Counting Tangled Labelings

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joint work with
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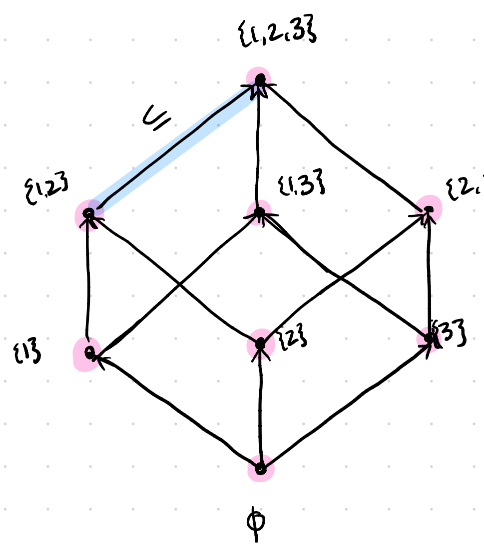
Poset Basics

Def A Partially Ordered Set, poset, is a pair $(P, \leq)$, where P is a set
& $\leq$ is a transitive
reflexive relation
antisymmetric

For example, $(\{\text{all subsets of } \{1,2,3\}\}, \subseteq)$

- $\{1\} \subseteq \{1\}$ reflexive
- $\{1\} \subseteq \{1\,2\} \subseteq \{1\,2\,3\}$
  $\Rightarrow \{1\} \subseteq \{1\,2\,3\}$ transitive
- If $s \subseteq p$
  and $p \subseteq s$, then $p = s$ Anti-symmetric.

We can represent these relationships using a Hasse Diagram:



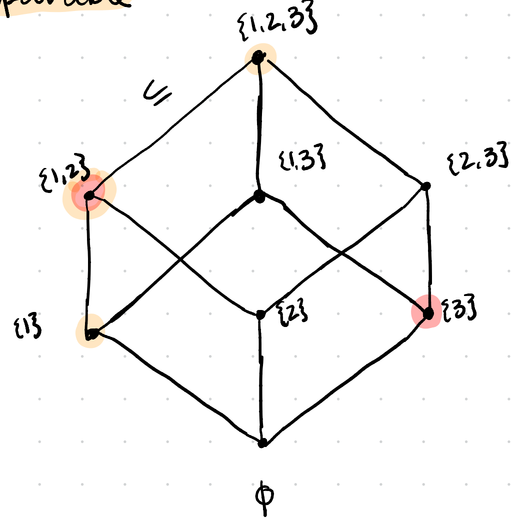
where the points represent the set &

each edge represents the relation, directed upwards.

Poset Basics

Elements that are connected by an upward path are comparable

Otherwise, they are incomparable

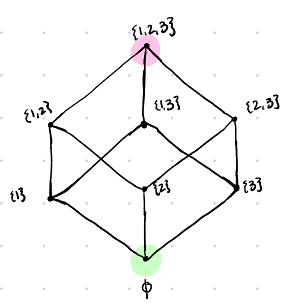


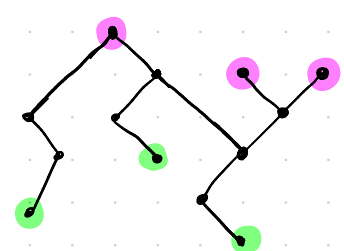
A Poset P has a maximum element k if $\forall\ a \in P$, $a \leq k$

A Poset P has a minimum element, l, if $\forall\ a \in P$, $l \leq a$.

An element $g \in P$ is maximal if there is no element $a \in P$ s.t. $g < a$.

An element $m \in P$ is minimal if there is no element $a \in P$ s.t. $a < m$.

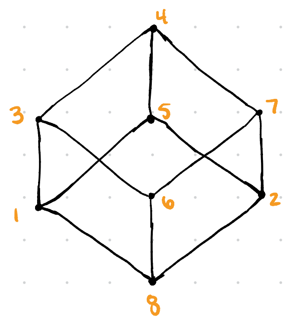


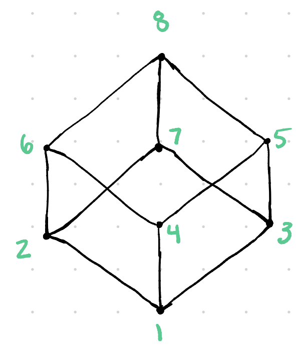


Linear Extensions of a Poset:

For P an n-element poset,
Let $\mathcal{L}(P)$ represent the set of all labelings of a poset with $\{1, \ldots, n\}$.

