Polynomials and DP Colorings of Graphs

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Graph Coloring

- Color vertices so that any vertices with an edge between them must get different colors.
- A proper m-coloring of a graph G is a labeling $c : V(G) \to [m]$, such that $c(u) \neq c(v)$ whenever u and v are adjacent in G.
- Minimum number of colors needed for such a coloring is called the chromatic number $\chi(G)$ of the graph G.
- Each vertex has the same list of colors $[m]$ available to it.

List Coloring

- List coloring was introduced independently by Vizing (1976) and Erdős, Rubin, and Taylor (1979), as a generalization of usual graph coloring.

List Coloring

- For graph G suppose each $v \in V(G)$ is assigned a list, $L(v)$, of colors. We refer to L as a list assignment. If all the lists associated with the list assignment L have size m, we say that L is an m-assignment.
- An L-coloring for G is a proper coloring, f, of G such that $f(v) \in L(v)$ for all $v \in V(G)$.
- When an L-coloring for G exists, we say that G is L-colorable or L-choosable.

List Chromatic Number

- The list chromatic number of a graph G, written $\chi_\ell(G)$, is the smallest m such that G is L-colorable whenever $|L(v)| \geq m$ for each $v \in V(G)$.

List Chromatic Number

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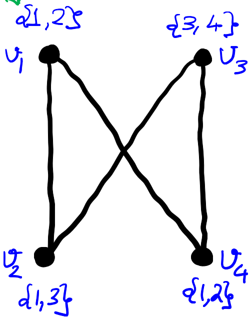
- Since usual coloring corresponds to a constant list assignment,

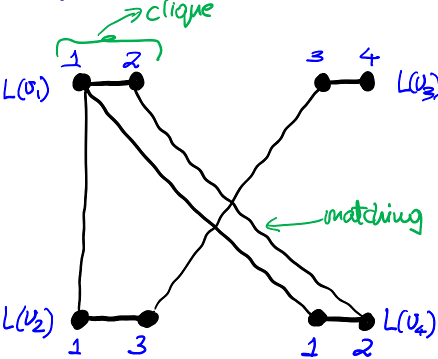
$$\chi(G) \leq \chi_\ell(G).$$

- The gap between $\chi(G)$ and $\chi_\ell(G)$ can be arbitrarily large: $\chi_\ell(K_{k,t}) = k + 1$ iff $t \geq k^k$.

A Different Perspective

List of colors





Proper L-coloring

$v_1 \to 2$

$v_2 \to 1$

$v_3 \to 3$

$v_4 \to 1$

corresponds to

Independent set of size 4 here.

$2 \in L(v_1)$, $1 \in L(v_2)$,
$3 \in L(v_3)$, $1 \in L(v_4)$

DP-Coloring

- In 2015, Dvořák and Postle introduced DP-coloring (they called it correspondence coloring) of graphs.
- Intuitively, DP-coloring considers the worst-case scenario of how many colors we need in the lists if we no longer can identify the names of the colors. Each vertex still gets a list of colors but identification of which colors are different can vary from edge to edge.

- A (DP-)cover of G is a pair $\mathcal{H} = (L, H)$ consisting of a graph H and a function $L : V(G) \to \mathcal{P}(V(H))$ satisfying:

  (1) the set $\{L(u) : u \in V(G)\}$ is a partition of $V(H)$;
  (2) for every $u \in V(G)$, the graph $H[L(u)]$ is complete;
  (3) if $E_H(L(u), L(v))$ is nonempty, then $u = v$ or $uv \in E(G)$;
  (4) if $uv \in E(G)$, then $E_H(L(u), L(v))$ is a matching (the matching may be empty).

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(DP-) Cover of a Graph

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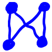
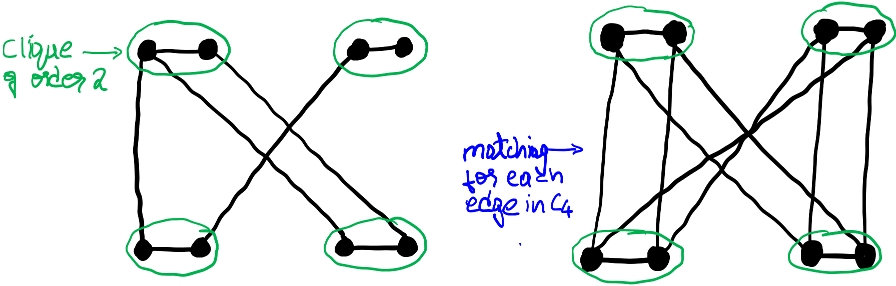
- See also "covering graphs", "Lifts". Studied since 1990s.

- Intuition:
  Blow up each vertex u in G into a clique of size $|L(u)|$;
  Add a matching (possibly empty) between any two such cliques for vertices u and v if uv is an edge in G.

(DP-) Cover of a Graph

- Intuition:
  Blow up each vertex u in G into a clique of size $|L(u)|$;
  Add a matching (possibly empty) between any two such cliques for vertices u and v if uv is an edge in G.
- A cover $\mathcal{H} = (L, H)$ is called m-fold if $|L(u)| = m$ for all u.
- Two 2-fold covers of C_4: 



DP-Chromatic Number of a Graph

- Given $\mathcal{H} = (L, H)$, a cover of G, an $\mathcal{H}$-coloring of G is an independent set in H of size $|V(G)|$. Equivalently, an independent transversal in $\mathcal{H}$.
- The DP-chromatic number of a graph G, $\chi_{DP}(G)$, is the smallest m such that G admits an $\mathcal{H}$-coloring for <u>every</u> m-fold cover $\mathcal{H}$ of G.

