Rank Polynomials of Fence Posets are Unimodal

(joint work with Mohan Ravichandran)

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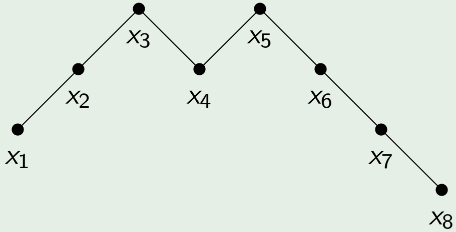
February 16, 2022

What are fences?

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s)$ be a composition of n. The fence poset of α, denoted $F(\alpha)$ is the poset on $x_1, x_2, \ldots, x_{n+1}$ with the order relations:

$$x_1 \preceq x_2 \preceq \cdots \preceq x_{\alpha_1+1} \succeq x_{\alpha_1+2} \succeq \cdots \succeq x_{\alpha_1+\alpha_2+1} \preceq x_{\alpha_1+\alpha_2+2} \preceq \cdots$$

Example ($\alpha = (2, 1, 1, 3)$)



For a composition of n, we get a poset of $n + 1$ nodes.

An ideal of a fence is a down-closed subset: $x \in I$, $y \preceq x \Rightarrow y \in I$.

$$\#I = rank(I)$$

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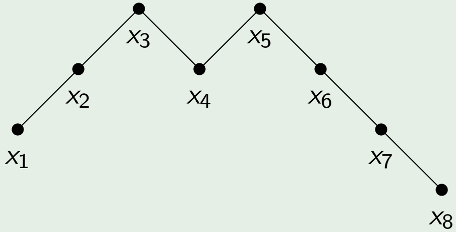

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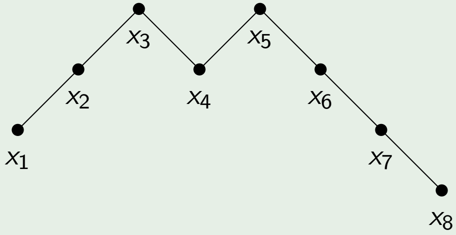

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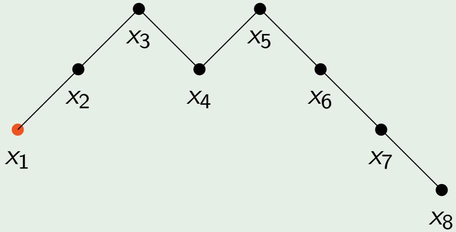

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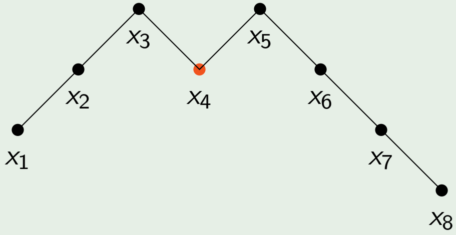

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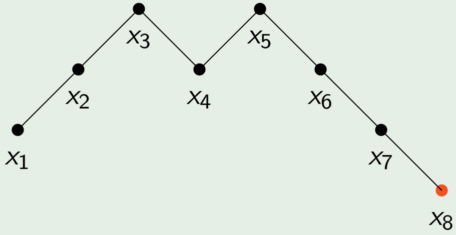

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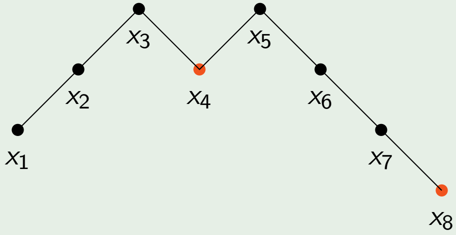

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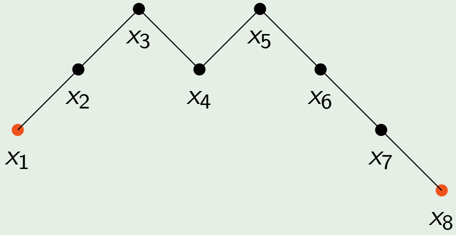