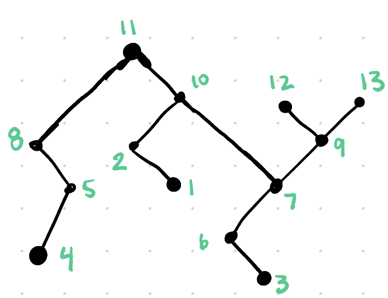
(note: there are $n!$ ways to label any poset)

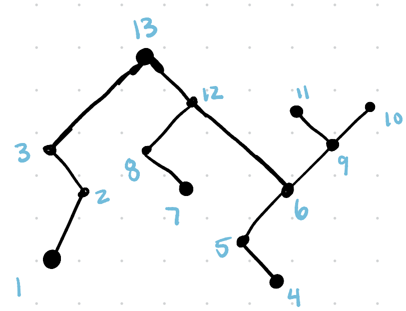


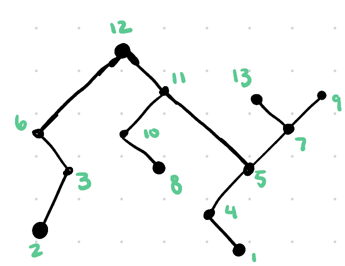


If this labeling "respects the order" of the original poset, it's called a linear extension.

$\left(\begin{array}{l} \text{If } \; a \leq_P b \; \text{ for elements } a, b \in P \\ \text{then } \; L(a) < L(b) \end{array} \right)$







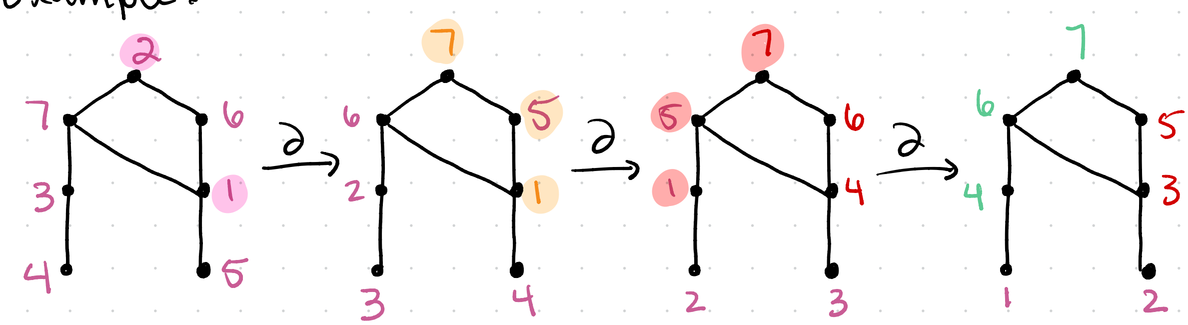
Extended Promotion $\partial : \mathcal{L}(P) \to \mathcal{L}(P)$

- Acts on the labelings of a poset P with n elements.
- Will "sort" any labeling into a linear extension after repeated applications.

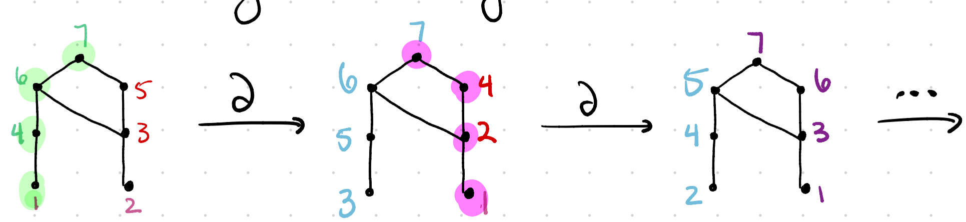
- ∂:
  1. relabel the element "1" as "n+1"
  2. Consider the labels of all elements greater than the element "n+1". Swap the smallest such label with "n+1".

