Box-Ball Systems and Robinson–Schensted–Knuth Tableaux

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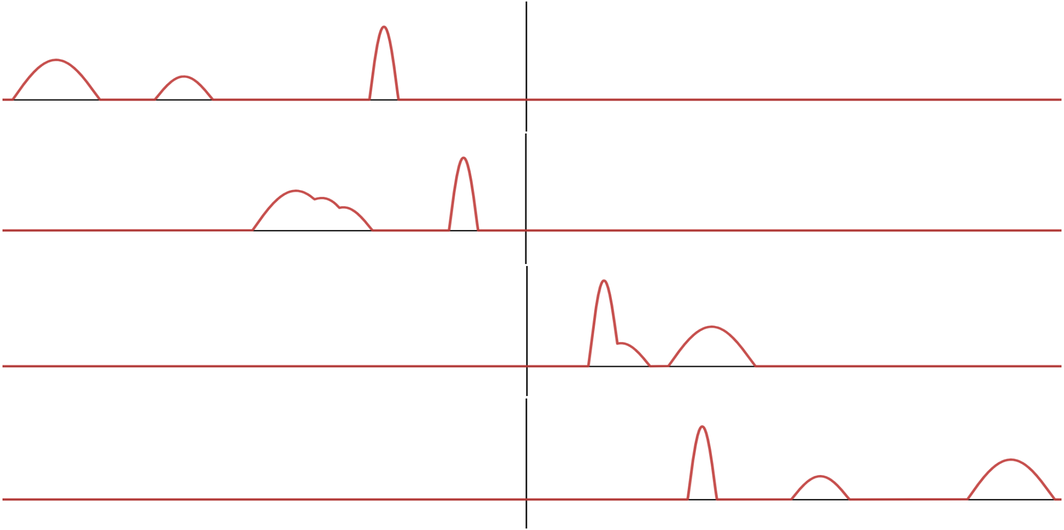
Solitary waves

(Desmos link)



Multicolor box-ball system, Takahashi 1993

A *box-ball system* (BBS) is a dynamical system with balls labeled by numbers 1 through n in an infinite strip of boxes. Balls take turns jumping to the rightmost empty box, starting with the smallest-numbered ball.

Example

One possible configuration of a box-ball system:

$\cdots$	4	5	2	3	6	1									$\cdots$

Box-ball move example (from $t = 0$ to $t = 1$)

$t = 0$	4	5	2	3	6	1			
	4	5	2	3	6		1		
	4	5		3	6	2	1		
	4	5			6	2	1	3	
		5	4		6	2	1	3	
			4	5	6	2	1	3	
$t = 1$			4	5		2	1	3	6

Box-ball system example ($t = 0$ through $t = 5$)

$t = 0$	4	5	2	3	6	1															
$t = 1$			4	5		2	1	3	6												
$t = 2$					4	5	2			1	3	6									
$t = 3$							4	2	5				1	3	6						
$t = 4$								4		2	5					1	3	6			
$t = 5$									4			2	5						1	3	6

Solitons and steady state

Definition

A *soliton* of a box-ball system is an increasing run of balls that moves at a speed equal to their length and is preserved by all future box-ball moves.

Example

The strings 4, 25, and 136 are solitons:

$t = 3$							4	2	5				1	3	6						
$t = 4$								4		2	5					1	3	6			
$t = 5$									4			2	5						1	3	6

After a finite number of BBS moves, the system reaches a *steady state* where:

- the system is decomposed into solitons, i.e., each ball belongs to one soliton
- the lengths of the solitons are weakly decreasing from right to left

Tableaux

Definition

A *tableau* is an arrangement of numbers $\{1, 2, ..., n\}$ into rows whose lengths are weakly decreasing.
A tableau is *standard* if its rows and columns are increasing.

Example

Standard Tableaux:

1	2	4
3	5	
6	7	

1	3	6
2	5	
4		

1	3	4
2		
5		
6		

Not a tableau:

1	2	
3	5	4

Nonstandard Tableau:

1	2	3
5	6	7
4		

Soliton decomposition

Definition

The *soliton decomposition* $\mathrm{SD}(w)$ of a permutation w is the tableau whose rows are the solitons stacked from right to left.