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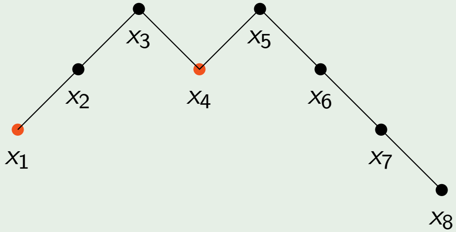

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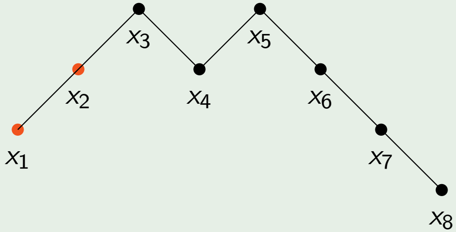

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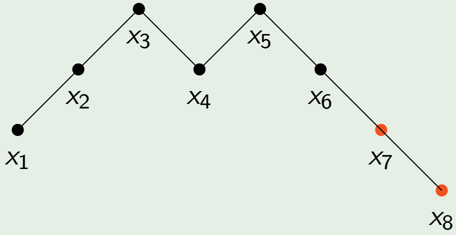

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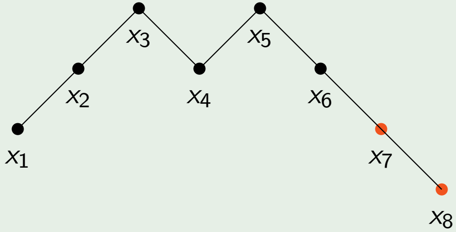

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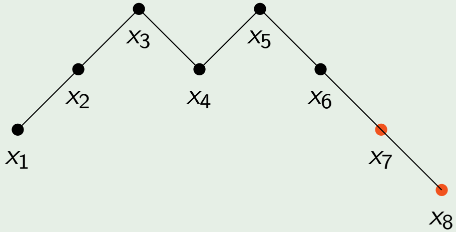

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$1+3q+5q^2+6q^3+6q^4+5q^5+3q^6+2q^7+q^8 \leftarrow$ Rank polynomial.

A q-deformation for rational numbers

Recently, a q-deformation rational numbers was introduced by Morier-Genoud and Ovsienko[1]. Their definition has a *convergence* property, which allows us to extend them to real numbers.

[1] Morier-Genoud and Ovsienko, "*q*-deformed rationals and *q*-continued fractions".

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For a given rational number r/s, we first write it as a continued fraction.

$$\frac{r}{s} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_{2m}}}}} = c_1 - \cfrac{1}{c_2 - \cfrac{1}{c_3 - \cfrac{1}{\ddots - \cfrac{1}{c_k}}}}$$

$$a_i \in \mathbb{Z},\ a_i \geq 1 \text{ for } i \geq 2 \qquad\qquad c_i \in \mathbb{Z},\ c_i \geq 2 \text{ for } i \geq 2$$

[1] Morier-Genoud and Ovsienko, "*q*-deformed rationals and *q*-continued fractions".

Then we replace the expansion terms with q-integers (q^{-1}-integers for a_{2k}), and the 1's with powers of q.

$$\left[\frac{r}{s}\right]_q := [a_1]_q + \cfrac{q^{a_1}}{[a_2]_{q^{-1}} + \cfrac{q^{-a_2}}{\ddots + \cfrac{q^{a_{2m-1}}}{[a_{2m}]_{q^{-1}}}}} = [c_1]_q - \cfrac{q^{c_1-1}}{[c_2]_q - \cfrac{q^{c_2-1}}{\ddots - \cfrac{q^{c_{k-1}-1}}{[c_k]_q}}}$$

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Also, when $\frac{r}{s} \geq 0$ the coefficients are non-negative.