     Repeat ② until the element labelled "n+1" is maximal in P
  3. Reduce all labels by 1

Example:



* Once sorted into a linear extension, the labeling will stay a linear extension.



Tangled Labelings

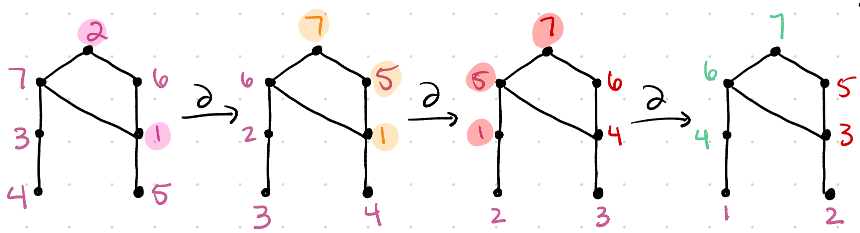
Colin Defant & Noah Kravitz (Promotion Sorting, 2023) showed that

$$\partial^{n-1}(L(P))$$

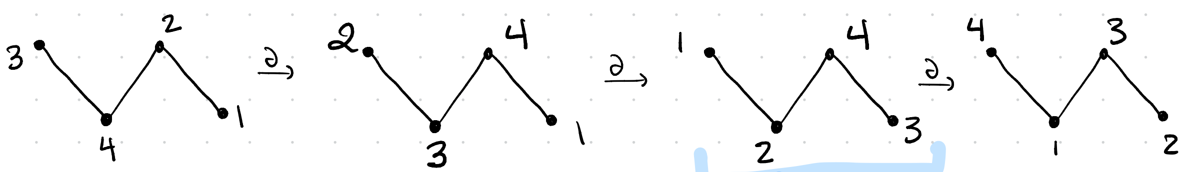
will always give a linear extension of P.

<u>Def</u> A labeling $L(P)$ of an n-element poset P is tangled if $\partial^{n-2}(L(P))$ is not a linear extension.

(in other words, it is the furthest from being sorted)



$\partial^3(L(P))$ is sorted, <u>not tangled</u>.



$n=4$;

$$\partial^{n-2} = \partial^2(L(P))$$

is not a linear extension, so this is <u>tangled</u>

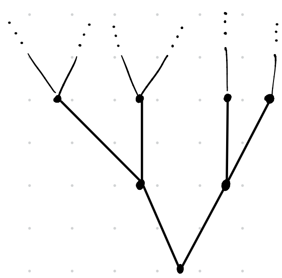
Previously:

(Defant & Kravitz, 2023)

Theorem: For a poset with r connected components, each component having a minimum element, there are
$$(n-r)(n-2)!$$
tangled labellings.

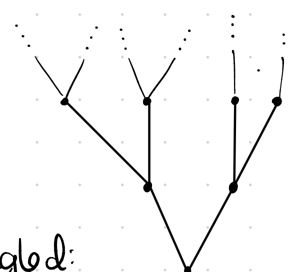
Particular case: For a single poset with a minimum element, $\hat{0}$, a labeling $\mathcal{L}$ is tangled if & only if
$$\mathcal{L}(\hat{0}) = n.$$

tangled:



n

not tangled:



$k \neq n$

These posets have exactly $(n-1)!$ tangled labelings.

Let $\Upsilon(P)$ be the set of all tangled labelings of a poset.