Example

$t = 3$							4	2	5				1	3	6						
$t = 4$								4		2	5					1	3	6			
$t = 5$									4			2	5						1	3	6

$$\mathrm{SD}(452361) = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array} \text{ with shape } (3,2,1).$$

RSK algorithm

The Robinson–Schensted–Knuth (RSK) insertion algorithm is a bijection

$$\pi \mapsto (P(\pi), Q(\pi))$$

from S_n onto pairs of size-n standard tableaux of equal shape.

Example

Let $w =$ **452361**. $P(w) = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}$ and $Q(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array}$.

RSK algorithm example

Let $w = \mathbf{452361}$.

$$P: \quad \mathbf{4} \qquad 4\ \mathbf{5} \qquad \begin{array}{ll} \mathbf{2} & 5 \\ 4 & \end{array} \qquad \begin{array}{ll} 2 & \mathbf{3} \\ 4 & 5 \end{array} \qquad \begin{array}{lll} 2 & 3 & \mathbf{6} \\ 4 & 5 & \end{array} \qquad \begin{array}{lll} \mathbf{1} & 3 & 6 \\ 2 & 5 & \\ 4 & & \end{array} \qquad P(w) = \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}$$

$$Q: \quad \mathbf{1} \qquad 1\ \mathbf{2} \qquad \begin{array}{ll} 1 & 2 \\ \mathbf{3} & \end{array} \qquad \begin{array}{ll} 1 & 2 \\ 3 & \mathbf{4} \end{array} \qquad \begin{array}{lll} 1 & 2 & \mathbf{5} \\ 3 & 4 & \end{array} \qquad \begin{array}{lll} 1 & 2 & 5 \\ 3 & 4 & \\ \mathbf{6} & & \end{array} \qquad Q(w) = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array}$$

Insertion and bumping rule for P

Insert x into the first row of P. If x is larger than every element in the first row, add x to the end of the first row. If not, replace the smallest number larger than x in row 1 with x. Insert this number into the row below following the same rules.

Recording rule for Q

For Q, insert $1, \ldots, n$ in order so that the shape of Q at each step matches the shape of P.

The Q tableau determines the dynamics of a box-ball system

Theorem (SUMRY 2021)

If $Q(\pi) = Q(w)$, then the box-ball systems of π and w are identical if we ignore the ball labels, in particular:

- π and w first reach steady state at the same time, and
- the soliton decompositions of π and w have the same shape

Example

$$\pi = 21435 \text{ and } w = 31425$$

$$Q(\pi) = Q(w) = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}$$

Both π and w first reach steady state at $t = 1$.

$$SD(\pi) = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 4 \\ \cline{1-1} 2 \\ \cline{1-1}\end{array} \qquad SD(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 4 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$$

Questions

- Given a Q tableau, find its steady-state time.
- Find an upper bound for steady-state time.

L-shaped soliton decompositions

The time when w first reaches steady state is called the *time to steady state* of w.

Theorem (SUMRY 2021)

If a permutation has an L-shaped soliton decomposition $SD = \begin{array}{|c|c|c|l}\hline 1 & & & \cdots \\ \hline & \multicolumn{3}{l}{} \\ \cline{1-1} & \multicolumn{3}{l}{} \\ \cline{1-1} \multicolumn{1}{c}{\vdots} & \multicolumn{3}{l}{} \end{array}$,

then its time to steady state is either $t = 0$ or $t = 1$.

Example

Such permutations include noncrossing involutions and column reading words of standard tableaux.

Both $\pi = 21435$ and $w = 31425$ have steady-state time $t = 1$.

$$SD(\pi) = \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 4 & \multicolumn{2}{c}{} \\ \cline{1-1} 2 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \quad SD(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 4 & \multicolumn{2}{c}{} \\ \cline{1-1} 3 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

$\pi = 21435 = (12)(34)$ and $w = 31425$ is the column reading word of $\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}$.

Maximum steady-state time

Theorem (UConn 2020)

If $n \geq 5$ and

$$Q(w) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline n & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \cdots \begin{array}{|c|c|}\hline n-2 & n-1 \\ \hline\end{array},$$

then the steady-state time of w is $n-3$.

Conjecture

For $n \geq 4$, the maximum time to steady state is $n-3$.