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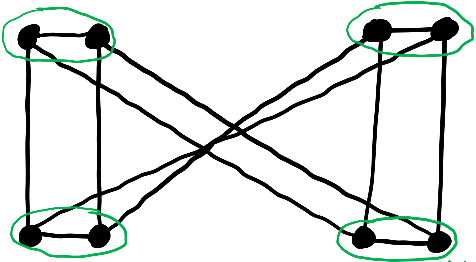
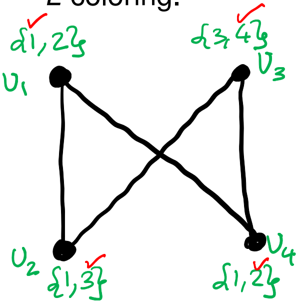
- $\chi_{DP}(C_4) > 2 = \chi_\ell(C_4)$:

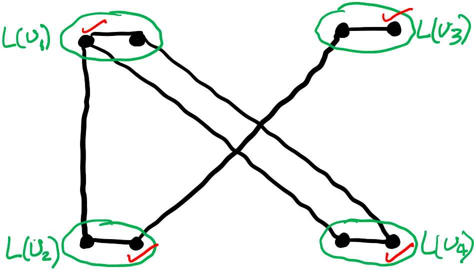


No independent set of size 4 in this 2-fold cover of C_4

DP-Coloring and List Coloring

- Given an m-assignment, L, for a graph G, it is easy to construct an m-fold cover $\mathcal{H}$ of G such that:
  G has an $\mathcal{H}$-coloring if and only if G has a proper L-coloring.





L-coloring

H-coloring

- $\chi(G) \leq \chi_\ell(G) \leq \chi_{DP}(G)$.

The Chromatic Polynomial

- Birkhoff 1912: For $m \in \mathbb{N}$, let $P(G, m)$ denote the number of proper colorings of G where the colors used come from $\{1, \ldots, m\}$.

- $P(G, m)$ is a polynomial in m of degree $|V(G)|$. We call $P(G, m)$ the chromatic polynomial of G.

- Explored deeply and widely in the past 100 years, and generalized in many different ways.

The List Color Function

- $P(G, L)$ be the number of proper L-colorings of G.
- Kostochka and Sidorenko 1990: The list color function $P_\ell(G, m)$ is the minimum value of $P(G, L)$ over all possible m-assignments L for G.
- In general, $P_\ell(G, m) \leq P(G, m)$.

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- In general, $P_\ell(G, m) \leq P(G, m)$.
- $P(K_{2,4}, 2) = 2$, and yet $P_\ell(K_{2,4}, 2) = 0$.
- $P_\ell(K_{3,26}, 3) \leq 3^8 2^{12} < 3^1 2^{26} \leq P(K_{3,26}, 3)$.

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Theorem (Kostochka, Sidorenko (1990); Kirov, Naimi (2016); K., Mudrock (2021))

1) $P_\ell(G, m) = P(G, m)$ for all m, if G is chordal.
2) $P_\ell(C_n, m) = P(C_n, m) = (m-1)^n + (-1)^n(m-1)$ for all m.
3) $P_\ell(C_n \vee K_k, m) = P(C_n \vee K_k, m)$ for all m.

The List Color Function

- $P_\ell(G, m) \le P(G, m)$. And for some G, $P_\ell(G, m) < P(G, m)$
- $P_\ell(G, m)$ need not be a polynomial, but it will equal the chromatic polynomial ultimately.

Theorem (Dong, Zhang (2022+); improving Wang, Qian, Yan (2017), Thomassen (2009), Donner (1992), question of Kostochka & Sidorenko (1990))

For any connected graph G with t edges, $P_\ell(G, m) = P(G, m)$ for $m > t - 1$.

The DP Color Function

- For $\mathcal{H} = (L, H)$, a cover of graph G, $P_{DP}(G, \mathcal{H})$ be the number of $\mathcal{H}$-colorings of G.
- K. and Mudrock 2021: The DP color function, $P_{DP}(G, m)$, is the minimum value of $P_{DP}(G, \mathcal{H})$ where the minimum is taken over all possible m-fold covers $\mathcal{H}$ of G.
- $P(C_4, 2) = P_\ell(C_4, 2) = 2$, and yet $P_{DP}(C_4, 2) = 0$.
- In general, $P_{DP}(G, m) \leq P_\ell(G, m) \leq P(G, m)$.

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- In general, $P_{DP}(G, m) \leq P_\ell(G, m) \leq P(G, m)$.

How is DP Color Function useful?

- Guaranteed number of DP-colorings regardless of the cover being used.

How is DP Color Function useful?

- Lower bound on both $P_\ell(G, m)$ and $P(G, m)$.

Theorem (Bernshteyn, Brazelton, Cao, Kang (2023))

For any triangle-free graph G with n vertices, t edges, $\Delta(G)$ large enough, and $m > (1 + o(1))\Delta(G)/\log \Delta(G)$, $P_{DP}(G, m) \geq (1 - \delta)^n (1 - \frac{1}{m})^t m^n$.

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Close to being sharp modulo the $(1-\delta)^n$ error term.

Proposition (K., Mudrock (2021))

For any graph G, $P_{DP}(G, m) \leq (1-\frac{1}{m})^{|E(G)|} m^{|V(G)|}$, for all m.

How is DP Color Function useful?

- Lower bound on both $P_\ell(G, m)$ and $P(G, m)$.

Theorem

- *Let G be a n-vertex planar graph.* $\chi_\ell(G), \chi_{DP}(G) \leq 5$.
  (Thomassen (2007a)) $P_\ell(G, 5) \geq 2^{n/9}$.

- *Let G be a n-vertex planar graph of girth at least 5.*
  $\chi_\ell(G), \chi_{DP}(G) \leq 3$.
  (Thomassen (2007b)) $P_\ell(G, 3) \geq 2^{n/10000}$.
  (Postle, Smith-Roberge (2022+)) $P_{DP}(G, 3) \geq 2^{n/292}$.
  (Dahlberg, K., Mudrock (2023+)) $P_{DP}(G, 3) \geq 3^{n/6}$.

How is DP Color Function useful?

- It can capture the behavior of extremal values:

How is DP Color Function useful?

Theorem (K., Mudrock, Sharma, Stratton (2023))

For any graphs G and H,

- $\chi_{DP}(G\square H) \leq \min\{\chi_{DP}(G) + col(H), \chi_{DP}(H) + col(G)\} - 1$.
- $\chi_{DP}(G\square K_{k,t}) = \chi_{DP}(G) + k$ *when* $t \geq (P_{DP}(G, \chi_{DP}(G) + k - 1))^k$.
- $\chi_{DP}(C_{2m+1}\square K_{k,t}) = k + 3$ *when* $t \geq \left(\frac{2k\ln(k+2)}{(k+1)!}\right)(P_{DP}(C_{2m+1}, k+2))^k$.
- $\chi_{DP}(C_{2m+1}\square K_{1,t}) = 4$ *iff* $t \geq \frac{P_{DP}(C_{2m+1},3)}{3} = \frac{2^{2m+1}-2}{3}$.
- $\chi_{DP}(C_{2m+2}\square K_{k,t}) = k + 3$ *when* $t \geq \left(\frac{2\ln(k+2)}{\lfloor (k+2)/2 \rfloor (k-1)!}\right)(P_{DP}(C_{2m+2}, k+2))^k$.
- $\chi_{DP}(C_{2m+2}\square K_{1,t}) = 4$ *iff* $t \geq P_{DP}(C_{2m+2}, 3) = 2^{2m+2} - 1$.

