Vexillary Double Stanley Symmetric Functions

joint work with:

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Tableaux

For $\lambda = (\lambda_1 \geq \dots \geq \lambda_\ell)$ an integer partition and $A \subseteq \mathbb{Z}$, let

$$SSYT_A(\lambda) = \left\{ \begin{array}{l} \text{semistandard Young tableaux of shape } \lambda \\ \text{with all entries belonging to } A. \end{array} \right\}$$

In this work, we'll usually take:

A	shorthand notation
$\mathbb{Z}_{>0}$	$SSYT_{>0}(\lambda)$
$\mathbb{Z}_{\leq 0}$	$SSYT_{\leq 0}(\lambda)$
$[k] = \{1, \dots, k\}$	$SSYT_k(\lambda)$

ex. of SSYT

$$\begin{array}{c|ccccccc} & \leq & \overset{2}{\leq} & \dots & & & \leq & \\ \hline & 1 & 1 & 1 & 2 & 2 & 4 & 5 \\ \wedge & 2 & 3 & 3 & 3 & 3 & 5 & \\ \wedge & 3 & 4 & 5 & 5 & 5 & 6 & \\ 4 \ \vdots & 6 & 6 & 7 & & & & \\ & 7 & & & & & & \\ \wedge & 8 & & & & & & \end{array}$$

(matrix coordinates)

$T(4,2) = 6$

row column

$$SSYT_3((2,1)) = \left\{ \begin{array}{cc} 1 & 1 \\ 2 & \end{array}, \begin{array}{cc} 1 & 1 \\ 3 & \end{array}, \begin{array}{cc} 1 & 2 \\ 2 & \end{array}, \begin{array}{cc} 1 & 2 \\ 3 & \end{array}, \begin{array}{cc} 1 & 3 \\ 2 & \end{array}, \begin{array}{cc} 1 & 3 \\ 3 & \end{array}, \begin{array}{cc} 2 & 2 \\ 3 & \end{array}, \begin{array}{cc} 2 & 3 \\ 3 & \end{array} \right\}$$

Schur functions

Let $x = (\dots, x_{-1}, x_0, x_1, \dots)$ be commuting variables
$y = (\dots, y_{-1}, y_0, y_1, \dots)$

$x_+ = (x_1, x_2, \dots)$

Assign each $T \in SSYT_A(\lambda)$ the weight

$$wt(T) = \prod_{(i,j)\in\lambda} x_{T(i,j)} = x_1^{\#\text{ of 1's}} \, x_2^{\#\text{ of 2's}} \cdots x_k^{\#\text{ of k's}}$$

- The Schur function s_λ is

$$s_\lambda(x_+) = s_\lambda(x_1, x_2, \dots) = \sum_{T \in SSYT_{>0}(\lambda)} wt(T)$$

Replace by "k" to get poly. in $x_1, \dots, x_k$.

ex.

1	1	1	2	2	4	5
2	3	3	3	3	5	
3	4	5	5	5	6	
6	6	7				
7						
8						

T

$$x_1^3 x_2^3 x_3^5 x_4^2 x_5^5 x_6^3 x_7^2 x_8$$

- char. of poly. irred. rep. of general linear groups
- distinguished basis for ring of symmetric functions

$f(x_+)$ is symmetric if it is invariant under any permutation of the variable indices.

Stanley Symmetric Function

$w = s_{a_1} \cdots s_{a_\ell}$

For $w \in S_n$ with $\ell = \ell(w) = \#\mathrm{inv}(w)$ and $R(w) = \left\{ \begin{array}{c}\text{reduced words} \\ \text{for } w\end{array} \right\}$,

$$F_w(x_+) = F_w(x_1, x_2, \ldots) = \sum_{(a_1,\ldots,a_\ell) \in R(w)} \ \sum_{\substack{1 \le i_1 \le \cdots \le i_\ell \\ a_j < a_{j+1} \Rightarrow i_j < i_{j+1}}} x_{i_1} \cdots x_{i_\ell} .$$

(Stanley 1984)
to compute $\# R(w)$

can also be defined using Schubert poly.
(next slides →)

THM. $$F_w(x_+) = \sum_\lambda c_\lambda \, s_\lambda(x_+)$$

"Schur-positive"

$$c_\lambda = \# \left\{ T \in SSYT(\lambda^t) : \begin{array}{c}\text{column word} \\ \text{of } T \in R(w)\end{array} \right\}$$

read column entries bottom to top, right to left.