Partial Results (SUMRY 2021):

1. Applying one box-ball move to a permutation produces the rightmost soliton.
2. If the shape of $Q(w)$ is $(n-3, 2, 1)$, the maximum steady-state time is $n-3$.

Box-Ball System Example ($t = 0$ through 5)

Let $w = 452361$. Then $Q(w) = $

1	2	5
3	4	
6		

and the steady-state time of w is $3 = n - 3$.

$t = 0$	4	5	2	3	6	1															
$t = 1$			4	5		2	1	3	6												
$t = 2$					4	5	2			1	3	6									
$t = 3$							4	2	5				1	3	6						
$t = 4$								4		2	5					1	3	6			
$t = 5$									4			2	5						1	3	6

Questions

- When is the soliton decomposition SD a standard tableau?
- Can we classify permutations with standard SD using pattern avoidance?

When is SD standard?

Example

SD(452361) =

1	3	6
2	5	
4		

SD(21435) =

1	3	5
4		
2		

SD(31425) =

1	2	5
4		
3		

Theorem (UConn 2020)

Given $w \in S_n$, the following are equivalent:

1. SD(w) is standard
2. SD(w) $= P(w)$
3. the shape of SD(w) is the same as the shape of $P(w)$

Definition

We say that a permutation w is *good* if the tableau SD(w) is standard.

$Q(w)$ determines whether w is good

Fact

Given a Q-equivalence class, either all permutations in it are good or all of them are not good.

Proof

1. The recording tableau Q determines the shape of $SD(w)$.
2. $SD(w)$ is standard if and only if $\operatorname{sh} SD(w) = \operatorname{sh} P(w)$

Suppose $Q(w) = Q(\pi)$. Then

$$\begin{aligned} \mathrm{SD}(w) \text{ is standard} &\implies \operatorname{sh} \mathrm{SD}(\pi) = \operatorname{sh} \mathrm{SD}(w) = \operatorname{sh} P(w) = \operatorname{sh} P(\pi) \\ &\implies \mathrm{SD}(\pi) \text{ is standard, that is, } \pi \text{ is also good} \end{aligned}$$

Definition (Good tableaux)

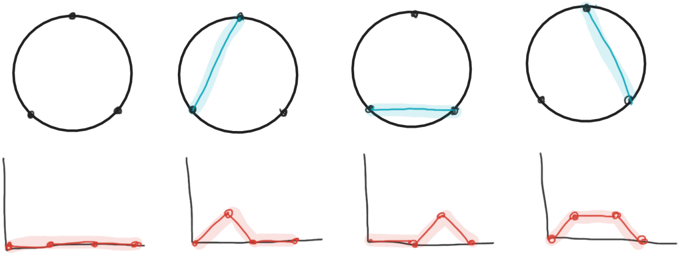
A standard tableau T is *good* if each permutation whose Q tableau equals T is good.

Good tableaux and Motzkin numbers

Conjecture

$\{Q(w) \mid w \in S_n$ and $\mathrm{SD}(w)$ is standard$\}$ are counted by the Motzkin numbers.

Other objects counted by Motzkin numbers:



$n = 3$

Consecutive pattern avoidance

Lemma (SUMRY 2021)

If T is a standard tableau which is good, then the tableau T' obtained by removing the largest k cells from T is also good.

Example

$T =$

<table><tr><td>1</td><td>2</td><td>3</td><td>7</td></tr><tr><td>4</td><td>5</td><td>6</td><td></td></tr><tr><td>8</td><td></td><td></td><td></td></tr></table>

is good.

$T' =$

<table><tr><td>1</td><td>2</td><td>3</td></tr><tr><td>4</td><td>5</td><td></td></tr></table>

is also good.

Consecutive pattern avoidance

Definition

A permutation σ is said to be a *consecutive pattern* of another permutation w if w has a consecutive subsequence whose elements are in the same relative order as σ.

Example

$w = 314592687$ contains $\sigma = 2413$ because the consecutive subsequence 5926 is ordered in the same way as $\sigma = 2413$.

Theorem (SUMRY 2021)

The good permutations are closed under consecutive pattern containment. That is, if a permutation is good, then any consecutive subpermutation is also good.

Knuth Relations

Suppose π, $w \in S_n$ and $x < y < z$.