A Natural Question

We know:

Theorem (Dong, Zhang (2022+); improving Wang, Qian, Yan (2017), Thomassen (2009), Donner (1992), question of Kostochka & Sidorenko (1990))

For any connected graph G with t edges, $P_\ell(G, m) = P(G, m)$ for $m > t - 1$.

- For every graph G, does $P_{DP}(G, m) = P(G, m)$ for sufficiently large m?

DP Color Function is different

Theorem (K., Mudrock (2021))

If G is a graph with girth that is even, then there is an N such that $P_{DP}(G, m) < P(G, m)$ whenever $m \geq N$.

Furthermore, for any integer $g \geq 3$ there exists a graph G with girth g and an N such that $P_{DP}(G, m) < P(G, m)$ whenever $m \geq N$.

Theorem (Dong, Yang (2022))

If G contains an edge e such that the length of a shortest cycle containing e in G is even, then there exists $N \in \mathbb{N}$ such that $P_{DP}(M, m) < P(M, m)$ whenever $m \geq N$.

A Follow-up Natural Question

- For which graphs G does $P_{DP}(G, m) = P(G, m)$ for all m?
- For which graphs G does there exist N such that $P_{DP}(G, m) = P(G, m)$ for all $m \geq N$?

Theorem (K., Mudrock (2021))

If G is chordal, then $P_{DP}(G, m) = P(G, m)$ for every m.

- a straightforward application of perfect elimination ordering.

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Theta Graphs

- A Generalized Theta graph $\Theta(l_1, \ldots, l_k)$ consists of a pair of end vertices joined by k internally disjoint paths of lengths $l_1, \ldots, l_k$. $\Theta(l_1, l_2, l_3)$ is simply called a Theta graph.
- $P(\Theta(l_1, \ldots, l_k), m) = \frac{\prod_{i=1}^{k}((m-1)^{l_i+1}+(-1)^{l_i+1}(m-1))}{(m(m-1))^{k-1}} + \frac{\prod_{i=1}^{k}((m-1)^{l_i}+(-1)^{l_i}(m-1))}{m^{k-1}}.$
- Widely studied for many graph theoretic problems and are the main subject of two classical papers on the chromatic polynomial by Sokal, which include the celebrated result that the zeros of the chromatic polynomials of the Generalized Theta graphs are dense in the whole complex plane with the possible exception of the unit disc around the origin (by including the join of Generalized Theta graphs with K_2 this extends to all of the complex plane).

Theta Graphs

Extending results of K. and Mudrock (2021),

Theorem (Halberg, K., Liu, Mudrock, Shin, Thomason (2021+))

Let $G = \Theta(l_1, l_2, l_3)$ and $2 \leq l_1 \leq l_2 \leq l_3$.

(1) If the parity of l_1 is different from both l_2 and l_3, then $P_{DP}(G, m) = P(G, m)$ for all m.

(2) If the parity of l_1 is the same as l_2 and different from l_3, then for $m \geq 2$: $P_{DP}(G, m) =$

$$\frac{1}{m}\left[(m-1)^{l_1+l_2+l_3} + (m-1)^{l_1} - (m-1)^{l_2+1} - (m-1)^{l_3} + (-1)^{l_3+1}(m-2)\right].$$

(3) If the parity of l_1 is the same as l_3 and different from l_2, then for $m \geq 2$: $P_{DP}(G, m) =$

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(4) If l_1, l_2 and l_3 all have the same parity, then for $m \geq 3$: $P_{DP}(G, m) =$

$$\frac{1}{m}\left[(m-1)^{l_1+l_2+l_3} - (m-1)^{l_1} - (m-1)^{l_2} - (m-1)^{l_3} + 2(-1)^{l_1+l_2+l_3}\right].$$

Two Fundamental Questions

- For which graphs G does there exist N such that $P_{DP}(G, m) = P(G, m)$ for all $m \geq N$?

- Given a graph G does there always exist an $N \in \mathbb{N}$ and a polynomial $p(m)$ such that $P_{DP}(G, m) = p(m)$ whenever $m \geq N$?

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Generalized Theta Graphs

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Let $G = \Theta(l_1, \ldots, l_k)$ where $k \geq 2$, $l_1 \leq \cdots \leq l_k$, and $l_2 \geq 2$.

(i) If there is a $j \in \{2, \ldots, k\}$ such that l_1 and l_j have the same parity, then there is an $N \in \mathbb{N}$ such that $P_{DP}(G, m) < P(G, m)$ for all $m \geq N$.

(ii) If l_1 and l_j have different parity for each $j \in \{2, \ldots, k\}$, then there is an $N \in \mathbb{N}$ such that $P_{DP}(G, m) = P(G, m)$ for all $m \geq N$.

- Statement (i) does not answer the question of whether $P_{DP}(G, m)$ equals a polynomial for sufficiently large m. To answer that question, we study the DP color function of a class of graphs that contains all Generalized Theta graphs.

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(ii) If l_1 and l_j have different parity for each $j \in \{2, \ldots, k\}$, then there is an $N \in \mathbb{N}$ such that $P_{DP}(G, m) = P(G, m)$ for all $m \geq N$.

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Graphs with a Feedback Vertex Set of Order One

- A feedback vertex set of a graph is a subset of vertices whose removal makes the resulting induced subgraph acyclic. Clearly, a Generalized Theta graph has a feedback vertex set of size one.

Theorem (Halberg, K., Liu, Mudrock, Shin, Thomason (2021+))

Let G be a graph with a feedback vertex set of order one. Then there exists N and a polynomial $p(m)$ such that $P_{DP}(G, m) = p(m)$ for all $m \geq N$.

What is the polynomial?