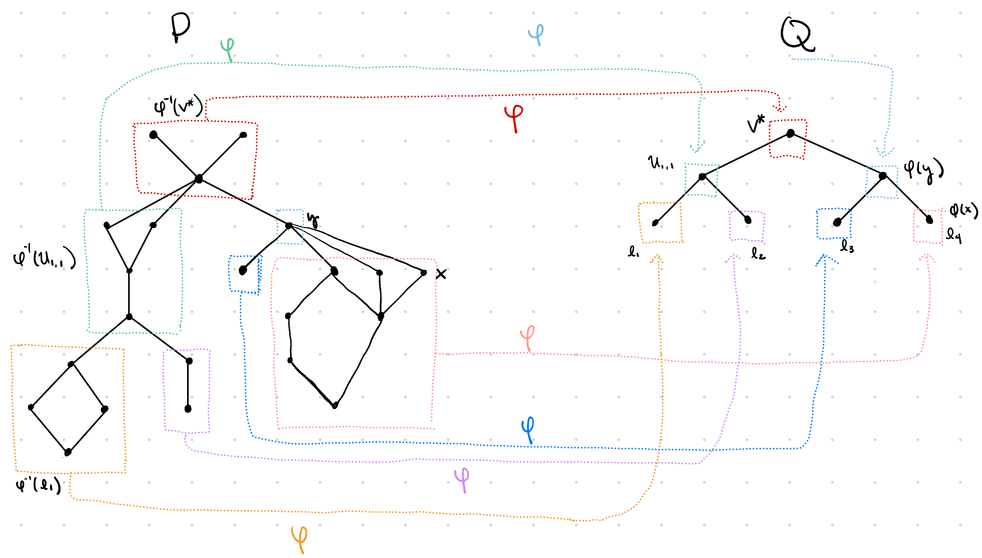
Defant & Kravitz's Conjecture:

For P an n-element poset,
$$(*) \quad \# \Upsilon(P) \leq (n-1)!$$

Previously:
(ctd.)

They proved $(*)'$ for a class of posets called __inflated rooted forests.__

- $\forall x \in Q$, $\varphi^{-1}(x)$ has a unique minimal elt. in P
- $\forall\ x, y \in P$ s.t. $\varphi^{-1}(x) \neq \varphi^{-1}(y)$, $x <_P y \iff \varphi(x) <_Q \varphi(y)$
- Q has a unique maximum, & every other elt. is covered by exactly one element.



Hodges' Conjecture:

For P an n-element poset with m minimal elements,

$$(*) \quad \#\mathcal{T}(P) \leq (n-m)(n-2)!$$

New!

__Theorem:__ (Boyer, Chau, Denker, K., Lee, Liang)

Inflated rooted forests satisfy $(*)$.

More Definitions!

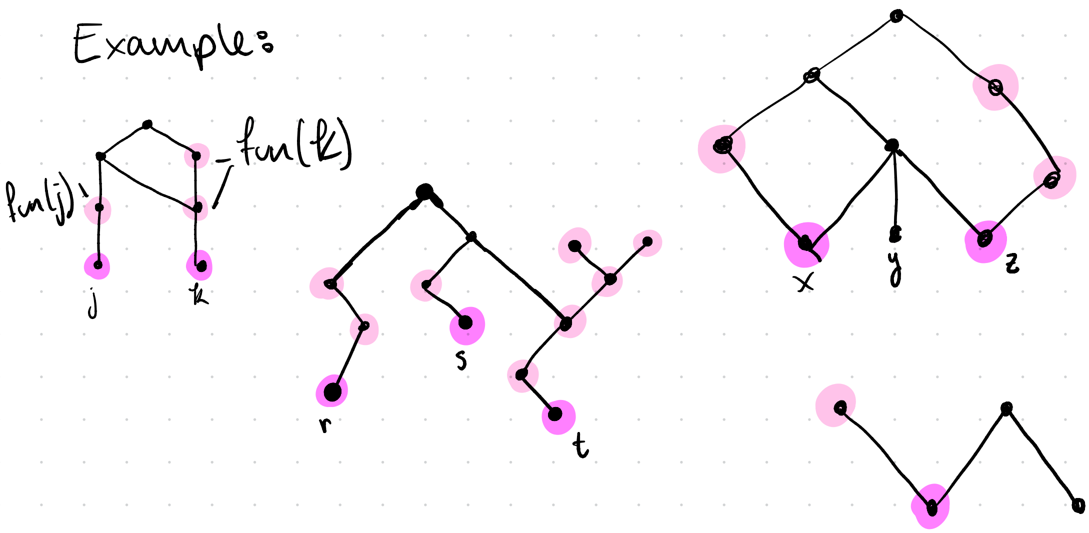
Let $x \in P$ be a minimal element. Its **funnel** is

$$\mathrm{fun}(x) = \{y >_P x : z <_P y \text{ and } z \text{ is minimal implies } z = x\}$$

(or, all of the elements whose lower order ideal contains only one minimal element, x.)