(PROOFs)
- Edelman-Greene 1987
- Morse-Schilling 2015

Schubert polynomials (Lascoux–Schützenberger 1982)

at most one descent

$w(i) > w(i+1)$

- Generalize Schur polynomials
  - $\{\text{partitions } \lambda\} \longleftrightarrow \{\text{Grassmannian } w_\lambda\}$

    $s_\lambda \longleftrightarrow \mathfrak{S}_{w_\lambda}$
- Coset representatives for $\mathbb{Z}[x_1,\ldots,x_n]/I \simeq H^*(Fl(\mathbb{C}^n), \mathbb{Z})$ complete flag variety

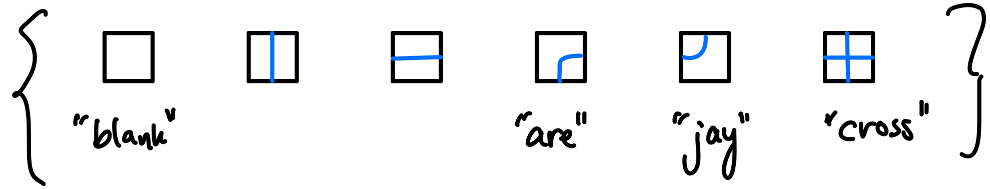
  $\{V_0=\{0\} \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n\}$ with $\dim V_j = j$
- Dual character of an associated Weyl module
- Have saturated Newton polytopes that are generalized permutahedra

⊛ LOTS OF COMBINATORIAL FORMULAS!!

we'll define them using one $\longrightarrow$

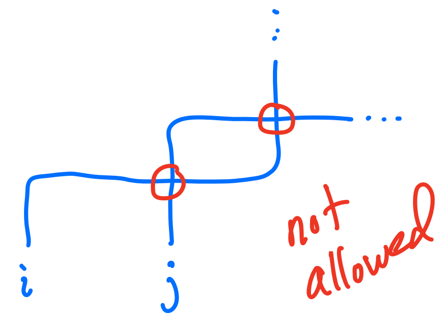
Schubert poly. via bumpless pipe dreams

A (reduced) bumpless pipe dream is a tiling of $n \times n$ grid using



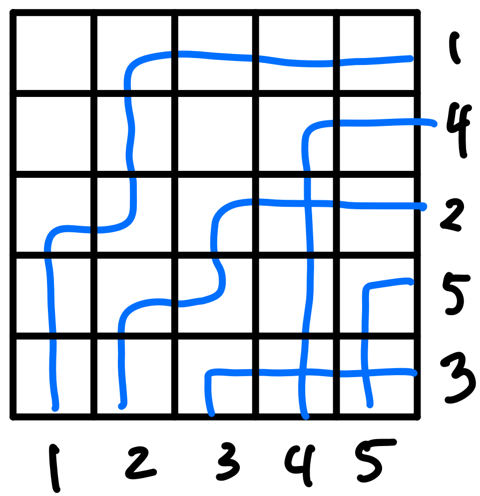
such that:

- n pipes enter from the bottom
- exit to the right
- no two pipes cross more than once.



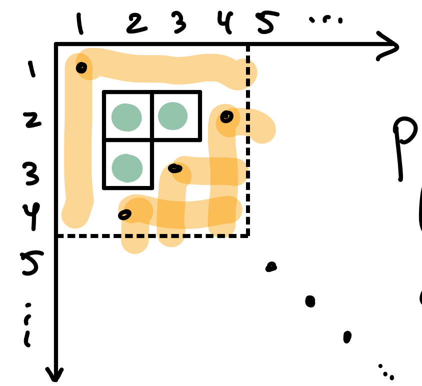
Pipes labeled $1, 2, \ldots, n$ from left to right as they enter.

Exit order determines a permutation



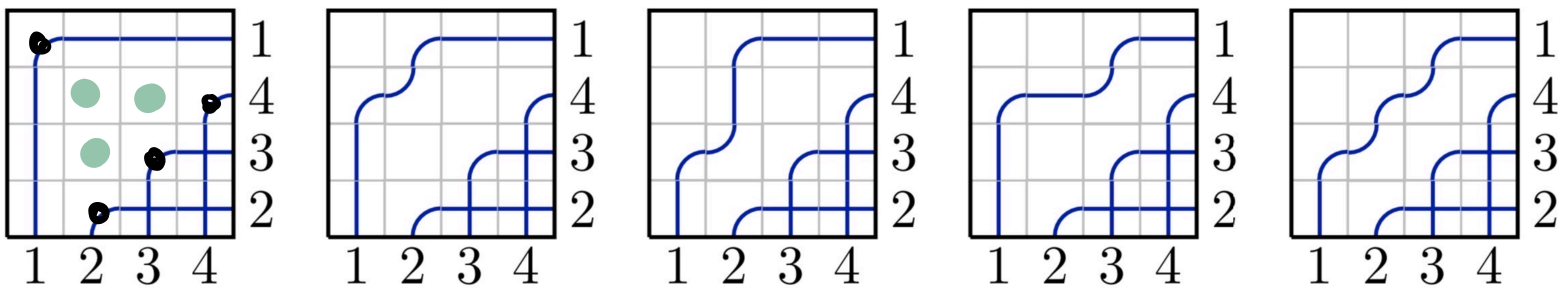
For fixed $w \in S_n$, let $BPD(w)$ denote the set of all associated BPD's.