Theorem (Halberg, K., Liu, Mudrock, Shin, Thomason (2021+))

Let G be a graph with a feedback vertex set of order one. Then there exists N and a polynomial $p(m)$ s.t. $P_{DP}(G, m) = p(m)$ for all $m \geq N$.

- There is no explicit formula for the polynomial $p(m)$ but we know its three highest degree terms are the same as $P(G, m)$.
- By extension of results of and answering a question of K. and Mudrock (2021),

Theorem (Mudrock, Thomason (2021))

For any graph G, $P(G, m) - P_{DP}(G, m) = O(m^{n-3})$ as $m \to \infty$.

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When does List Color Ftn equal Chromatic Poly?

- Given any graph G, the list color function number of G, denoted $\nu_\ell(G)$, is the smallest $m \geq \chi(G)$ such that $P_\ell(G, m) = P(G, m)$.
- The list color function threshold of G, denoted $\tau_\ell(G)$, is the smallest $k \geq \chi(G)$ such that $P_\ell(G, m) = P(G, m)$ for all $m \geq k$.

When does List Color Ftn equal Chromatic Poly?

- Given any graph G, the list color function number of G, denoted $\nu_\ell(G)$, is the smallest $m \geq \chi(G)$ such that $P_\ell(G, m) = P(G, m)$.

- The list color function threshold of G, denoted $\tau_\ell(G)$, is the smallest $k \geq \chi(G)$ such that $P_\ell(G, m) = P(G, m)$ for all $m \geq k$.

- By Donner's 1992 result, we know that both $\nu_\ell(G)$ and $\tau_\ell(G)$ are finite for any graph G. Furthermore, $\chi(G) \leq \chi_\ell(G) \leq \nu_\ell(G) \leq \tau_\ell(G)$.

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Theorem (Thomassen (2009))

$\tau_\ell(G) \leq |V(G)|^{10} + 1.$

Theorem (Wang, Qian, Yan (2017))

$\tau_\ell(G) \leq (|E(G)| - 1)/\ln(1 + \sqrt{2}) + 1.$

Theorem (Dong, Zhang (2022+))

$\tau_\ell(G) \leq (|E(G)| - 1).$

When does List Color Ftn equal Chromatic Poly?

- Two well-known open questions on the list color function can be stated as:
  - Kirov and Naimi 2016: For every graph G, is it the case that $\nu_\ell(G) = \tau_\ell(G)$?
  - Thomassen 2009: Is there a universal constant μ such that for any graph G, $\tau_\ell(G) - \chi_\ell(G) \leq \mu$?

When does List Color Ftn equal Chromatic Poly?

- Kirov and Naimi 2016: For every graph G, is it the case that $\nu_\ell(G) = \tau_\ell(G)$?

  A question of stickiness: Do the list color function and the corresponding chromatic polynomial of a graph stay the same after the first point at which they are both nonzero and equal?

  Still Open. But corresponding DP color function question has been answered negatively.

When does List Color Ftn equal Chromatic Poly?

- Thomassen 2009: Is there a universal constant μ such that for any graph G, $\tau_\ell(G) \leq \chi_\ell(G) + \mu$?

  The answer is no in a very strong sense.

Theorem (K., Kumar, Mudrock, Rewers, Shin, To (2022+))

There is a constant $C > 0$ such that for each $l \geq 16$, $\tau_\ell(K_{2,l}) - \chi_\ell(K_{2,l}) = \tau_\ell(K_{2,l}) - 3 \geq C\sqrt{l}$.

When does List Color Ftn equal Chromatic Poly?

- Threshold Extremal functions:
  $\delta_{max}(t) = \max\{\tau_\ell(G) - \chi_\ell(G) : |E(G)| \leq t\}$
  $\tau_{max}(t) = \max\{\tau_\ell(G) : |E(G)| \leq t\}$

Theorem (Wang et al. (2017) and K. et al. (2022+))
$C_1\sqrt{t} \leq \delta_{max}(t) \leq C_2 t$ for large enough t
$C_3\sqrt{t} \leq \tau_{max}(t) \leq C_2 t$ for large enough t

- What is the asymptotic behavior of $\delta_{max}(t)$?
  What is the asymptotic behavior of $\tau_{max}(t)$? In particular, is $\tau_{max}(t) = \omega(\sqrt{t})$?

  Since $\chi_\ell(G) = O(\sqrt{|E(G)|})$ as $|E(G)| \to \infty$,
  if $\tau_{max}(t) = \omega(\sqrt{t})$ as $t \to \infty$, then $\delta_{max}(t) \sim \tau_{max}(t)$ as $t \to \infty$.

When does DP Color Ftn equal Chromatic Poly?

- Given any graph G, the DP color function number of G, denoted $\nu_{DP}(G)$, is the smallest $m \geq \chi(G)$ such that $P_{DP}(G, m) = P(G, m)$.
  If $P(G, m) - P_{DP}(G, m) > 0$ for all m, we let $\nu_{DP}(G) = \infty$.

- The DP color function threshold of G, denoted $\tau_{DP}(G)$, is the smallest $k \geq \chi(G)$ such that $P_{DP}(G, m) = P(G, m)$ whenever $m \geq k$.
  If $P(G, m) - P_{DP}(G, m) > 0$ for infinitely many m, we let $\tau_{DP}(G) = \infty$.

When does DP Color Ftn equal Chromatic Poly?

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- $\chi(G) \leq \chi_\ell(G) \leq \chi_{DP}(G) \leq \nu_{DP}(G) \leq \tau_{DP}(G)$.

When does DP Color Ftn equal Chromatic Poly?

- We can now ask two natural questions about the DP color function:
  - For every graph G, is it the case that $\nu_{DP}(G) = \tau_{DP}(G)$?
  - When is $\tau_{DP}(G)$ finite?
    Find any universal bounds on τ_{DP}.
    Mostly wide open. Some results with Becker, Hewitt, Maxfield, Mudrock, Spivey, Thomason, Wagstrom (2021+).

When does DP Color Ftn equal Chromatic Poly?

- Kirov and Naimi 2016: For every graph G, is it the case that $\nu_\ell(G) = \tau_\ell(G)$?

  Still Open.

- For every graph G, is it the case that $\nu_{DP}(G) = \tau_{DP}(G)$? No!