1. π and w differ by a Knuth relation of the **first kind** (K_1) if
$$\pi = x_1 \dots yxz \dots x_n \text{ and } w = x_1 \dots yzx \dots x_n \text{ or vice versa}$$
2. π and w differ by a Knuth relation of the **second kind** (K_2) if
$$\pi = x_1 \dots xzy \dots x_n \text{ and } w = x_1 \dots zxy \dots x_n \text{ or vice versa}$$

In addition, π and w differ by a Knuth relation of **both kinds** (K_B) if they differ by K_1 and they differ by K_2, that is,

$$\pi = x_1 \dots y_1 xzy_2 \dots x_n \text{ and } w = x_1 \dots y_1 zxy_2 \dots x_n \text{ or vice versa}$$

where $x < y_1, y_2 < z$

Example $\quad \mathbf{326}154 \sim^{K_1} \mathbf{362}154 \qquad 362\mathbf{154} \sim^{K_B} 362\mathbf{514}$

We say that π and w are *Knuth equivalent* if they differ by a finite sequence of Knuth relations.

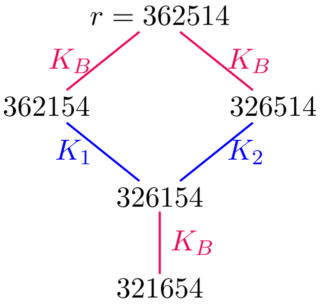
Facts (Knuth)

- There is a path of Knuth moves from w to the row reading word of $P(w)$.
- Two permutations have the same P tableau if and only if they are in the same Knuth equivalence class.

Example

The Knuth equivalence class of the row reading word $r = 362514$ of

1	4
2	5
3	6



Soliton decompositions and Knuth moves

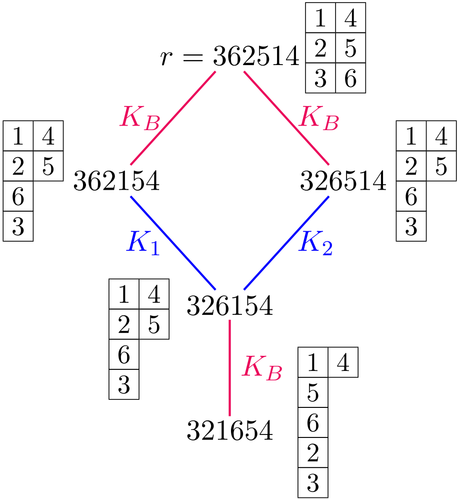
The soliton decomposition is preserved by non-K_B Knuth moves, but one K_B move changes the soliton decomposition.

Theorem (UConn Math REU 2020)

Let r denote the row reading word of $P(w)$.

- $\mathrm{SD}(r) = P(r)$.
- If there exists a path of *non*-K_B Knuth moves from w to r, then $\mathrm{SD}(w) = P(w)$.
- If there exists a path from w to r containing an *odd* number of K_B moves, then $\mathrm{SD}(w) \neq P(w)$.

Soliton decompositions in the Knuth equivalence class of 362154



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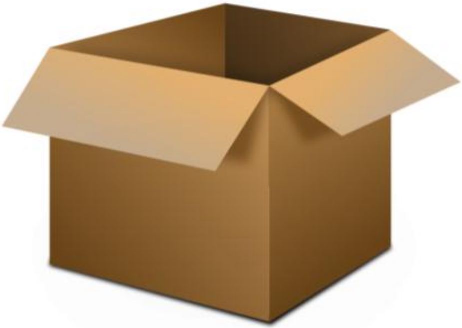

A localized version of Greene's theorem

Definition (A localized version of longest k-increasing subsequences)

Let $\mathrm{i}(u) :=$ the length of a longest increasing subsequence of u.

For $w \in S_n$ and $k \geq 1$, let $\mathrm{I}_k(w) = \max_{w=u_1|\cdots|u_k} \sum_{j=1}^{k} \mathrm{i}(u_j)$, where the maximum is taken over ways of writing w as a concatenation $u_1 \mid \cdots \mid u_k$ of consecutive subsequences.

Example

Let $w = 5623714$. For short, we write $\mathrm{I}_k := \mathrm{I}_k(w)$. Then

$\mathrm{I}_1 = \mathrm{i}(w) = 3$ (since the longest increasing subsequences are 567, 237, and 234),

$\mathrm{I}_2 = 5$ (witnessed by $56|23714$ or $56237|14$),

$\mathrm{I}_3 = 7$ (witnessed uniquely by $56|237|14$), and

$\mathrm{I}_k = 7$ for all $k \geq 3$.