Example

$$\frac{32}{9} = 3 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4}}} = 4 - \cfrac{1}{3 - \cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{2}}}}$$

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$$\left[\frac{32}{9}\right]_q = \frac{1+3q+5q^2+6q^3+6q^4+5q^5+3q^6+2q^7+q^8}{1+2q+2q^2+2q^3+q^4+q^5}.$$

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In general, if r/s corresponds to $[a_1, a_2, \ldots, a_{2m}]$, we have

$$\left[\frac{r}{s}\right]_q = \frac{\text{Rank polynomial for } (a_1 - 1, a_2, a_3, \ldots, a_{2m} - 1)}{\text{Rank polynomial for } (0, a_2 - 1, a_3, \ldots, a_{2m} - 1)}$$

A closer look at rank sequences for fences

$$\begin{array}{rcl}
(2,1,1,3) & \rightarrow & (1,3,5,6,6,5,3,2,1) \\
(3,1,1,2) & \rightarrow & (1,2,3,5,6,6,5,3,1) \\
(1,2,1,3) & \rightarrow & (1,3,5,6,6,5,4,2,1) \\
(1,1,2,3) & \rightarrow & (1,3,5,7,7,5,4,2,1) \\
(2,2,3) & \rightarrow & (1,2,4,5,6,6,4,2,1) \\
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\end{array}$$

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\end{aligned}$$

Conjecture (Morier-Genoud, Ovsienko, 2020)

The rank polynomials of fence posets are unimodal.

What more can we say?

Consider $(2, 1, 1, 3) \to (1, 3, 5, 6, 6, 5, 3, 2, 1)$.

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We have $1 \leq 1 \leq 2 \leq 3 \leq 3 \leq 5 \leq 5 \leq 6 \leq 6$.

We call such a sequence bottom-interlacing:

$$a_n \leq a_0 \leq a_{n-1} \leq a_1 \leq \ldots \leq a_{\lfloor n/2 \rfloor}. \qquad \text{(BI)}$$

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We call similarly have top-interlacing sequences:

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We call similarly have top-interlacing sequences:

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For example, the rank sequence $(1, 2, 4, 5, 6, 6, 4, 2, 1)$ of $(2, 2, 3)$ is top interlacing:

$$1 \leq 1 \leq 2 \leq 2 \leq 4 \leq 4 \leq 5 \leq 6 \leq 6.$$

What more can we say?

$$
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(2,1,1,3) &\rightarrow (1,3,5,6,6,5,3,2,1) \rightarrow \text{BI} \\
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Conjecture (McConville, Sagan, Smyth, 2021[2])

Suppose $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s)$.

(a) If $s = 1$ then $r(\alpha) = (1, 1, \ldots, 1)$ is symmetric.

(b) If s is even, then $r(\alpha)$ is bottom interlacing.

(c) If $s \geq 3$ is odd we have:

  (i) If $\alpha_1 > \alpha_s$ then $r(\alpha)$ is bottom interlacing.

  (ii) If $\alpha_1 < \alpha_s$ then $r(\alpha)$ is top interlacing.

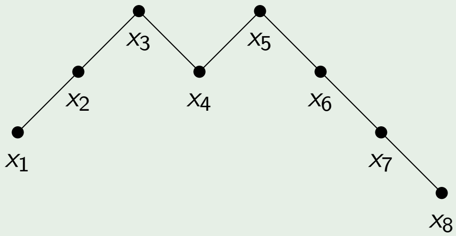
  (iii) If $\alpha_1 = \alpha_s$ then $r(\alpha)$ is symmetric, bottom interlacing, or top interlacing depending on whether $r(\alpha_2, \alpha_3, \ldots, \alpha_{s-1})$ is symmetric, top interlacing, or bottom interlacing, respectively.

[2] McConville, B. E. Sagan, and Smyth, *On a rank-unimodality conjecture of Morier-Genoud and Ovsienko*.

Idea

What if we close up the fence?