Theorem (K., Maxfield, Mudrock, Thomason (2022+))

If G is $\Theta(2,3,3,3,2)$ or $\Theta(2,3,3,3,3,3,2,2)$, then $P_{DP}(G,3) = P(G,3)$ and there is an N such that $P_{DP}(G,m) < P(G,m)$ for all $m \geq N$.

- Only two counterexamples!

Polynomial Method

In a survey article, Terrence Tao describes the polynomial method as:

“strategy is to capture the arbitrary set of objects in the zero set of a polynomial whose degree is in control; for instance the degree may be bounded by a function of the number of the objects.”

Then we use algebraic tools to understand this zero set.

This paradigm has been used for breakthrough results in arithmetic combinatorics, additive combinatorics, number theory, graph theory, discrete geometry, and more.

Combinatorial Nullstellentsatz

- How many zeros can a n-variable polynomial on a field $\mathbb{F}$ have?

Lemma

Let $f \in \mathbb{F}[x_1, \ldots, x_n]$. For each i, let the degree of f in x_i be at most t_i, and suppose S_i is a set of more than t_i distinct values from $\mathbb{F}$. If $f(x_1, \ldots, x_n) = 0$ for $(x_1, \ldots, x_n) \in \prod_{i=1}^n S_i$, then f is the zero polynomial.

Can we do better? Instead of controlling the individual degree of each variable, work with the total degree of the polynomial.

Combinatorial Nullstellentsatz

Theorem (Combinatorial Nullstellensatz; Alon (1999))

Suppose that $f \in \mathbb{F}[x_1, \ldots, x_n]$, and the degree of f is at most $\sum_{i=1}^{n} t_i$. For each $i \in \{1, \ldots, n\}$, suppose that S_i is a set of elements in $\mathbb{F}$ with $|S_i| > t_i$.

If $[\prod_{i=1}^{n} x_i^{t_i}]_f \neq 0$, then $f(s_1, \ldots, s_n) \neq 0$ for some $(s_1, \ldots, s_n) \in \prod_{i=1}^{n} S_i$.

- $[\prod_{i=1}^{n} x_i^{t_i}]_p$ denotes the element of $\mathbb{F}$ that is the coefficient of the monomial $\prod_{i=1}^{n} x_i^{t_i}$ in the expanded form of $p \in \mathbb{F}[x_1, \ldots, x_n]$.

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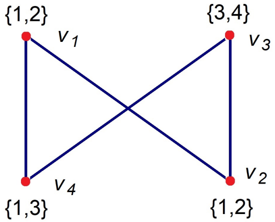
- Combinatorial Nullstellensatz has been applied to numerous problems in additive combinatorics, number theory, discrete geometry, graph theory since 1980s.

Graph Polynomial

- The graph polynomial of G with $V(G) = \{v_1, \ldots, v_n\}$ is $f_G(x_1, x_2, \ldots, x_n) = \prod_{v_i v_j \in E(G),\, j>i} (x_i - x_j)$.



$f_G(x_1, x_2, x_3, x_4) = (x_1 - x_2)(x_2 - x_3)(x_3 - x_4)(x_1 - x_4)$

- f_G is homogenous of degree $|E(G)|$.
- If L is a list assignment for G with $L(v) \subset \mathbb{R}$ for $v \in V(G)$, then a proper L-coloring of G exists if and only if there is a $(c_1, \ldots, c_n) \in \prod_{i=1}^{n} L(v_i)$ such that $f_G(c_1, \ldots, c_n) \neq 0$.
- $f_G(1, 2, 4, 3) = (-1)(-2)(1)(-2) = -4$ (In fact, $\chi_\ell(C_4) \leq 2$)

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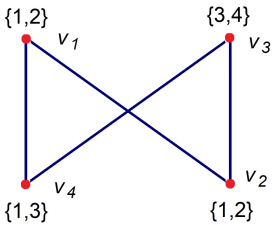

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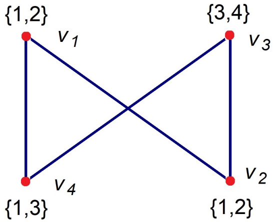

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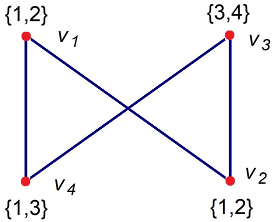

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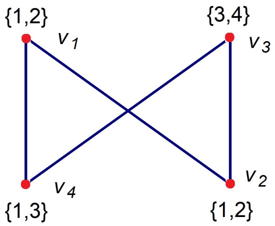
Combinatorial Nullstellensatz and List Coloring



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- Suppose $S_1 = S_2 = \{1, 2\}$, $S_3 = \{3, 4\}$, and $S_4 = \{1, 3\}$.
- Since $[x_1 x_2 x_3 x_4]_f = -2 \neq 0$, the CN tells us there is an element in $\prod_{i=1}^{4} S_i$ for which f is nonzero.
- Alon-Tarsi (1990) famously gave a combinatorial interpretation of this non-zero coefficient of the graph polynomial. *A fundamental method for bounding the list chromatic number: $\chi_\ell(G) \leq AT(G)$.*

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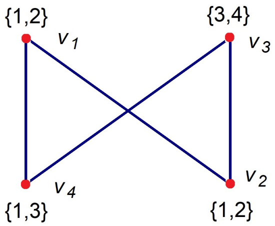

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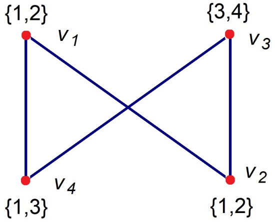

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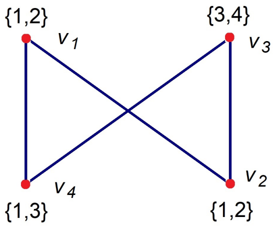

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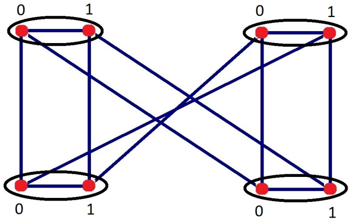
Combinatorial Nullstellensatz and DP Coloring

- We saw $\chi_\ell(C_4) \leq AT(C_4) \leq 2$, but we know $\chi_{DP}(C_4) > 2$.
- Intuitively, the issue with applying the Combinatorial Nullstellensatz in the DP-context is that which "colors" are <u>different</u> can vary from edge to edge.