A localized version of Greene's theorem

Definition (A localized version of longest k-decreasing subsequences)

Let $\mathrm{D}(u) \coloneqq 1 + |\{\text{descents of } u\}|$.

For $w \in S_n$ and $k \geq 1$, let $\mathrm{D}_k(w) = \max_{w = u_1 \sqcup \cdots \sqcup u_k} \sum_{j=1}^{k} \mathrm{D}(u_j)$, where the maximum is taken over ways to write w as the union of disjoint subsequences u_j of w.

Example

Let $w = 5623714$. For short, we write $\mathrm{D}_k \coloneqq \mathrm{D}_k(w)$. Then

$\mathrm{D}_1 = \mathrm{D}(w) = 1 + |\text{descents of } 5623714| = 1 + |\{2, 5\}| = 3$,

$\mathrm{D}_2 = 6$ (one can take subsequences 531 and 6274, among other partitions),

$\mathrm{D}_3 = 7$ (one can take subsequences 52, 631, and 74, among other partitions), and

$\mathrm{D}_k = 7$ for all $k \geq 3$.

A localized version of Greene's theorem

Theorem (Lewis–Lyu–Pylyavskyy–Sen 2019)

Suppose $w \in S_n$. *Let* $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3, \dots)$ *denote* $\operatorname{sh}\mathrm{SD}(w)$. *Let* $M = (M_1, M_2, M_3, \dots)$ *denote the conjugate of* Λ. *Then, for any* k,

$$\mathrm{I}_k(w) = \Lambda_1 + \Lambda_2 + \ldots + \Lambda_k,$$
$$\mathrm{D}_k(w) = M_1 + M_2 + \ldots + M_k.$$

Example

Let $w = 5623714$. Then $\operatorname{sh}\mathrm{SD}(w) = (\mathrm{I}_1, \mathrm{I}_2 - \mathrm{I}_1, \mathrm{I}_3 - \mathrm{I}_2) = (3, 2, 2)$. We can verify this by computing the soliton decomposition $\mathrm{SD}(w)$, which turns out to be the (non-standard) tableau

$$\begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 7 \\ \cline{1-2} 5 & 6 \\ \cline{1-2} \end{array}\ .$$

Note: $\operatorname{sh}\mathrm{SD}(w) = (3, 2, 2)$ is smaller than $\operatorname{sh}P(w) = (3, 3, 1)$ in the dominance order.

Examples: permutations with L-shaped SD

L-shaped SD which is not a column reading word:

$w = 3217654 = (13)(47)(56)$ is a noncrossing involution.

$$P(w) = Q(w) = \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 & 6 \\ \hline 7 & \\ \cline{1-1} \end{array} \text{ and } \mathrm{SD}(w) = \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 5 & \\ \cline{1-1} 6 & \\ \cline{1-1} 7 & \\ \cline{1-1} 2 & \\ \cline{1-1} 3 & \\ \cline{1-1} \end{array}$$

An involution which is neither noncrossing nor a column reading word:

$\pi = 5274163 = (15)(37)$ has a crossing.

$$P(\pi) = Q(\pi) = \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 4 & \\ \cline{1-2} 5 & 7 & \\ \cline{1-2} \end{array} \text{ and } \mathrm{SD}(\pi) = \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 4 & & \\ \cline{1-1} 2 & & \\ \cline{1-1} 7 & & \\ \cline{1-1} 5 & & \\ \cline{1-1} \end{array}$$

Good permutations are not closed under classical pattern containment

Starting with $n = 5$, a good permutation in S_n may have a substring which is not good.

Example

- The permutation 25143 is good, but its subpermutation 2143 is not good.
- The permutation 35142 is good, but its subpermutation 3142 is not good.
- Let $w = 42513$, which is a good permutation, and let $\sigma = 4253$ be a substring of w. The standardization of σ is 3142, which is not good.

(Therefore, the good permutations cannot be characterized by a set of classical avoided patterns.)

Permutations connected by K_B moves and have the same SD

Two permutations with the same SD which are connected by K_B moves: