Super FiboCatalan Numbers and their Lucas Analogues

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The Fibonacci Numbers

- 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...
- $F_n =$ tilings of a strip of length $n-1$ with squares and dominos
- $F_n = F_{n-1} + F_{n-2}$

The Fibonomials

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- Is $\binom{n}{k}_F$ an integer? Yes! (Benjamin and Plott, Sagan and Savage, and Bennet, Carrillo, Machacek and Sagan)

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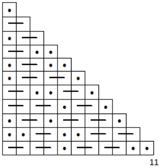

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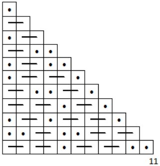
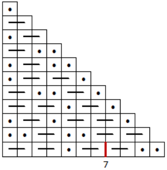


To prove $\binom{12}{7}_F$ is an integer, we will partition the set of tilings into disjoint subsets $S_1, S_2, \ldots, S_k$ such that $\frac{|S_m|}{F_7!F_5!}$.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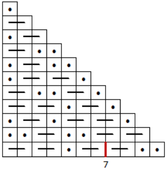
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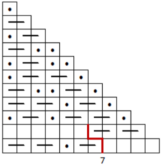
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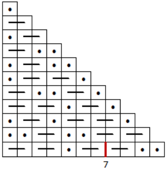

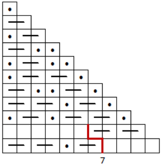

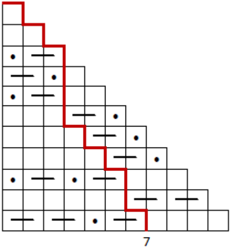

The empty boxes to the left of the path can be filled in $F_7!$ ways and the empty boxes to the right of the path can be filled in $F_5!$ ways.

The Lucas polynomials $\{n\}$ are defined in variables s and t as $\{0\} = 0$, $\{1\} = 1$ and for $n \geq 2$ we have:

$$\{n\} = s\{n-1\} + t\{n-2\}.$$

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When $s = t = 1$ we have the Fibonacci sequence. Note that $\{n\}$ is the generating function for tilings of a strip of length $n-1$ with squares s and dominoes t.

The *lucanomials* are defined by

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \frac{\{n\}!}{\{k\}!\{n-k\}!}$$

where $\{n\}! = \{n\}\{n-1\}\cdots\{2\}\{1\}$.

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The lucanomials are polynomials with non-negative integer coefficients. (BCMS)

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$$C_{n,F} = \frac{1}{F_{n+1}}\binom{2n}{n}_F = \frac{F_{2n}!}{F_{n+1}!F_n!}$$

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- Gessel and Xin (2005) give a combinatorial interpretation of these numbers.

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Their proof is based on fact that

$$\frac{6(2n)!}{(n+2)!n!} = 4C_n - C_{n+1}.$$

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$$S(m,n) = \frac{(2m)!(2n)!}{m!n!(m+n)!}$$

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Gessel (1992) called these numbers the *super Catalan numbers* since

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Note that

$$\frac{1}{2}S(2,n) = \frac{6(2n)!}{(n+2)!n!}$$

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Note that when $r = 1$ we have

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Chen and Wang have given a combinatorial interpretation for $S(m, m + s)$ for $0 \leq s \leq 3$.

In my work, I have defined the *super FiboCatalan numbers* as

$$S(m,n)_F = \frac{F_{2m}!F_{2n}!}{F_m!F_n!F_{m+n}!}$$

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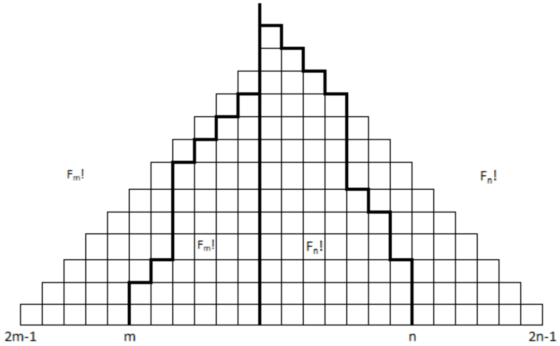
$$S(m,n)_F = \frac{F_{2m}!F_{2n}!}{F_m!F_n!F_{m+n}!}$$

and the *generalized FiboCatalan numbers* as

$$J_{r,F}\frac{F_{2n}!}{F_n!F_{n+r+1}!}$$

where $J_{r,F} = \frac{F_{2r+1}!}{F_r!}$.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m
n
2n-1

