Flattened Stirling Permutations and Type B Set Partitions

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Outline

- Brief history of Stirling permutations
- Definitions
  - Flattened statistic
  - Type B set partitions
- Sketch of our Main Theorem: A bijection between flattened Stirling permutations and type B set partitions.
- Future work: call for collaborators!

Notation:

For $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$

- let $[n] := \{1, 2, \ldots, n\}$
- let $[n]_2 := \{1, 1, 2, 2, \ldots, n, n\}$ be the multiset with each element in $[n]$ appearing twice
- let S_n denote the set of **permutations** of $[n]$.

Descents of permutations

Given a permutation on $[n]$ in one-line notation

$$\pi = \pi_1 \pi_2 \cdots \pi_n$$

we say that π has a **descent at** i if $\pi_i > \pi_{i+1}$.

The number of descents of π is denoted by $des(\pi)$.

Example

$$\pi \;=\; 8 \searrow 2 \nearrow 5 \searrow 4 \searrow 1 \nearrow 3 \nearrow 6 \nearrow 7 \;\in\; S_8$$

has descents at index 1, 3, and 4 and $des(\pi) = 3$.

History - A result of Euler

Theorem (Euler, 1749)

$$\sum_{m=0}^{\infty} m^n t^m = \frac{A_n(t)}{(1-t)^{n+1}}$$

where

$$A_n(t) = \sum_{\pi \in S_n} t^{des(\pi)}$$

which is called the Eulerian polynomial.

Example S_3 =	{123,	132,	213,	231,	312,	321}
number of descents	0	1	1	1	1	2

- $A_3(t) = 1t^0 + 4t^1 + 1t^2$

History - A result of Gessel and Stanley

Theorem (Euler, 1749)

$\sum_{m=0}^{\infty} m^n t^m = \frac{A_n(t)}{(1-t)^{n+1}}$ where $A_n(t) = \sum_{\pi \in S_n} t^{des(\pi)}$ which is called the Eulerian polynomial.

Theorem (Gessel and Stanley, 1978)

$$\sum_{m=0}^{\infty} S(m+n,n)t^m = \frac{\mathcal{Q}_n(t)}{(1-t)^{2n+1}}$$

where $S(m+n,n)$ are Stirling numbers of the second kind, and

$$\mathcal{Q}_n(t) = ?$$

which is called the Stirling polynomial and $\mathcal{Q}_n$ is the set of Stirling permutations of order n.

History - A result of Gessel and Stanley

Theorem (Gessel and Stanley, 1978)

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where $S(m+n,n)$ are Stirling numbers of the second kind, and

$$\mathcal{Q}_n(t) = \sum_{w \in \mathcal{Q}_n} t^{des(w)}$$

which is called the Stirling polynomial and $\mathcal{Q}_n$ is the set of Stirling permutations of order n.

*Recall that the Stirling numbers of the second kind, $S(n,k)$, count the number of set partitions of $[n]$ into k parts.

Stirling Permutations

A **Stirling permutation** of order n is a permutation on the multiset $[n]_2 = \{1, 1, 2, 2, \ldots, n, n\}$ such that

- numbers between the i values must be greater than i,

we refer to such values as being **nested** between i.

Example

Note

$$123321 \in \mathcal{Q}_3$$

because

1. $i = 1$, $2, 3 > 1$: 123321
2. $i = 2$, $3 > 2$: 123321
3. $i = 3$, no numbers are between 3: 123321

Stirling Permutations

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Example

$$123231 \notin \mathcal{Q}_3$$

1. Note that the value 2 lies between the 3's and so this is not a Stirling permutation.

Stirling Permutations

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- numbers between the i values must be greater than i,

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Theorem (Gessel and Stanley, 1978)

For $n \geq 1$,

$$|\mathcal{Q}_n| = (2n-1)!!$$

where $\ell!!$ denotes the the double factorial, which is the product of all the integers from 1 up to ℓ that have the same parity (odd or even) as ℓ.

Motivation for our work

- Like Euler, many have studied (discrete) statistics of permutations, such as descents.
- Recent work studied flattened permutations with k runs:
  - Olivia Nabawanda, Fanja Rakotondrajao, and Alex Bamunoba. *Run distribution over flattened partitions.* Journal of Integer Sequences, 23:Article 20.9.6, 10 2020.
- Our work extends this study to Stirling permutations.