Example ($\alpha = (2, 1, 1, 3)$)



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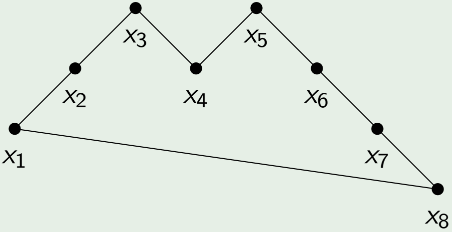

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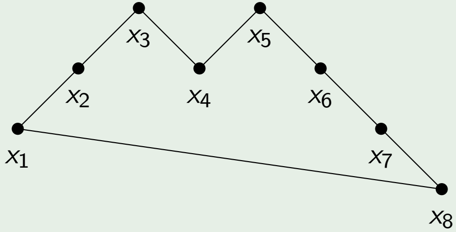

The *circular* fence has rank sequence $(1, 2, 3, 4, 4, 3, 2, 1)$.

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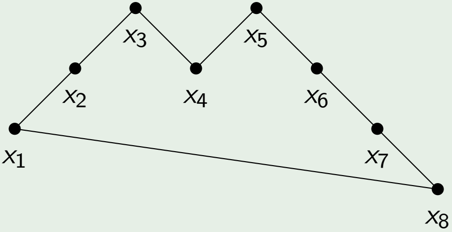

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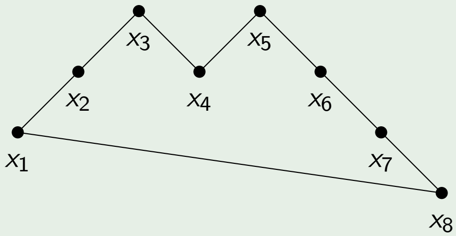

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It is symmetric. Is this always so?

Answer: Yes, but it is not trivial to prove.

Theorem (Kantarcı Oğuz, Ravichandran, 2021[3])

Rank polynomials of circular fence posets are symmetric.

---

[3] Kantarcı Oğuz and Ravichandran, *Rank Polynomials of Fence Posets are Unimodal*.

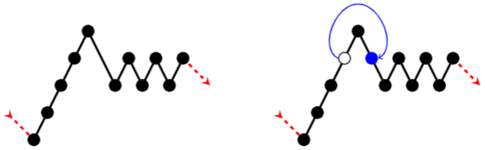
[4] Elizalde and B. Sagan, *Partial rank symmetry of distributive lattices for fences*.

Theorem (Kantarcı Oğuz, Ravichandran, 2021[3])

Rank polynomials of circular fence posets are symmetric.

Our proof:

We have one case that is trivially symmetric: $(k, 1, 1, \ldots, 1)$.



We show that moving a node from one segment to the next does not break symmetry.

[3] Kantarcı Oğuz and Ravichandran, *Rank Polynomials of Fence Posets are Unimodal*.

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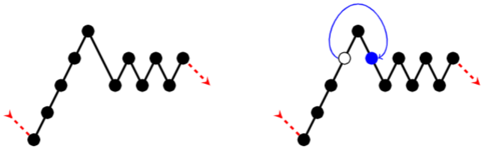
Our proof:

We have one case that is trivially symmetric: $(k, 1, 1, \ldots, 1)$.



We show that moving a node from one segment to the next does not break symmetry.

>> Recent bijective proof by Sagan and Elizalde[4].

[3]Kantarcı Oğuz and Ravichandran, *Rank Polynomials of Fence Posets are Unimodal*.

[4]Elizalde and B. Sagan, *Partial rank symmetry of distributive lattices for fences*.

The next step

There are several natural ways to associate a circular fence to a given fence.