The generalized FiboCatalan number for $r = 0$ is an integer:

$$J_{0,F}\frac{F_{2n}!}{F_n!F_{n+0+1}!} = \frac{F_1!}{F_0!}\frac{F_{2n}!}{F_n!F_{n+1}!} = C_{n,F} = S(1,n)_F.$$

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The generalized FiboCatalan number for $r = 1$ is:

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Are these numbers integers?

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Lemma 2: $F_{kn}F_{n+2} - F_{kn+2}F_n = (-1)^n F_{(k-1)n}$.

Theorem

(KK) $F_{2n+1}F_{2n}C_{n,F} - F_{n+1}F_nC_{n+1,F} = (-1)^n F_nF_{2n+1}\frac{F_{2n}!}{F_{n+2}!F_n!}$.

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Proof.

$$\begin{aligned}
F_{2n+1}F_{2n}C_{n,F} - F_{n+1}F_nC_{n+1,F} \\
&= \frac{F_{2n+1}F_{2n}F_{2n}!}{F_{n+1}F_n!F_n!} - \frac{F_{n+1}F_nF_{2n+2}!}{F_{n+2}F_{n+1}!F_{n+1}!} \\
&= F_{2n+1}F_{2n}F_{n+2}\frac{F_{2n}!}{F_{n+2}!F_n!} - F_{2n+2}F_{2n+1}F_n\frac{F_{2n}!}{F_{n+2}!F_n!} \\
&= F_{2n+1}[F_{2n}F_{n+2} - F_{2n+2}F_n]\frac{F_{2n}!}{F_{n+2}!F_n!} \\
&= F_{2n+1}(-1)^nF_n\frac{F_{2n}!}{F_{n+2}!F_n!}
\end{aligned}$$

It is well know that $F_{2n} = F_nF_{n+1} + F_nF_{n-1}$, thus the left side of Theorem 1 is equal to

$$F_{2n+1}[F_nF_{n+1} + F_nF_{n-1}]C_{n,F} - F_{n+1}F_nC_{n+1,F}$$

and is therefore divisible by F_n. Thus

$$\begin{aligned} F_{2n+1}F_{n+1}C_{n,F} + F_{2n+1}F_{n-1}C_{n,F} - F_{n+1}C_{n+1,F} \\ = (-1)^n F_{2n+1}\frac{F_{2n}!}{F_{n+2}!F_n!} \\ = (-1)^n \frac{1}{F_{n+2}}\binom{2n+1}{n}_F. \end{aligned}$$

Corollary

For $n \geq 1$,

$$F_{2n+1}\frac{F_{2n}!}{F_{n+2}!F_n!} = \frac{1}{F_{n+2}}\binom{2n+1}{n}_F$$

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Note this is not true for binomial coefficients! $n = 2$ fails to give an integer, for example.

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A well-known fact about Fibonacci numbers is that

$$gcd(F_n, F_m) = F_{gcd(m,n)}$$

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If $gcd(2n+1, n+2) = 3$ then $gcd(F_{2n+1}, F_{n+2}) = F_3 = 2$, so F_{n+2} divides $2C_{n,F}$.

Corollary

For $n \geq 1$, the generalized FiboCatalan number for $r = 1$,

$$\frac{2F_{2n}!}{F_{n+2}!F_n!} = \frac{1}{F_{n+2}} 2C_{n,F}$$

is an integer.

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Theorem

For $n \geq 1$, the super FiboCatalan number is an integer for $m = 2$. I.e.,

$$S(2, n)_F = \frac{6F_{2n}!}{F_n!F_{n+2}!}$$

is an integer.

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When $s = 0$ we have:

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which is an integer.