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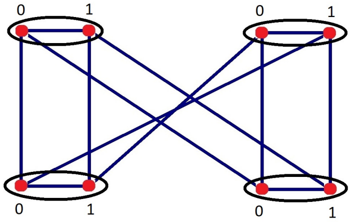

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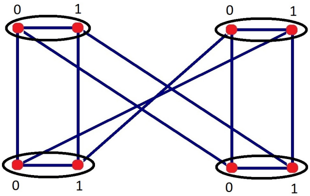
- To (partially) overcome this issue we view graph polynomials as having coefficients in some finite field.

Prime Covers

- Given a graph G and a function $f : V(G) \to \mathbb{N}$, we say $\mathcal{H} = (L, H)$ is an f-cover of G if $|L(u)| = f(u)$ for each $u \in V(G)$. We say that G is f-DP-colorable if G is $\mathcal{H}$-colorable whenever $\mathcal{H}$ is an f-cover of G.
- An f-cover $\mathcal{H} = (L, H)$ of G is a prime cover of G of order t whenever t is a power of a prime and $\max_{v \in V(G)} f(v) \leq t$.
- When the choice of t is implicitly known, we simply say prime cover or prime f-cover.
- If $\mathcal{H} = (L, H)$ is a prime cover of G of order t, we assume that $L(v) \subseteq \{(v, j) : j \in \mathbb{F}_t\}$ for each $v \in V(G)$.



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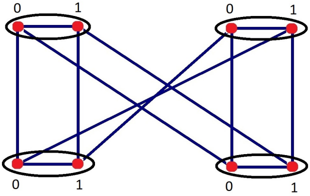

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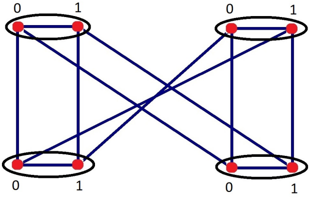

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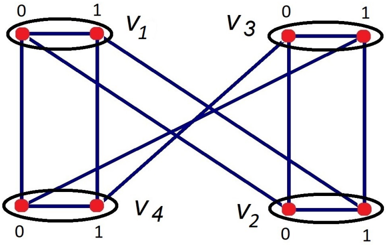
Saturation Functions

- $V(G) = \{v_1, \ldots, v_n\}$.
- $\mathcal{H} = (L, H)$ be a prime cover of G of order t.
- For each $v_i v_j \in E(G)$, the saturation function associated with $E_H(L(v_i), L(v_j))$ is denoted $\sigma^{\mathcal{H}}_{v_i v_j}$.



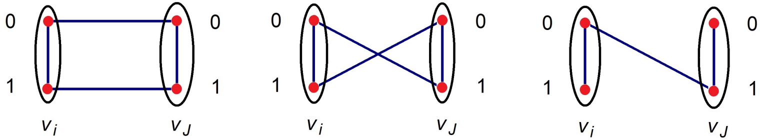
For example, $\sigma^{\mathcal{H}}_{v_3 v_4}(0) = 1$ and $\sigma^{\mathcal{H}}_{v_3 v_4}(1) = 0$.

Good Saturation Functions

- We say that $\sigma^{\mathcal{H}}_{v_i v_j}$ is good if there is a $\beta \in \mathbb{F}_t$ such that for each a in the domain of $\sigma^{\mathcal{H}}_{v_i v_j}$

$$a - \sigma^{\mathcal{H}}_{v_i v_j}(a) = \beta$$

  where subtraction is performed in $\mathbb{F}_t$.
- For a 2-fold cover, each saturation function associated with a matching, must be good!



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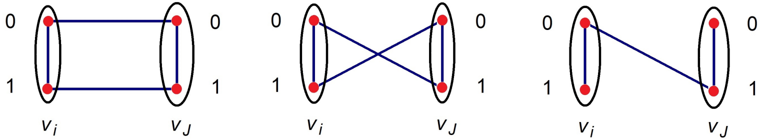
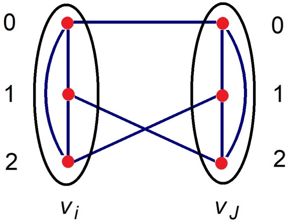


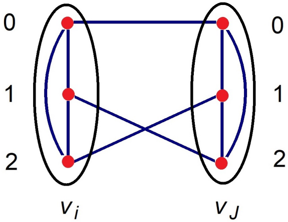
Good Covers

- Suppose $\mathcal{H} = (L, H)$ is a prime cover of G of order t.
- We say that $\mathcal{H}$ is a good prime cover of order t if for each $v_i v_j \in E(G)$ with $j > i$, the associated saturation function $\sigma^{\mathcal{H}}_{v_i v_j}$ is good.
- For example, we know every 2-fold cover is a good prime cover of order 2.
- It **cannot** be said that every 3-fold cover is a good prime cover of order 3.



Good Covers

- Suppose $\mathcal{H} = (L, H)$ is a prime cover of G of order t.
- We say that $\mathcal{H}$ is a good prime cover of order t if for each $v_i v_j \in E(G)$ with $j > i$, the associated saturation function $\sigma^{\mathcal{H}}_{v_i v_j}$ is good.
- For example, we know every 2-fold cover is a good prime cover of order 2.
- It **cannot** be said that every 3-fold cover is a good prime cover of order 3.



Key Observations

- Suppose G is a graph with $V(G) = \{v_1, \ldots, v_n\}$ and $\mathcal{H} = (L, H)$ is a good prime cover of G of order t.
- For each $v_i v_j \in E(G)$ with $j > i$, there is a $\beta_{i,j} \in \mathbb{F}_t$ such that $a - \sigma^{\mathcal{H}}_{v_i v_j}(a) - \beta_{i,j} = 0$ for each a in the domain of $\sigma^{\mathcal{H}}_{v_i v_j}$.
- Let $\hat{f}(x_1, \ldots, x_n) = \prod_{v_i v_j \in E(G),\, j > i} (x_i - x_j - \beta_{ij})$.
- An $\mathcal{H}$-coloring of G exists if there is a $(p_1, \ldots, p_n) \in \prod_{i=1}^{n} P_i$ such that $\hat{f}(p_1, p_2, \ldots, p_n) \neq 0$, where $P_i = \{j \in \mathbb{F}_t : (v_i, j) \in L(v)\}$.
- Note that if $\sum_{i=1}^{n} t_i = |E(G)|$, then $\left[\prod_{i=1}^{n} x_i^{t_i}\right]_{\hat{f}} = \left[\prod_{i=1}^{n} x_i^{t_i}\right]_{f_G}$. So, the Combinatorial Nullstellensatz can be applied.