Runs of a Stirling permutation

Given $w \in \mathcal{Q}_n$, the **runs** of w are the maximal contiguous weakly increasing subwords of w. We write

$$w = \sigma_1 \sigma_2 \cdots \sigma_r$$

where σ_i is a run.

Example

$$\underbrace{112299}_{\sigma_1}\ \underbrace{388}_{\sigma_2}\ \underbrace{3(10)(10)(11)(11)}_{\sigma_3}\ \underbrace{46677}_{\sigma_4}\ \underbrace{455}_{\sigma_5} \in \mathcal{Q}_{11}$$

Leading Terms of the Runs

Define $\sigma_{i,1}$ to be the initial value the run σ_i and call these values the **leading terms of the runs** of w.

Example

When

$$w = \underbrace{112299}_{\sigma_1}\ \underbrace{388}_{\sigma_2}\ \underbrace{3(10)(10)(11)(11)}_{\sigma_3}\ \underbrace{46677}_{\sigma_4}\ \underbrace{455}_{\sigma_5} \in \mathcal{Q}_{11}$$

the leading terms of the runs are:

- $\sigma_{1,1} = 1$
- $\sigma_{2,1} = 3$
- $\sigma_{3,1} = 3$
- $\sigma_{4,1} = 4$
- $\sigma_{5,1} = 4$

Flattened Stirling Permutations

If the leading terms of the runs are in weakly increasing order, i.e.

$$\sigma_{1,1} \leq \sigma_{2,1} \leq \cdots \leq \sigma_{r,1},$$

then we say the Stirling permutation w is **flattened**. Let $\text{flat}(\mathcal{Q}_n)$ denote the set of flattened Stirling permutations of order n.

Example

If $w = \underbrace{1233}_{\sigma_1}\ \underbrace{2}_{\sigma_2}\ \underbrace{1}_{\sigma_3} \in \mathcal{Q}_3$

then

$$\sigma_{1,1} \overset{?}{\leq} \sigma_{2,1} \overset{?}{\leq} \sigma_{3,1}$$

Flattened Stirling Permutations

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Example

If $w = \underbrace{1233}_{\sigma_1}\ \underbrace{2}_{\sigma_2}\ \underbrace{1}_{\sigma_3} \in \mathcal{Q}_3$

then

$$1 \ \leq \ 2 \not\leq 1.$$

Thus, $123321 \notin \text{flat}(\mathcal{Q}_3)$.

Flattened Stirling Permutations

If the leading terms of the runs are in weakly increasing order, i.e.

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then

$$\sigma_{1,1} \leq \sigma_{2,1} \leq \sigma_{3,1} \leq \sigma_{4,1} \leq \sigma_{5,1}$$

Flattened Stirling Permutations

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then

$$1\ \leq\ 3\ \leq\ 3\ \leq\ 4\ \leq\ 4.$$

Thus, $w \in \text{flat}(\mathcal{Q}_{11})$.

Research Question

How many flattened Stirling permutations of order n are there?

Computationally, we found:

n	1	2	3	4	5	6	7	8	9	10
$\lvert \text{flat}(\mathcal{Q}_n) \rvert$	1	2	6	24	116	648	4088	28640	219920	1832224

Table: Number of flattened Stirling permutations

Note that the cardinalities of $\text{flat}(\mathcal{Q}_n)$ are identically to the Dowling numbers, described in OEIS A007405.

"This is the number of type B set partitions, see R. Suter."
- Per W. Alexandersson, Dec 19. 2022

Goal: Give a bijection between flattened Stirling permutations of order n and type B set partitions.

Set partitions

A **set partition** of $[-n, n] = \{-n, -n+1, \dots, -1, 0, 1, \dots, n-1, n\}$ is a collection of sets

$$\pi = \{\beta_0, \beta_1, \beta_2, \dots, \beta_k\}$$

where β_i is a subset of $[-n, n]$ for all $1 \leq i \leq k$, such that

$$\cup_{i=0}^{k} \beta_i = [-n, n]$$

and are pair wise disjoint:

$$\beta_i \cap \beta_j = \emptyset$$

whenever $i \neq j$.

Note that each subset β_i in π is called a **block of** π.

Type B Set Partitions (Adler, 2016)

A set partition π of $[-n, n]$ is called a **type B set partition** if in addition it satisfies:

1. For any block $\beta \in \pi$, then $-\beta \in \pi$.
   We call β and $-\beta$ a **block pair** of π.

2. There is exactly one block β of π satisfying $\beta = -\beta$.
   We call this block, the **zero-block of π** as 0 must be an element of this block.