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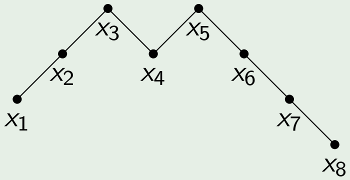
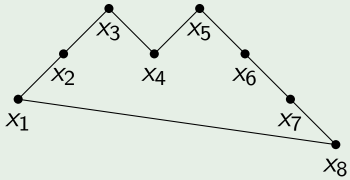
Example (Adding the relation $x_1 \succeq x_8$)





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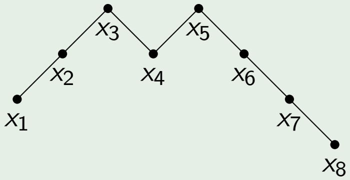

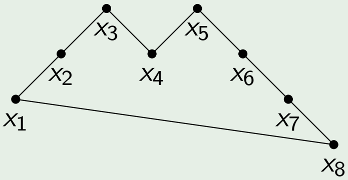

$$\sum_I q^{\mathrm{rank}(I)} = \sum_{\{I \mid x_1 \in I \Rightarrow x_8 \in I\}} q^{\mathrm{rank}(I)} + \sum_{\{I \mid x_1 \in I, x_8 \notin I\}} q^{\mathrm{rank}(I)}$$

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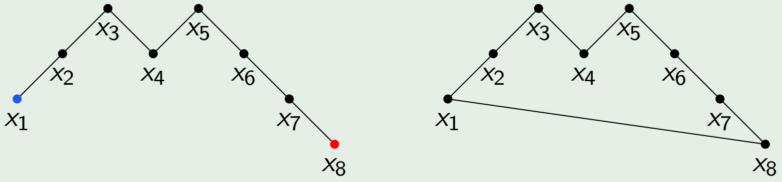

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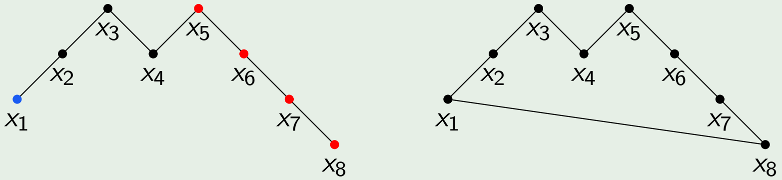

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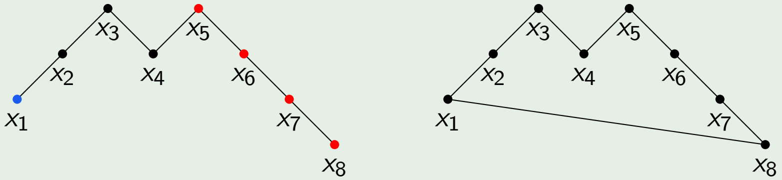

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circular rank polynomial

$q \times$ rank polynomial for $(1,1)$

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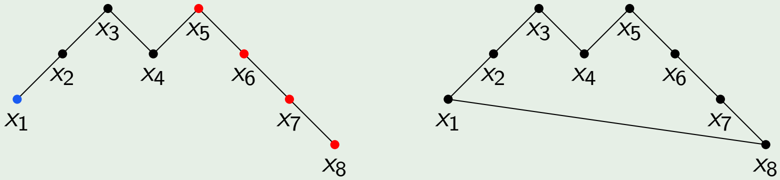

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circular rank polynomial *(symmetric)*	$q \times$ rank polynomial for $(1,1)$ *(smaller, shifted center)*

What does this tell us about the rank polynomial?

symmetric piece	$(1, 2, 3, 5, 5, 5, 3, 2, 1)$	$b_0 = b_n,\ b_1 = b_{n-1}, \ldots$
$+$	$+$	
smaller piece, shifted center	$(0, 1, 2, 1, 1, 0, 0, 0, 0)$	$c_0 \geq c_n,\ c_1 \geq c_{n-1}, \ldots$
$=$	$=$	
$\sum_I q^{\text{rank}(I)}$	$(1, 3, 5, 6, 6, 5, 3, 2, 1)$	$a_0 \geq a_n,\ a_1 \geq a_{n-1}, \ldots$

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This gives us half of the equations for being bottom interlacing:

$$a_n \leq a_0, \quad a_{n-1} \leq a_1, \quad a_{n-2} \leq a_2 \quad a_{n-3} \leq a_3, \ldots$$

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We need a way to shift the pairings to (a_0, a_{n-1}), (a_1, a_{n+1}), $\dots$ to get the rest of the inequalities.