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Combinatorial Nullstellensatz for DP Coloring

Theorem (K., Mudrock (2020))

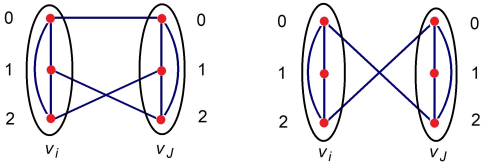
Let $\mathcal{H} = (L, H)$ be a good prime cover of order t of a graph G. Suppose that $f_G \in \mathbb{F}_t[x_1, \ldots, x_n]$. If $[\prod_{i=1}^n x_i^{t_i}]_{f_G} \neq 0$ and $|L(v_i)| > t_i$ for each $i \in [n]$, then there is an $\mathcal{H}$-coloring of G.

With applications to:

- f-DP-coloring of a cone of a connected bipartite graph.
- DP-coloring analogue of a Theorem of Akbari et al. (2006) on a sufficient condition for f-choosability in terms of unique colorability.
- Completely determine the DP-chromatic number of squares of all cycles.
- Algebraic sufficient condition for 3-DP-colorability.

Three-fold Covers

- Suppose that $\mathcal{H}$ is a prime cover of G of order 3.
- If $\sigma^{\mathcal{H}}_{v_i v_j}$ is bad, there is a $\beta_{i,j} \in \mathbb{F}_3$ so that $a + \sigma^{\mathcal{H}}_{v_i v_j}(a) = \beta_{i,j}$.



- So, for any $v_i v_j \in E(G)$ there is a $c_{i,j}, \beta_{i,j} \in \mathbb{F}_3$ so that

$$a + (-1)^{c_{i,j}} \sigma^{\mathcal{H}}_{v_i v_j}(a) = \beta_{i,j}$$

for each a in the domain of $\sigma^{\mathcal{H}}_{v_i v_j}$.

Three-fold Covers

Theorem (K., Mudrock (2020))

Suppose G is a graph with $\chi_{DP}(G) \geq 2$ and $V(G) = \{v_1, \ldots, v_n\}$. Let $\mathcal{F} \subseteq \mathbb{F}_3[x_1, \ldots, x_n]$ be the set of at most $2^{|E(G)|}$ polynomials given by:
$\mathcal{F} = \left\{ \prod_{v_i v_j \in E(G),\, j>i} (x_i + b_{i,j} x_j) : b_{i,j} \in \{-1, 1\} \right\}.$
If for each $f \in \mathcal{F}$ there exists $(t_1, t_2, \ldots, t_n) \in \prod_{i=1}^{n} \{0, 1, 2\}$ such that $[\prod_{i=1}^{n} x_i^{t_i}]_f \neq 0$, then $\chi_{DP}(G) \leq 3$.

- The number of polynomials in set $\mathcal{F}$ can be reduced to $2^{|E(G)|-|V(G)|+1}$, when G is a connected graph containing a cycle.

Combinatorial Nullstellensatz for DP-color Function

Theorem (Alon, Füredi (1993))

Let $\mathbb{F}$ be an arbitrary field, let $A_1, A_2, \ldots, A_n$ be any non-empty subsets of $\mathbb{F}$, and let $B = \prod_{i=1}^{n} A_i$. Suppose that $P \in \mathbb{F}[x_1, \ldots, x_n]$ is a polynomial of degree d that does not vanish on all of B. Then, the number of points in B for which P has a non-zero value is at least $\min \prod_{i=1}^{n} q_i$ where the minimum is taken over all integers q_i such that $1 \leq q_i \leq |A_i|$ and $\sum_{i=1}^{n} q_i \geq -d + \sum_{i=1}^{n} |A_i|$.

with $\hat{f}(x_1, \ldots, x_n) = \prod_{v_i v_j \in E(G),\, j>i} (x_i + (-1)^{c_{ij}} x_j - \beta_{ij})$ gives:

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Theorem (Dahlberg, K., Mudrock (2023+))

Let G be a graph with $\chi_{DP}(G) \leq 3$. Suppose that $|V(G)| = n$, $|E(G)| = l$, and $2n \geq l$. Then, $P_{DP}(G, 3) \geq 3^{n-l/2}$.

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Corollary

Let G be an n-vertex planar graph of girth at least 5. Then, $P_{DP}(G, 3) \geq 3^{n/6}$.

- Previous best bounds: $P_\ell(G, 3) \geq 2^{n/10000}$ (Thomassen (2007b)), and $P_{DP}(G, 3) \geq 2^{n/292}$ (Postle, Smith-Roberge (2022+)).

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Corollary

There are infinitely many graphs G for which $\chi_{DP}(G) = 3$, $P_{DP}(G, 3) = P(G, 3)$, and there is an $N_G \in \mathbb{N}$ such that $P_{DP}(G, m) < P(G, m)$ whenever $m \geq N_G$.

- Previously only two such graphs were known.

Combinatorial Nullstellensatz for DP-color Function

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Thank You!

Questions?

- For which graphs G does $\exists N$ such that $P_{DP}(G,m) = P(G,m)$ for all $m \geq N$? That is, when is $\tau_{DP}(G)$ finite?
- Given a graph G does there always exist an $N \in \mathbb{N}$ and a polynomial $p(m)$ such that $P_{DP}(G,m) = p(m)$ whenever $m \geq N$?
- Given a graph G and $p \in \mathbb{N}$, what is the value of $\tau_{DP}(K_p \vee G)$?
- What is the asymptotic behavior of $\delta_{max}(t)$ and $\tau_{max}(t)$? In particular, is $\tau_{max}(t) = \omega(\sqrt{t})$?
- For fixed n what is the asymptotic behavior of $\tau_\ell(K_{n,l})$ as $l \to \infty$?
- Kirov and Naimi 2016: For every graph G, is it the case that $\nu_\ell(G) = \tau_\ell(G)$? That is, if $P_\ell(G,m) = P(G,m)$ for some $m \geq \chi(G)$, does it follow that $P_\ell(G,m+1) = P(G,m+1)$?