A minimal $x \in P$ is a **basin** if $\mathrm{fun}(x) \neq \emptyset$

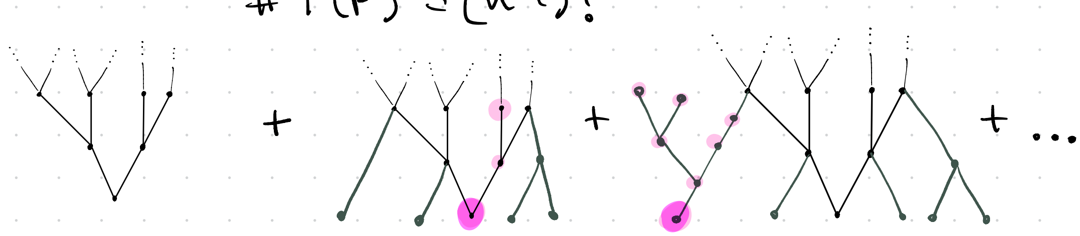
Example:



Theorem: (Boyer, Chan, Denker, K., Lee, Liang)

For an n-element poset with b basins, if $b \le 1$,

$$\#\mathcal{T}(P) \le (n-1)! \quad (*)$$



*Lemma | (Boyer, Chau, Denker, K., Lee, Liang)

For P a poset on n elements, $L: P \to [n]$, L is tangled if & only if the following conditions are both met:

① $L^{-1}(n)$ is minimal in P

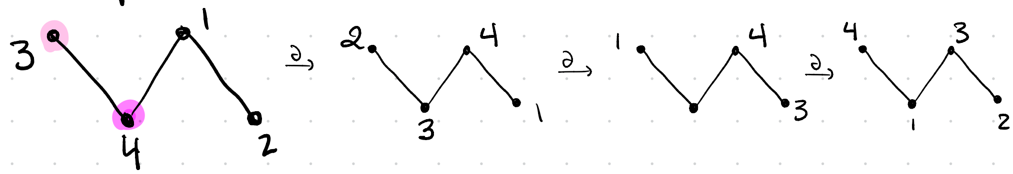
② $(\partial^{n-2}L)^{-1}(1) >_P L^{-1}(n)$

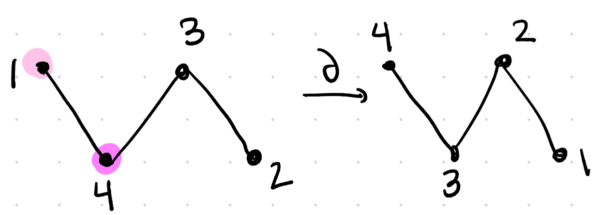
Essentially: "n-1" labelled element falls above minimal "n" element at ∂^{n-2}

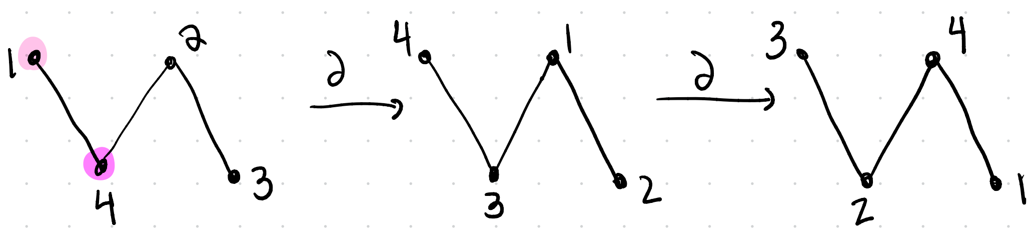
$\Updownarrow$

L is a tangled labeling.