When $s = 2$,

$$\begin{aligned} S(m, m+2)_F &= \frac{F_{2m}!F_{2m+4}!}{F_m!F_{m+2}!F_{2m+2}!} \\ &= \frac{F_{2m}!F_{2m+4}F_{2m+3}F_{2m+2}!}{F_{m+2}F_{m+1}F_m!F_m!F_{2m+2}!} \\ &= \frac{1}{F_{m+1}} \frac{F_{2m}!}{F_m!F_m!} \frac{F_{2m+4}}{F_{m+2}} F_{2m+3} \\ &= F_{2m+3} C_{m,F} \frac{F_{2(m+2)}}{F_{m+2}}. \end{aligned}$$

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Since $F_{2n} = F_n F_{n-1} + F_n F_{n+1}$ then F_n divides F_{2n} so F_{m+2} divides $F_{2(m+2)}$. Therefore,

$$F_{2m+3} C_{m,F} \frac{F_{2(m+2)}}{F_{m+2}}$$

is an integer.

When $s = 3$,

$$
\begin{aligned}
S(m, m+3)_F &= \frac{F_{2m}!F_{2(m+3)}!}{F_m!F_{m+3}!F_{2m+3}!} \\
&= \frac{F_{2m}!F_{2m+6}F_{2m+5}F_{2m+4}F_{2m+3}!}{F_m!F_m!F_{m+1}F_{m+2}F_{m+3}F_{2m+3}!} \\
&= C_{m,F}\frac{F_{2m+6}}{F_{m+3}}\frac{F_{2m+4}}{F_{m+2}}F_{2m+5}
\end{aligned}
$$

which again is an integer since F_n divides F_{2n}.

The Lucas analogue of the generalized FiboCatalan number for $r = 0$ is equal to $C_{\{n\}}$ which is equal to $\frac{1}{\{2\}} S\{1, n\}$:

$$J_{\{0\}} \frac{\{2n\}!}{\{n\}!\{n+0+1\}!} = \frac{\{1\}!}{\{0\}!} \frac{\{2n\}!}{\{n\}!\{n+1\}!} = C_{\{n\}} = \frac{1}{\{2\}} \ S\{1, n\}.$$

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$$J_{\{0\}} \frac{\{2n\}!}{\{n\}!\{n+0+1\}!} = \frac{\{1\}!}{\{0\}!} \frac{\{2n\}!}{\{n\}!\{n+1\}!} = C_{\{n\}} = \frac{1}{\{2\}} \ S\{1, n\}.$$

The Lucas analogue of the generalized FiboCatalan number for $r = 1$ is:

$$\begin{aligned} J_{\{1\}} \frac{\{2n\}!}{\{n\}!\{n+1+1\}!} &= \frac{\{3\}!}{\{1\}!} \frac{\{2n\}!}{\{n\}!\{n+2\}!} \\ &= \{3\}! \frac{\{2n\}!}{\{n\}!\{n+2\}!} \\ &= \frac{\{2\}}{\{4\}} S(2, n)_F. \end{aligned}$$

These polynomials are polynomials with non-negative integer coefficients.

Theorem

$$\{2n+1\}\{2n\}C_{\{n\}} - \{n+1\}\{n\}C_{\{n+1\}} = (-1)^n t^n \{2\}\{n\}\{2n+1\}\frac{\{2n\}!}{\{n+2\}!\{n\}!}.$$

Corollary

For $n \geq 1$,

$$\{2n+1\}\{2\}\frac{\{2n\}!}{\{n+2\}!\{n\}!} = \{2\}\frac{1}{\{n+2\}}\left\{ \begin{array}{c} 2n+1 \\ n \end{array} \right\}$$

is a polynomial with non-negative integer coefficients.

Theorem

$S\{m, m+s\}$ is a polynomial with non-negative integer coefficients for $0 \le s \le 4$.

Theorem

$S\{m, m+s\}$ is a polynomial with non-negative integer coefficients for $0 \le s \le 4$.