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Tools for DP Color Function - I

Classic tools like:

Lemma (from Whitney's Broken Circuit Theorem (1932))

G be a connected graph on n vertices and s edges with girth g. Suppose $P(G, m) = \sum_{i=0}^{n} (-1)^i a_i m^{n-i}$. Then, for $i = 0, 1, \ldots, g-2$
$a_i = \binom{s}{i}$ and $a_{g-1} = \binom{s}{g-1} - t$,
where t is the number of cycles of length g contained in G.

- Inclusion-Exclusion type arguments.
- AM-GM inequality, and its generalization the Rearrangement Inequality.
- Probabilistic arguments/ Random constructions.

Tools for DP Color Function - II

Proposition (K., Mudrock (2021))
$P_{DP}(G,m) \leq \frac{m^n (m-1)^{|E(G)|}}{m^{|E(G)|}}$ *for all m.*

- Expected number of independent transversals in a random *m*-fold cover.

Tools for DP Color Function - II

Proposition (K., Mudrock (2021))

$P_{DP}(G,m) \leq \frac{m^n(m-1)^{|E(G)|}}{m^{|E(G)|}}$ for all m.

- This upper bound is the same as the lower bound on $P(G,m)$ when G is bipartite, as claimed by the well-known Sidorenko's conjecture on counting homomorphisms from bipartite graphs.

Corollary (K., Mudrock (2021))

For any connected graph G,
$P_{DP}(G,m) = \frac{m^{|V(G)|}(m-1)^{|E(G)|}}{m^{|E(G)|}}$ for all m if and only if G is a tree.

Tools for DP Color Function - II

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Lemma (K., Mudrock (2021))

Let G be a graph with $e \in E(G)$.
If $m \geq 2$ and $P(G - \{e\}, m) < \frac{m}{m-1} P(G, m)$,
then $P_{DP}(G, m) < P(G, m)$.

Tools for DP Color Function - III

- Let $\mathcal{H} = (L, H)$ be an m-fold cover of G. We say that $\mathcal{H}$ has a canonical labeling if it is possible to name the vertices of H so that $L(u) = \{(u, j) : j \in [m]\}$ and $(u, j)(v, j) \in E(H)$ for each $j \in [m]$ whenever $uv \in E(G)$.
- When $\mathcal{H}$ has a canonical labeling, G has an $\mathcal{H}$-coloring if and only if G has a proper m-coloring.
- Trees have a canonical labeling.
- Using canonical labeling, we can develop tools to handle graphs that are close to being a forest.

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Tools for DP Color Function - III

- A sharp bound when removing an edge gives us a canonical labeling.

Lemma (K., Mudrock (2021))

Let $\mathcal{H} = (L, H)$ be an m-fold cover of G with $m \geq 2$. Suppose $e = uv \in E(G)$. Let $H' = H - E_H(L(u), L(v))$ so that $\mathcal{H}' = (L, H')$ is an m-fold cover of $G - \{e\}$. If $\mathcal{H}'$ has a canonical labeling, then

$$P_{DP}(G, \mathcal{H}) \geq P(G-e, m) - \max\left\{P(G-e, m) - P(G, m), \frac{P(G,m)}{m-1}\right\}$$

Moreover, there exists an m-fold cover of G, $\mathcal{H}^* = (L, H^*)$, s.t.

$$P_{DP}(G, \mathcal{H}^*) = P(G-e, m) - \max\left\{P(G-e, m) - P(G, m), \frac{P(G,m)}{m-1}\right\}.$$

- Next, a sharp bound when removing an induced P_3.

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Tools for DP Color Function - III

Lemma (K., Mudrock (2021))

Let $\mathcal{H} = (L, H)$ be an m-fold cover of G with $m \geq 3$. Let e_1, e_2 be the edges of an induced path P of length two.
Let $G_0 = G - \{e_1, e_2\}$, $G_1 = G - e_1$, $G_2 = G - e_2$, and G^* be the graph obtained by making P into K_3. Suppose $\mathcal{H}'$, the m-fold cover of G_0 induced by $\mathcal{H}$, has a canonical labeling. Let

$$A_1 = P(G_0, m) - P(G, m), A_2 = P(G_0, m) - P(G_2, m) + \frac{1}{m-1}P(G, m),$$

$$A_3 = P(G_0, m) - P(G_1, m) + \frac{1}{m-1}P(G, m),$$

$$A_4 = \frac{1}{m-1}\left(P(G_1, m) + P(G_2, m) + P(G^*, m) - P(G, m)\right), \text{ and}$$

$$A_5 = \frac{1}{m-1}\left(P(G_1, m) + P(G_2, m) - \frac{1}{m-2}P(G^*, m)\right).$$

Then, $P_{DP}(G, \mathcal{H}) \geq P(G_0, m) - \max\{A_1, A_2, A_3, A_4, A_5\}$.
Moreover, there exists an m-fold cover of G that achieves the equality.

Tools for DP Color Function - IV

- Clique-gluing and the closely related clique-sum are fundamental graph operations which have been used to give a structural characterization of many families of graphs.
- A simple example is that chordal graphs are precisely the graphs that can be formed by clique-gluings of cliques. While the most famous example would be Robertson and Seymour's seminal Graph Minor Structure Theorem characterizing minor-free families of graphs.

Tools for DP Color Function - IV

- We build a toolbox for studying K_p-gluings of graphs: Choose a copy of K_p contained in each G_i and form a new graph G ($\in \bigoplus_{i=1}^{n}(G_i, p)$), called a K_p-gluing of $G_1, \ldots, G_n$, from the union of $G_1, \ldots, G_n$ by arbitrarily identifying the chosen copies of K_p.

Tools for DP Color Function - IV

- Given vertex disjoint graphs $G_1, \ldots, G_n$, we define amalgamated cover, a natural analogue of "gluing" m-fold covers of each G_i together so that we get an m-fold cover for $G \in \bigoplus_{i=1}^{n}(G_i, p)$.

- We define separated covers, a natural analogue of "splitting" an m-fold cover of $G \in \bigoplus_{i=1}^{n}(G_i, p)$ into separate m-fold covers for each G_i.

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- We apply these ideas together with other tools to build a theory of DP Color Function of Clique-gluings of graphs and how the DP Color Function of such graphs compares with the corresponding chromatic polynomial.
  But that's another talk.