Example:

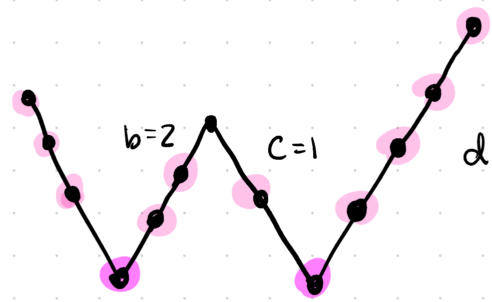






W - Posets | (explicit enumeration of $\#\mathcal{T}(P)$)

Given $a,b,c,d \in \mathbb{Z}_{\geq 0}$, the W-poset is $W_{a,b,c,d}$ s.t. there are $a+b+c+d+3$ elements, ordered as below for $W_{3,2,1,4}$



Theorem: (Boyer, Chau, Denker, K., Lee, Liang)
For $P = W_{a,b,c,d}$, a W-poset on $n=a+b+c+d+3$ elements: let

$$X = \binom{n-2}{a} \sum_{i=0}^{b-1} \sum_{j=0}^{d} (d-j+1) \binom{i+j+c-1}{i,j}$$

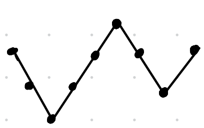
&

$$Z = \binom{n-2}{d} \sum_{i=0}^{c-1} \sum_{j=0}^{a} (a-j+1) \binom{i+j+b-1}{i,j}.$$

Then,

$$\#\mathcal{T}(W_{a,b,c,d}) = (n-2)(n-2)! - a!b!c!d!\,(X + Z).$$

Ex: $W_{2,2,1,1}$ has 34,412 tangled labelings.



$n=9$

Shoelace Posets

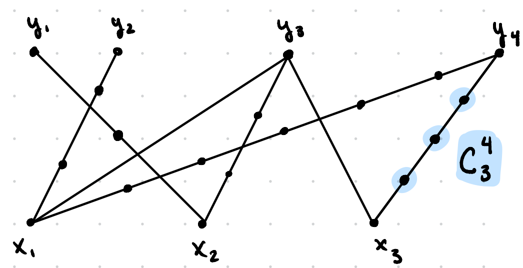
A generalization of the W-Posets.

Def A shoelace poset P is a collection of minimal elements $\{x_1, \dots, x_\ell\}$ & maximal elements $\{y_1, \dots, y_m\}$, along with a set $\mathcal{S}(P) \subseteq \{x_1, \dots, x_\ell\} \times \{y_1, \dots, y_m\}$ such that:

① $\forall (i,j) \in [\ell] \times [m]$, x_i & y_j are comparable $\iff (x_i, y_j) \in \mathcal{S}(P)$

② $\forall (x_i, y_j) \in \mathcal{S}(P)$, the open interval $(x_i, y_j)_P$ is a (possibly empty) chain, denoted C_i^j.

Example



$$\mathcal{S}(P) = \{(x_1, y_2), (x_1, y_3), (x_1, y_4), (x_2, y_1), (x_2, y_3), (x_3, y_3), (x_3, y_4)\}$$

C_1^3 is empty

C_3^3 is empty

Corollary to Theorem ⊛: (Bayer, Chan, Denker, K., Lee, Liang)

For a shoelace poset P on n elements, with m minimal elements,

$$\# \mathcal{T}(P) \leq (n-m)(n-2)!$$

(or, shoelaces satisfy (⊛))

(*) Shoelaces, continued

Notation: We say that L is an x-labeling of P if $L(x) = n-1$. Let $\mathcal{T}_x(P)$ denote the set of all tangled x-labelings of P.

<u>Theorem:</u> (Bayer, Chan, Denker, K., Lee, Liang)