When $s = 0$ we have:

$$S\{m, m\} = \frac{\{2m\}!\{2m\}!}{\{m\}!\{m\}!\{2m\}!} = \left\{ \begin{array}{c} 2m \\ m \end{array} \right\}$$

which is a polynomial with non-negative integer coefficients.

When $s = 1$,

$$S\{m, m+1\} = \frac{\{2m\}!\{2m+2\}!}{\{m\}!\{m+1\}!\{2m+1\}!} \quad (1)$$

$$= \frac{\{2m+2\}\{2m\}!}{\{m+1\}!\}m\}!} \quad (2)$$

$$= \{2m+2\}C_{\{m\}} \quad (3)$$

which is a polynomial with non-negative integer coefficients.

When $s = 2$,

$$\begin{aligned} S\{m, m+2\} &= \frac{\{2m\}!\{2m+4\}!}{\{m\}!\{m+2\}!\{2m+2\}!} \\ &= \frac{\{2m\}!\{2m+4\}\{2m+3\}\{2m+2\}!}{\{m+2\}\{m+1\}\{m\}!\{m\}!\{2m+2\}!} \\ &= \frac{1}{\{m+1\}} \frac{\{2m\}!}{\{m\}!\{m\}!} \frac{\{2m+4\}}{\{m+2\}} \{2m+3\} \\ &= \{2m+3\} C_{\{m\}} \frac{\{2(m+2)\}}{\{m+2\}}. \end{aligned}$$

When $s = 2$,

$$\begin{aligned} S\{m, m+2\} &= \frac{\{2m\}!\{2m+4\}!}{\{m\}!\{m+2\}!\{2m+2\}!} \\ &= \frac{\{2m\}!\{2m+4\}\{2m+3\}\{2m+2\}!}{\{m+2\}\{m+1\}\{m\}!\{m\}!\{2m+2\}!} \\ &= \frac{1}{\{m+1\}} \frac{\{2m\}!}{\{m\}!\{m\}!} \frac{\{2m+4\}}{\{m+2\}} \{2m+3\} \\ &= \{2m+3\} C_{\{m\}} \frac{\{2(m+2)\}}{\{m+2\}}. \end{aligned}$$

Since $\{2n\} = t\{n\}\{n-1\} + \{n\}\{n+1\}$ then $\{n\}$ divides $\{2n\}$ so $\{m+2\}$ divides $\{2(m+2)\}$. Therefore,

$$\{2m+3\} C_{\{m\}} \frac{\{2(m+2)\}}{\{m+2\}}$$

is a polynomial with non-negative integer coefficients.

When $s = 3$,

$$\begin{aligned} S\{m, m+3\} &= \frac{\{2m\}!\{2(m+3)\}!}{\{m\}!\{m+3\}!\{2m+3\}!} \\ &= \frac{\{2m\}!\{2m+6\}\{2m+5\}\{2m+4\}\{2m+3\}!}{\{m\}!\{m\}!\{m+1\}\{m+2\}\{m+3\}\{2m+3\}!} \\ &= C_{\{m\}} \frac{\{2m+6\}}{\{m+3\}} \frac{\{2m+4\}}{\{m+2\}} \{2m+5\} \end{aligned}$$

which again is a polynomial with non-negative integer coefficients since $\{n\}$ divides $\{2n\}$.

- Open Problems:
- The rest of the cases!
- A combinatorial interpretation of the super FiboCatalan numbers.
- Type B_n and others.
- Other identities

In addition, the super Catalan numbers satisfy a number of interesting binomial identities, such as this identity of von Szily (1894):

$$S(m,n) = \sum_{k\in\mathbb{Z}} (-1)^k \binom{2m}{m+k}\binom{2n}{n+k}.$$

Is there an analogue for the super FiboCatalan numbers?

Mikic recently proved the following alternating convolution formula for the super Catalan numbers:

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}S(k,l)S(2n-k,l)=S(n,l)S(n+l,n)$$

for all non-negative integers n and l. Mikic also proved a similar identity for the Catalan numbers:

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}C_k C_{2n-k}=C_n\binom{2n}{n}.$$

Is there an analogue for the super FiboCatalan numbers?