Notation:

- Π_n^B denotes type B set partitions of $[-n, n]$, and
- $\Pi_{n,m}^B$ denotes the type B set partitions with m exactly block pairs.

Example for Type B Set Partition

Example

Note that

$$\pi = \{\{0, 1, -1, 2, -2\}, \{3, -4\}, \{-3, 4\}, \{5\}, \{-5\}\}$$

is a type B set partition.

- Note π is a set partition.
- For $\beta = \{3, -4\} \in \pi$, then $-\beta = \{-3, 4\} \in \pi$.
- For $\beta = \{5\} \in \pi$, then $-\beta = \{-5\} \in \pi$.
- Zero-block: $\beta = \{0, 1, -1, 2, -2\}$, and $\beta = -\beta$.

Our Major Finding:

Theorem
For $n \geq 1$, the set Π^B_{n-1} is in bijection with the set $\mathrm{flat}(\mathcal{Q}_n)$.

Remainder of talk:

1. Next we introduce notation for type B set partitions, which we adapt from Adler.
2. Using this notation, we give a sketch of the bijection from Π^B_{n-1} to $\mathrm{flat}(\mathcal{Q}_n)$.

Notation for Proof, Thanks to Adler!

Encode $\pi \in \Pi^B_{n,m}$ as a sequence of subwords

$$\pi_0 | \pi_1 | \cdots | \pi_m$$

from the alphabet $0, 1, \ldots, n$.

- To begin keep the zero-block, and the block from each block pair that contains the smallest positive value.

Example

$$\pi = \left\{ \begin{array}{c} \{0\}, \{1\}, \{-1\}, \{4\}, \{-4\}, \{2, 7, -8\}, \{-2, -7, 8\}, \\ \{3, 5, 6, -9, -10\}, \{-3, -5, -6, 9, 10\} \end{array} \right\}.$$

Notation for Proof, Thanks to Adler!

Encode $\pi \in \Pi^B_{n,m}$ as a sequence of subwords

$$\pi_0 | \pi_1 | \cdots | \pi_m$$

from the alphabet $0, 1, \ldots, n$.

- To begin keep the zero-block, and the block from each block pair that contains the smallest positive value.

Example

$$\pi = \left\{ \begin{matrix} \{0\}, \{1\}, \cancel{\{-1\}}, \{4\}, \cancel{\{-4\}}, \{2, 7, -8\}, \cancel{\{-2, -7, 8\}}, \\ \{3, 5, 6, -9, -10\}, \cancel{\{-3, -5, -6, 9, 10\}} \end{matrix} \right\}.$$

Notation for Proof, Thanks to Adler!

Encode $\pi \in \Pi^B_{n,m}$ as a sequence of subwords

$$\pi_0 | \pi_1 | \cdots | \pi_m$$

from the alphabet $0, 1, \ldots, n$.

- Order the blocks (from left to right) by smallest nonnegative element.
- If $a < 0$, then write as $\overline{a}$. (*barred* elements)

Example

$$\{0\}, \{1\}, \{4\}, \{2, 7, -8\}, \{3, 5, 6, -9, -10\}.$$

Notation for Proof, Thanks to Adler!

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- Order the blocks (from left to right) by smallest nonnegative element.
- If $a < 0$, then write as $\overline{a}$. (*barred* elements)

Example

$$\{0\}, \{1\}, \{2, 7, \overline{8}\}, \{3, 5, 6, \overline{9}, \overline{10}\}, \{4\}$$

Notation for Proof, Thanks to Adler!

Encode $\pi \in \Pi^B_{n,m}$ as a sequence of subwords

$$\pi_0 | \pi_1 | \cdots | \pi_m$$

from the alphabet $0, 1, \ldots, n$.

- Form π_0 by listing positive elements of the zero-block in increasing order.

Example

$$\{0\}, \{1\}, \{2, 7, \overline{8}\}, \{3, 5, 6, \overline{9}, \overline{10}\}, \{4\}$$

$$\pi_0 = 0$$

Notation for Proof, Thanks to Adler!

Encode $\pi \in \Pi^B_{n,m}$ as a sequence of subwords

$$\pi_0 | \pi_1 | \cdots | \pi_m$$

from the alphabet $0, 1, \ldots, n$.

- Form π_i from the remaining blocks (in order listed) by
  1. placing the barred elements of the block in increasing order,
  2. followed by placing the non-barred elements in increasing order.