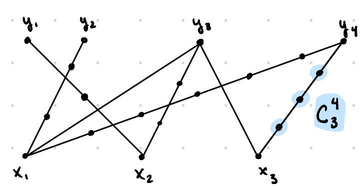
For a shoelace poset P on n elements,

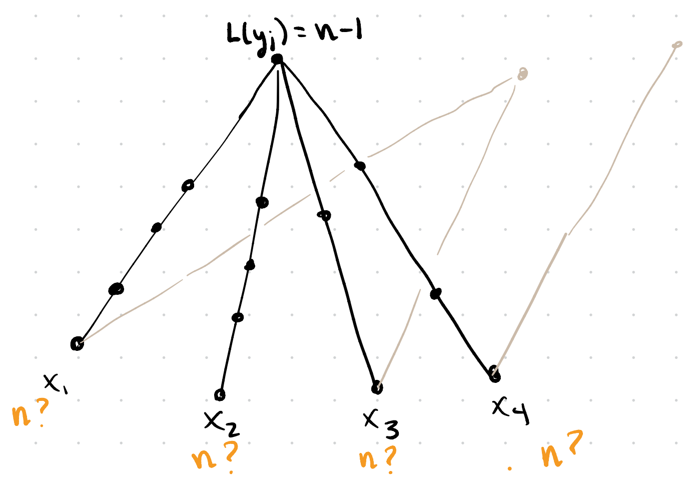
$$(*) \qquad \# \mathcal{T}_x(P) \leq (n-2)! \qquad \forall x \in P$$

[note: if x is a minimal element, then our previous lemma implies $\# \mathcal{T}_x(P) = 0$, so,

$$\# \mathcal{T}(P) = \sum_{x \in P} \# \mathcal{T}_x(P) = \sum_{\substack{x \in P \text{ s.t.} \\ x \text{ is not} \\ \text{minimal}}} \# \mathcal{T}_x(P) \overset{(*)}{\leq} (n-m)(n-2)! \quad]$$

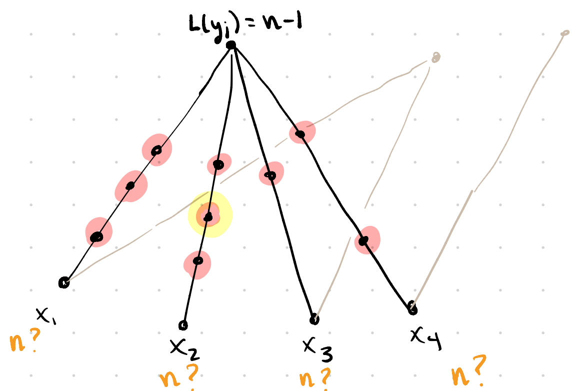
Crux of the proof: Case where $L(y_j) = n-1$



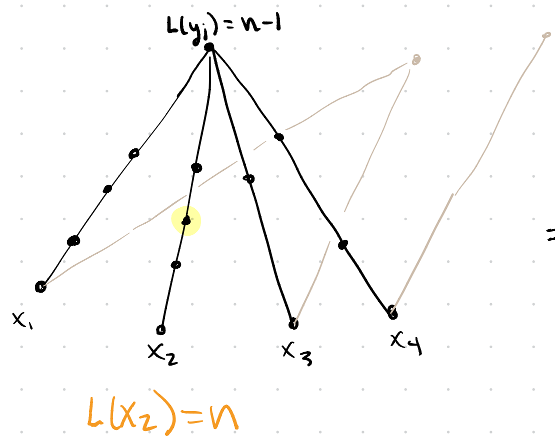


$$\# \mathcal{T}_x(P) \leq 4(n-2)!$$
$$\leq b(n-2)!$$

Shoelaces, tied up



"n" The placement of "n" is entirely dependent on the minimal label on $[C_i^j]_{i=1}^b$



$$\# \tilde{T}_x(P) \leq \frac{b(n-2)!}{b}$$

$$\leq (n-2)!$$

Conjecture: (Boyer, Chan, Denker, K., Lee, Liang)

For an n-element poset P,

$$(*) \quad \# T_x(P) \leq (n-2)! \quad \forall \ x \in P.$$

- checked computationally up to $n = 9$ *

Generating Functions

We have also been searching for patterns in generating functions.

<u>Def</u>

A labeling $L: P \to [n]$ is said to be **k-sorted** if

$\partial^{k}(L(P))$ is a linear extension of P

& $\partial^{k-1}(L(P))$ is not, for $k > 0$.

"0-sorted" = linear extension

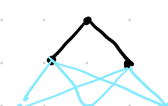
"$n-1$-sorted" = tangled labeling.