Let us associate another circular fence to our fence.

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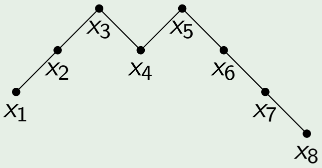
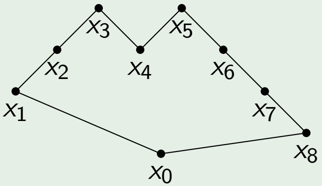
Example (Connecting x_8 and x_1 by a minimal node x_0)





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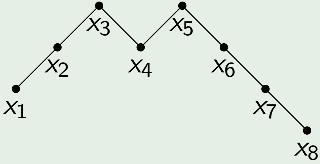

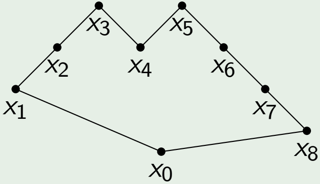

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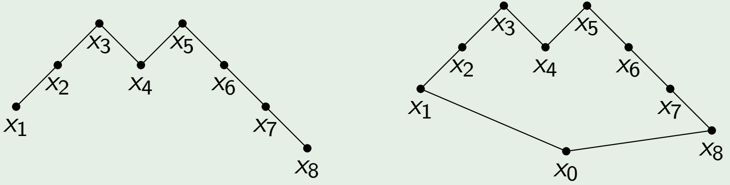

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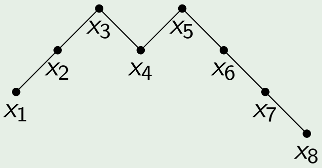

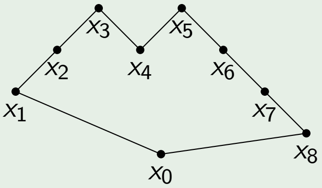

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$q \times$ rank polynomial for $(2,1,1,3)$	circular rank polynomial *(symmetric, shifted center)*	rank polynomial for (0) *(smaller, shifted center)*

On the rank polynomial side

symmetric piece larger	$(1,2,3,5,6,6,5,3,2,1)$	$b_0 = b_{n+1},\ b_1 = b_n, \dots$
$-$	$-$	
smaller piece, shifted center	$(1,1,0,0,0,0,0,0,0)$	$c_0 \geq c_n,\ c_1 \geq c_{n-1}, \dots$
$=$	$=$	
$(0, a_0, a_1, \dots, a_n)$	$(0,1,3,5,6,6,5,3,2,1)$	$0 \leq a_n,\ a_0 \leq a_{n-1}\ \dots$

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This gives us the other half of the bottom-interlacing equations:

$$a_n \leq a_0, \quad a_{n-1} \leq a_1, \quad a_{n-2} \leq a_2, \quad a_{n-3} \leq a_3, \dots$$

$$+$$

$$a_0 \leq a_{n-1}, \quad a_1 \leq a_{n-2}, \quad a_2 \leq a_{n-3}, \dots$$

$$=$$

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Theorem (Kantarcı Oğuz, Ravichandran, 2021)

Rank polynomials of fence posets are unimodal.

In particular, for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s)$ *we have:*

(a) *If* $s = 1$ *then* $r(\alpha) = (1, 1, \ldots, 1)$ *is symmetric.*
(b) *If* s *is even, then* $r(\alpha)$ *is bottom interlacing.*
(c) *If* $s \geq 3$ *is odd we have:*
  (i) *If* $\alpha_1 > \alpha_s$ *then* $r(\alpha)$ *is bottom interlacing.*
  (ii) *If* $\alpha_1 < \alpha_s$ *then* $r(\alpha)$ *is top interlacing.*
  (iii) *If* $\alpha_1 = \alpha_s$ *then* $r(\alpha)$ *is symmetric, bottom interlacing, or top interlacing depending on whether* $r(\alpha_2, \alpha_3, \ldots, \alpha_{s-1})$ *is symmetric, top interlacing, or bottom interlacing, respectively.*

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Are they also unimodal?