Example

$$\begin{array}{ccccc} \{0\} & \{1\} & \{2,7,\overline{8}\} & \{3,5,6,\overline{9},\overline{10}\} & \{4\} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \pi_0 = 0 & \pi_1 = 1 & \pi_2 = \overline{8}\ 2\ 7 & \pi_3 = \overline{9}\ \overline{10}\ 3\ 5\ 6 & \pi_4 = 4 \end{array}$$

Notation for Proof, Thanks to Adler!

Put the $\pi_0, \pi_1, \ldots, \pi_m$ into a set of blocks separated by dividers to get

$$\pi_0|\pi_1|\cdots|\pi_m \ . \tag{1}$$

Example

Thus

$$\pi = \left\{ \begin{matrix} \{0\}, \{1\}, \{-1\}, \{4\}, \{-4\}, \{2,7,-8\}, \{-2,-7,8\}, \\ \{3,5,6,-9,-10\}, \{-3,-5,-6,9,10\} \end{matrix} \right\}$$

is encoded by:

$$\pi_0|\pi_1|\pi_2|\pi_3|\pi_4 = 0 \mid 1 \mid \overline{8}\ 2\ 7 \mid \overline{9}\ \overline{10}\ 3\ 5\ 6 \mid 4.$$

Sketch of Proof ($\Rightarrow$)

Theorem

For $n \geq 1$, the set Π^B_{n-1} is in bijection with the set $\text{flat}(\mathcal{Q}_n)$.

Proof: ($\Rightarrow$) Given a type B set partition,

- Adler's notation to get $\pi_0|\pi_1|\cdots|\pi_m$
- partition each π_i further into $N_i P_i$
- bump up each of the number's magnitude
- duplicate each number and nest accordingly
- eliminate the bars and dividers

Sketch of (⇒) Through Example

Example

Proof: (⇒) Given a type B set partition,

$$\pi = \left\{ \begin{matrix} \{0\}, \{1\}, \{-1\}, \{4\}, \{-4\}, \{2, 7, -8\}, \{-2, -7, 8\}, \\ \{3, 5, 6, -9, -10\}, \{-3, -5, -6, 9, 10\}\} \end{matrix} \right\} \in \Pi^B_{10,4}$$

- Encoding the type B set partition using the adapted notation from Adler

to get:

$$0 \mid 1 \mid \overline{8}\ 2\ 7 \mid \overline{9}\ \overline{10}\ 3\ 5\ 6 \mid 4.$$

Sketch of (⇒) Through Example

Example

Take

$$0 \mid 1 \mid \overline{8}\ 2\ 7 \mid \overline{9}\ \overline{10}\ 3\ 5\ 6 \mid 4$$

- further partition π_i into $\pi_i = N_i P_i$, where:
  - N_i consists of the barred elements in π_i and
  - P_i consists of the positive values in π_i

to get:

$$\underbrace{0}_{P_0} \mid \underbrace{1}_{P_1} \mid \underbrace{\overline{8}}_{N_2}\underbrace{27}_{P_2} \mid \underbrace{\overline{9}(\overline{10})}_{N_3}\underbrace{356}_{P_3} \mid \underbrace{4}_{P_4}.$$

Sketch of ($\Rightarrow$) Through Example

Example

Take

$$\underbrace{0}_{P_0} \mid \underbrace{1}_{P_1} \mid \underbrace{\overline{8}}_{N_2}\underbrace{27}_{P_2} \mid \underbrace{\overline{9}(\overline{10})}_{N_3}\underbrace{356}_{P_3} \mid \underbrace{4}_{P_4}$$

- Increase the magnitude of each letter by

$$h(S) = \begin{cases} \{|i| + 1 \ : \ i \in S\} & \text{if } S \neq \emptyset \\ \emptyset & \text{if } S = \emptyset \end{cases}$$

  where
  - P_i follows the function,
  - N_i follows the function and puts a bar over the new number

to get:

$$\underbrace{1}_{P_0} \mid \underbrace{2}_{P_1} \mid \underbrace{\overline{9}}_{N_2}\underbrace{38}_{P_2} \mid \underbrace{(\overline{10})(\overline{11})}_{N_3}\underbrace{467}_{P_3} \mid \underbrace{5}_{P_4}.$$

Sketch of ($\Rightarrow$) Through Example

Example

Take

$$\underbrace{1}_{P_0} \mid \underbrace{2}_{P_1} \mid \underbrace{\overline{9}}_{N_2}\ \underbrace{38}_{P_2} \mid \underbrace{(\overline{10})(\overline{11})}_{N_3} \underbrace{467}_{P_3} \mid \underbrace{5}_{P_4}$$

