions.

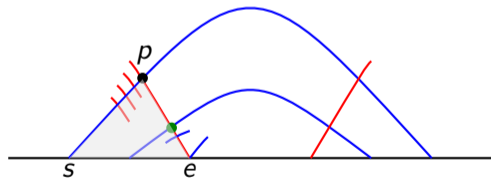


Figure: Base Diagram ($F_a = 4, F_b = 3$)

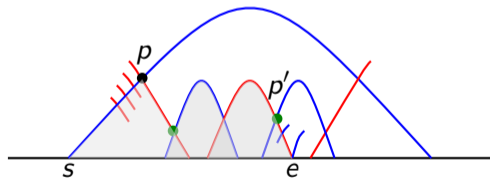


Figure: Extended Diagram ($\tilde{F}_a = 4, \tilde{F}_b = 3$)

Extended Diagram bound

Proposition

Fix $m \geq 0$ and crossing numbers (F_a, F_b) . Let L_d^2 and L_d^1 denote the frequencies of the diagram and the extended diagram. The exact formula depends on the color of the arc beginning at s :

- If the arc at s is blue, then

$$\frac{L_d^2}{L_d^1} = \sum_{t,s=d+1}^{\infty} \frac{P_{(0,t)}(F_a, 1; d) \cdot P_{(s,0)}(1, t-1; d) \cdot P_{(0,F_b+1)}(s-1, 1; d)}{P_{(0,F_b+1)}(F_a, 1; d)}.$$

- If the arc at s is red, then

$$\frac{L_d^2}{L_d^1} = \sum_{t,s=d+1}^{\infty} \frac{P_{(t,0)}(1, F_a; d) \cdot P_{(0,s)}(t-1, 1; d) \cdot P_{(F_b+1,0)}(1, s-1; d)}{P_{(F_b+1,0)}(1, F_a; d)}.$$

Note that this ratio does not depend on the face diagram that is chosen.

Proof of Main Theorem

Proposition

Take any $d \geq 1$ possibly with $d \rightarrow \infty$ as $n \rightarrow \infty$, but $d = o(n)$. Then for any face diagram,

$$\frac{L_d^2}{L_d^1} \lesssim \frac{1}{d^2}.$$

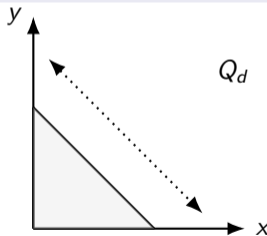


Figure: The lattice path must cross an additional two times for the extended face diagram.

Solving for $P_a(x, y)$

Define a discrete Laplacian as follows:

$$(\Delta f)(x, y) = \frac{1}{3}(f(x + 1, y) + f(x - 1, y + 1) + f(x, y - 1) - 3f(x, y)).$$

Lemma

Consider the lattice \mathbb{Z}^2 with step set $(1, 0), (-1, 1), (0, -1)$, equivalently the equilateral lattice in coordinates $z = xe_1 + ye_2$, $e_1 = (1, 0)$, $e_2 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Define

$$G_\infty(x, y) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{e^{i(x\theta+y\phi)} - 1}{\frac{1}{3}(e^{i\theta} + e^{-i\theta+i\phi} + e^{-i\phi}) - 1} d\theta d\phi.$$

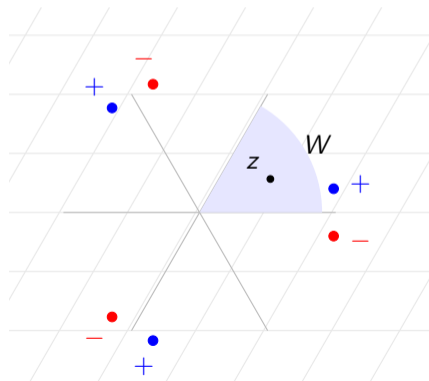
Then the integral converges absolutely for all $(x, y) \in \mathbb{Z}^2$, $G_\infty(0, 0) = 0$, and

$$\Delta G_\infty(x, y) = \delta_{(0,0)}(x, y)$$

Reflection Principle

We solve for $G_W(z, z_0)$. That is, $\Delta G_W(z, z_0) = \delta_{z_0}(z)$ and $G_W(z, z_0) = 0$ when z belong to one of the $k\pi/3$ rays for $k \in \mathbb{Z}$. Then $P_a(z) = \frac{1}{3} G_W(z, a - v_j)$.

$$G_W(z, z_0) = \sum_{\gamma \in D_3} \text{sgn}(\gamma) G_\infty(z - \gamma(z_0)),$$



Future directions:

- Analogs for other Lie types $Sp(4)$, $G(2)$ and higher rank ($SL(4)$ constructed in [GPP⁺25]).
- Probabilities using the measure from the triple-dimer model.

Thank you!



Christian Gaetz, Oliver Pechenik, Stephan Pfannerer, Jessica Striker, and Joshua P Swanson.

Rotation-invariant web bases from hourglass plabic graphs.

Inventiones mathematicae, pages 1–102, 2025.



Greg Kuperberg.

Spiders for rank 2 lie algebras.

Communications in mathematical physics, 180(1):109–151, 1996.



Julianna S. Tymoczko.

A simple bijection between standard $3 \times n$ tableaux and irreducible webs for \mathfrak{sl}_3 .

Algebras and Representation Theory, 10(5):519–543, 2007.