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Are they also unimodal? **Answer:** Not always.

For the circular poset $(1, a, 1, a)$ we get a small dip in the middle:

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We also know that if there is a problem with unimodality, it only happens in the middle.

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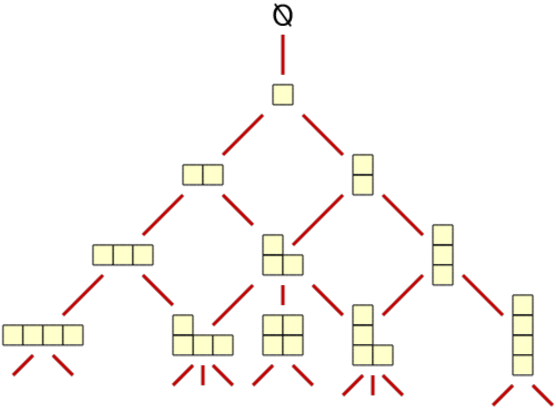
Conjecture (Kantarcı Oğuz, Ravichandran, 2021)

For any $\alpha \neq (1, k, 1, k)$ or $(k, 1, k, 1)$ for some k, the rank sequence $\overline{R}(\alpha; q)$ is unimodal.

Another Perspective

We can also see fences as intervals in the Young's lattice.

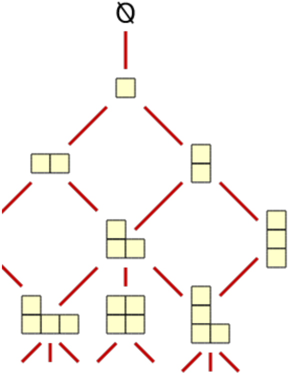
Young's Lattice is the lattice of Ferrers diagrams of Partitions ordered by inclusion.



(Image from Wikipedia, created by David Eppstein)

For any partition, we can look at the generating function of the partitions that lay under it.

$$G(\lambda; q) := \sum_{\mu \subset \lambda} q^{|\mu|}$$



$$G\left(\ydiagram{2,1}\,; q\right) = q^3 + 2q^2 + q + 1$$

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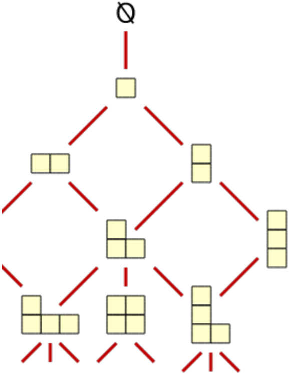

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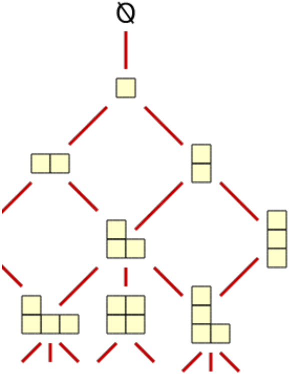

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We can also look at the interval between two partitions.

$$G(\lambda/\nu; q) := \sum_{\nu \subset \mu \subset \lambda} q^{|\mu| - |\nu|}$$

$$G\left(\ydiagram{2,1,1} / \ydiagram{1,1}; q\right) = q^2 + 2q + 1$$

Unimodality of these polynomials were considered by Stanton in 1990[5].

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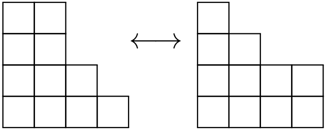

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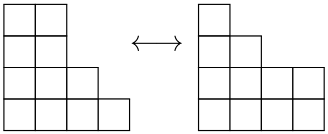

Conjecture (Stanton,1990)

The polynomials corresponding to self-dual partitions are unimodal.