- duplicate each letter and nest accordingly

$$f(N_i) = \begin{cases} s_1\ s_1\ s_2\ s_2\ \cdots\ s_k\ s_k & \text{if } N_i = \{|s_1| < |s_2| < |s_3| < \cdots < |s_k|\} \\ \emptyset & \text{if } N_i = \emptyset \end{cases}$$

$$g(P_i) = \begin{cases} s_1\ s_2\ s_2\ s_3\ s_3\ \cdots\ s_k\ s_k\ s_1 & \text{if } P_i = \{s_1 < s_2 < \cdots < s_k\} \\ \emptyset & \text{if } P_i = \emptyset \end{cases}$$

to get:

$$1\ 1 \mid 2\ 2 \mid \overline{9}\ \overline{9}\ 3\ 8\ 8\ 3 \mid (\overline{10})\ (\overline{10})\ (\overline{11})\ (\overline{11})\ 4\ 6\ 6\ 7\ 7\ 4 \mid 5\ 5.$$

Sketch of ($\Rightarrow$) Through Example

Example

Take

$$1\ 1 \mid 2\ 2 \mid \overline{9}\ \overline{9}\ 3\ 8\ 8\ 3 \mid (\overline{10})\ (\overline{10})\ (\overline{11})\ (\overline{11})\ 4\ 6\ 6\ 7\ 7\ 4 \mid 5\ 5$$

- eliminate bars over the barred elements and the bars dividing the blocks

to get:

$$1122993883(10)(10)(11)(11)46677455.$$

This is a flattened Stirling permutation as we computed before:

$$w = \underbrace{112299}_{\sigma_1}\ \underbrace{388}_{\sigma_2}\ \underbrace{3(10)(10)(11)(11)}_{\sigma_3}\ \underbrace{46677}_{\sigma_4}\ \underbrace{455}_{\sigma_5} \in \mathrm{flat}(\mathcal{Q}_{11})$$

Sketch of Proof ($\Leftarrow$)

Proof: ($\Leftarrow$) Given a flattened Stirling permutation,

- start at right most element of w and partition it into N_i^*s and P_i^*s
- delete the right most occurrence of each of the letters
- bump down the magnitude for each letter
- note Adler's notation is a type B set partition

Sketch of ($\Leftarrow$) Through Example

Example

Given a flattened Stirling permutation,

$$w = 1122993883(10)(10)(11)(11)46677455.$$

- Consider the value w_{2n}.
- Find the second occurrence of that value s.t. $w_j = w_{2n}$. Label the subword $w_j \cdots w_{2n}$ as P_1^*.

to get:

$$1122993883(10)(10)(11)(11)466774\underbrace{55}_{P_1^*}.$$

Sketch of ($\Leftarrow$) Through Example

Example

Take

$$1122993883(10)(10)(11)(11)466774\underbrace{55}_{P_1^*}.$$

- Look at w_{j-1}.
  1. If $w_{j-1} < w_j$, then $N_1^* = \emptyset$
  2. If $w_{j-1} > w_j$, then find $w_{j-m} \cdots w_{j-1}$ such that $w_x > w_j$ for all $j - m \leq x \leq j - 1$.
     Label the subword $w_{j-m} \cdots w_{j-1}$ as N_1^*.

to get:

$$1122993883(10)(10)(11)(11)466774\underbrace{\emptyset}_{N_1^*}\underbrace{55}_{P_1^*}.$$

Sketch of $(\Leftarrow)$ Through Example

Example

Take $1122993883(10)(10)(11)(11)466774\underbrace{\emptyset}_{N_1^*}\underbrace{55}_{P_1^*}$.

- Continue constructing P_i^* and N_i^* in this way.