Let $or(L)$ denote the promotion order of a labeling, or, how many applications of promotion it takes for L to be sorted.

# of labelings that are i-sorted

$$f(P) = \sum_{L: P \to [n]} q^{or(L)} = \sum_{i=0}^{n-1} a_i q^i$$

P



T_3

* We determined the generating functions of $T_k \oplus P$,
  - ↳ provided a unified proof of some enumeration problems from Defant & Kravitz, & Hodges.
* We computed the generating functions for the ordinal sum of antichains, & showed log-concavity of the coefficients.
  - ↳ defined a partial order on these ordinal sums of various sizes of r antichains, & found connections to the weak Bruhat order.

Thank you for listening!

* Colin Defant & Noah Kravitz, "Promotion Sorting" Order 40.1 (2023).

* Eliot Hodges, "On Promotion and Quasi-tangled Labelings of Posets." Annals of Combinatorics (2024).

$$\# T_x(P) \leq (n-2)!$$

An Application

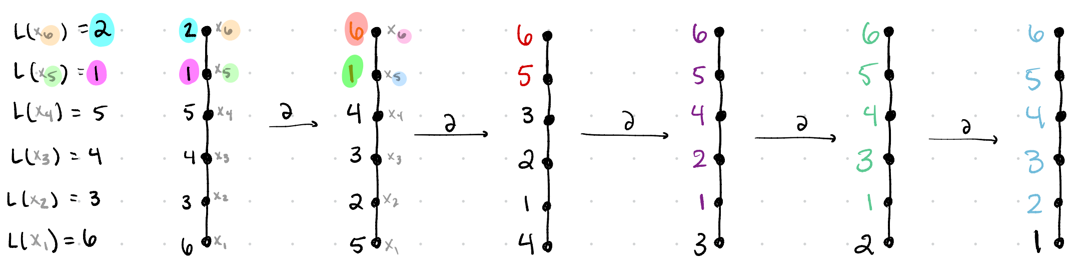
$\star\ \partial$:

① relabel the element "1" as "n+1"

② Consider the labels of all elements greater than the element "n+1". Swap the smallest such label with "n+1",

Repeat ② until the element labelled "n+1" is maximal in P

③ Reduce all labels by 1



$\hat{\sigma}(L) = (5\ 6\ 2\ 3\ 4\ 1)$ index: $1\ 2\ 3\ 4\ 5\ 6$ $\quad (5\ 2\ 3\ 4\ 1\ 6)$ $1\ 2\ 3\ 4\ 5\ 6$ $\quad (2\ 3\ 4\ 1\ 5\ 6)$ $\quad (2\ 3\ 1\ 4\ 5\ 6)$ $(2\ 1\ 3\ 4\ 5\ 6)$ $(1\ 2\ 3\ 4\ 5\ 6)$

What if I only keep track of how $\hat{\sigma}(\mathcal{L})$ changes?

at index:

$$\mathcal{L} = \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ 6 & 3 & 4 & 5 & 1 & 2 \end{array}$$

$\hat{\sigma}(\mathcal{L}) = (5\ \ 6\ \ 2\ \ 3\ \ 4\ \ 1)$

$\hat{\sigma}(\partial(\mathcal{L})) = (5\ \ 2\ \ 3\ \ 4\ \ 1\ \ 6)$

$(2\ \ 3\ \ 4\ \ 1\ \ 5\ \ 6)$

$(2\ \ 3\ \ 1\ \ 4\ \ 5\ \ 6)$

$(2\ \ 1\ \ 3\ \ 4\ \ 5\ \ 6)$

$(1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6)$

This is not an accident!

Extended Promotion on finite chain posets is in bijection with the bubble sort algorithm on sequences of integers!

$$\hat{\sigma}(\partial(L)) = \mathbf{B}(\hat{\sigma}(L))$$

> is a total order: every element is comparable!

This might lead you to ask, can we recover a different sorting algorithm for non-totally ordered objects?

* Can we even describe the behavior of ∂ in a more complete way?

* How long does it take to "sort" an arbitrary poset?

* Are some labelings "worse" (farther from being sorted) than others?