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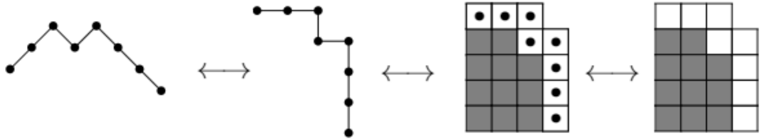
TABLE I

Partition	i	Values	Partition	i	Values
8 8 4 4	15	31 30 31	11 11 6 6	21	67 66 67
10 9 4 4	17	46 45 46	14 13 4 4	21	76 75 76
10 10 4 4	17	46 45 46	16 12 4 4	23	91 90 91
12 10 4 4	19	61 60 61	14 14 4 4	21	76 75 76
12 11 4 4	19	61 60 61	12 12 8 4	23	81 80 81
12 12 4 4	19	61 60 61	12 10 8 6	23	82 81 82
14 11 4 4	21	76 75 76	8 8 8 6 4 2	23	141 140 141
11 11 6 5	21	67 66 67	8 8 6 6 4 4	23	144 143 144
14 12 4 4	21	76 75 76			

(Table from "Unimodality and Young's Lattice", Stanton)

Given a fence, we can see it as a difference of two partitions α/ν.

Example $((2,1,1,3) \to (4,4,4,4,3)/(3,3,3,2))$



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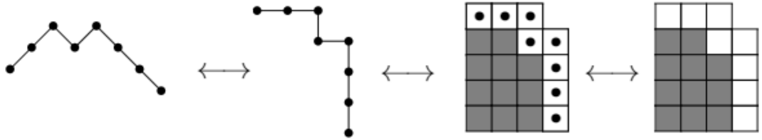

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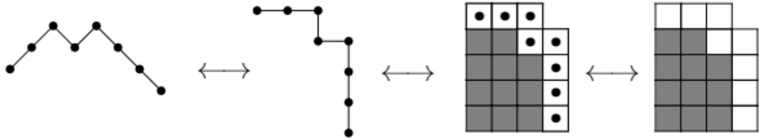

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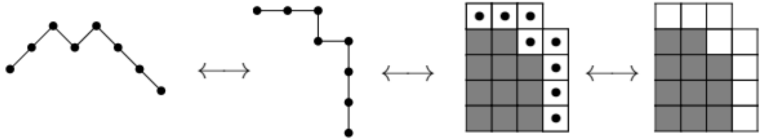
Rank polynomials actually correspond to a special class of differences called *ribbon diagrams*, where we have no 2×2 box.

Given a fence, we can see it as a difference of two partitions α/ν.

Example ($(2,1,1,3) \to (4,4,4,4,3)/(3,3,3,2)$)



Note that the ideals of the fence coincide with the partitions that lie between α and ν, so $G(\lambda/\nu)$ agrees with the rank polynomial.

Rank polynomials actually correspond to a special class of differences called *ribbon diagrams*, where we have no 2×2 box.

Polynomials corresponding to ribbon diagrams are unimodal.

Thank you for listening!

Further Reading

- Kantarcı Oğuz, E. & Ravichandran, M. Rank Polynomials of Fence Posets are Unimodal. (2021)
- Morier-Genoud, S. & Ovsienko, V. q-deformed rationals and q-continued fractions. *Forum Math. Sigma*. **8** pp. Paper No. e13, 55 (2020).
- McConville, T., Sagan, B. & Smyth, C. On a rank-unimodality conjecture of Morier-Genoud and Ovsienko. *Discrete Math.*. **344** pp. 13 (2021).
- Elizalde, S. & Sagan, B. Partial rank symmetry of distributive lattices for fences. (2022)
- Stanton, D. Unimodality and Young's lattice. *J. Comb. Theory, Ser. A.* **54**, 41-53 (1990)