1. Look at w_ℓ and find the second occurrence of it s.t. $w_j = w_\ell$. Label the subword $w_j \cdots w_\ell$ as P_i^*.
2. If $w_{j-1} < w_j$, then $N_i^* = \emptyset$
3. If $w_{j-1} > w_j$, then find $w_{j-m} \cdots w_{j-1}$ such that $w_x > w_j$ for all $j - m \leq x \leq j - 1$.
   Label the subword $w_{j-m} \cdots w_{j-1}$ as N_i^*.

to get:

$$\underbrace{11}_{P_5^*}\underbrace{\emptyset}_{N_4^*}\underbrace{22}_{P_4^*}\underbrace{99}_{N_3^*}\underbrace{3883}_{P_3^*}\underbrace{(10)(10)(11)(11)}_{N_2^*}\underbrace{466774}_{P_2^*}\underbrace{\emptyset}_{N_1^*}\underbrace{55}_{P_1^*}.$$

Sketch of $(\Leftarrow)$ Through Example

Example

Take $\underbrace{11}_{P_5^*}\underbrace{\emptyset}_{N_4^*}\underbrace{22}_{P_4^*}\underbrace{99}_{N_3^*}\underbrace{3883}_{P_3^*}\underbrace{(10)(10)(11)(11)}_{N_2^*}\underbrace{466774}_{P_2^*}\underbrace{\emptyset}_{N_1^*}\underbrace{55}_{P_1^*}$.

For each P_k^* and N_k^*

- delete the right most occurrence of each letter,
- for P_k^*, replace each remaining letter (w_i) with $(w_i - 1)$,
- for N_k^*, replace each remaining letter, w_i, with $\overline{(w_i - 1)}$,
- draw a divider to the right of these letters in the P_k^*

to get:

$$\underbrace{0}_{P_0}\,|\,\underbrace{1}_{P_1}\,|\,\underbrace{\overline{8}}_{N_2}\underbrace{27}_{P_2}\,|\,\underbrace{\overline{9}(\overline{10})}_{N_3}\underbrace{356}_{P_3}\,|\,\underbrace{4}_{P_4}.$$

Sketch of ($\Leftarrow$) Through Example

Example

Take

$$\underbrace{0}_{P_0} \mid \underbrace{1}_{P_1} \mid \underbrace{\overline{8}}_{N_2}\underbrace{27}_{P_2} \mid \underbrace{\overline{9}(\overline{10})}_{N_3}\underbrace{356}_{P_3} \mid \underbrace{4}_{P_4}.$$

- Eliminate the P_i's and N_i's below the blocks.
- Notice that this is the Adler notation.

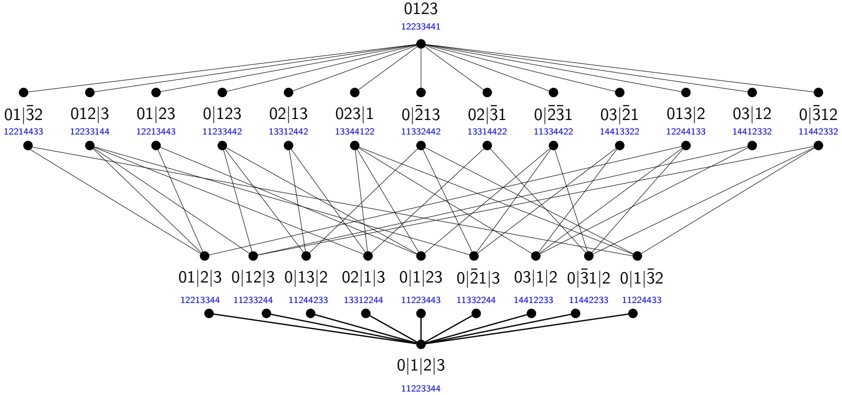
$$0 \mid 1 \mid \overline{8}\ 2\ 7 \mid \overline{9}\ \overline{10}\ 3\ 5\ 6 \mid 4$$

Adler's notation is a type B set partition.

Therefore, the set Π^B_{n-1} is in bijection with the set $\mathrm{flat}(\mathcal{Q}_n)$. $\square$

Picture

Illustrating the bijection between Π_3^B and flat$(\mathcal{Q}_4)$.



Cover relations are given by combining blocks.

Open Problem

Benedict W. J. Irwin gives the following conjecture for the Dowling numbers OEIS A007405, which we rephrase in terms of flattened Stirling permutations of order n.

Conjecture

Let M_n be an $n \times n$ matrix whose elements are

$$m_{ij} = \begin{cases} 1 & \text{if } i < j-1 \\ -1 & \text{if } i = j-1 \\ \binom{n-i}{j-i} & \text{otherwise.} \end{cases}$$

Then

$$|\text{flat}(\mathcal{Q}_n)| = \det(M_n).$$

Interested in collaborating? Email me!

Thank You!

- UW-Milwaukee Math Department
- MSU Combinatorics and Graph Theory Seminar
- You for being a wonderful audience!

Kim's Website

arXiv Article

Contact email: kjharry@uwm.edu