BPD(w) generated by "droop moves":



plot $(i, w(i))$ e.g 1432

out of moves...

BPD(1432) = 

$$\mathrm{wt}(B) = \prod_{(i,j)\in B(\square)} (x_i - y_j) \;;$$

has weight $= (x_1-y_1)(x_2-y_1)(x_2-y_3)$

$$\boxed{\mathfrak{S}_w(x_+,y_+) = \sum_{B\in BPD(w)} \mathrm{wt}(B)}$$

in $x_1, x_2, \dots, y_1, y_2, \dots$

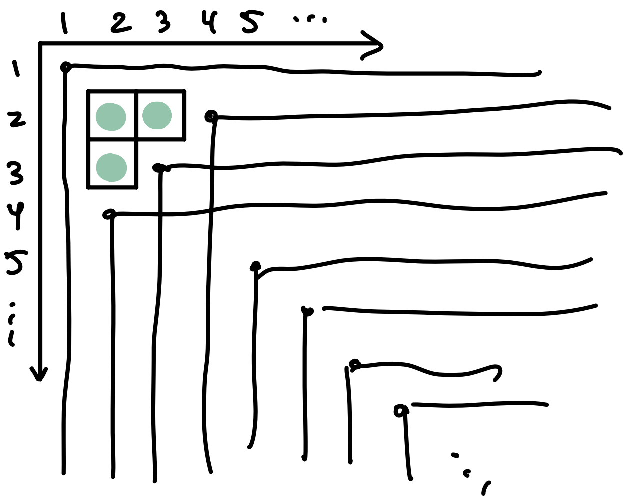
; set $y_i = 0$ to get

$$\boxed{\mathfrak{S}_w(x_+) = \mathfrak{S}_w(x_+, 0).}$$

Stanley via Schubert

$w \in S_n \hookrightarrow S_{n+1}$ by appending fixed point:

$1432 \equiv 14325 \equiv 143256 \ldots$

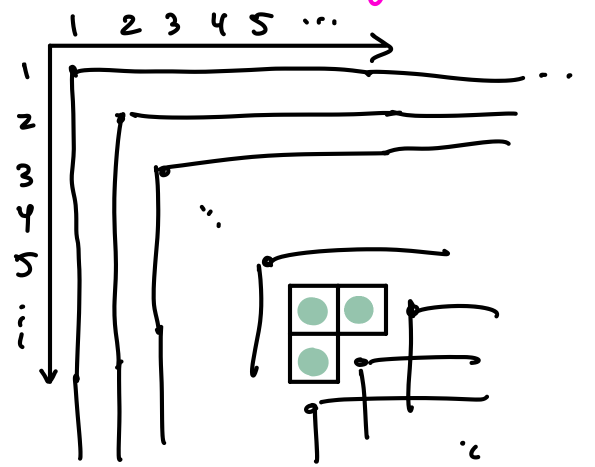


Notice this doesn't affect $\mathfrak{S}_w(x,y)$.

So $\mathfrak{S}_w(x_1,\ldots,x_n) = \mathfrak{S}_w(x_1,x_2,\ldots)$

(Pre-pending fixed points does affect $\mathfrak{S}_w(x,y)$.)

$$\boxed{F_w(x_+) = \lim_{k\to\infty} \mathfrak{S}_{12\ldots k \oplus w}(x_+,0)}$$



$S_{\mathbb{Z}}$ Permutations

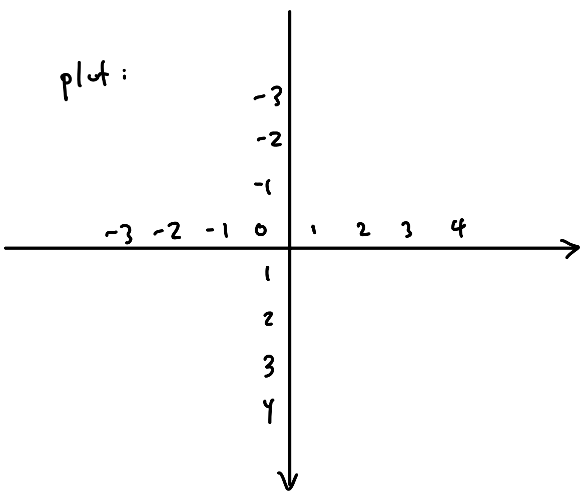
$$S_{\mathbb{Z}} = \{\text{bijections } w:\mathbb{Z}\to\mathbb{Z} \mid \operatorname{supp}(w) = \{k : w(k)\neq k\} \text{ finite}\}$$

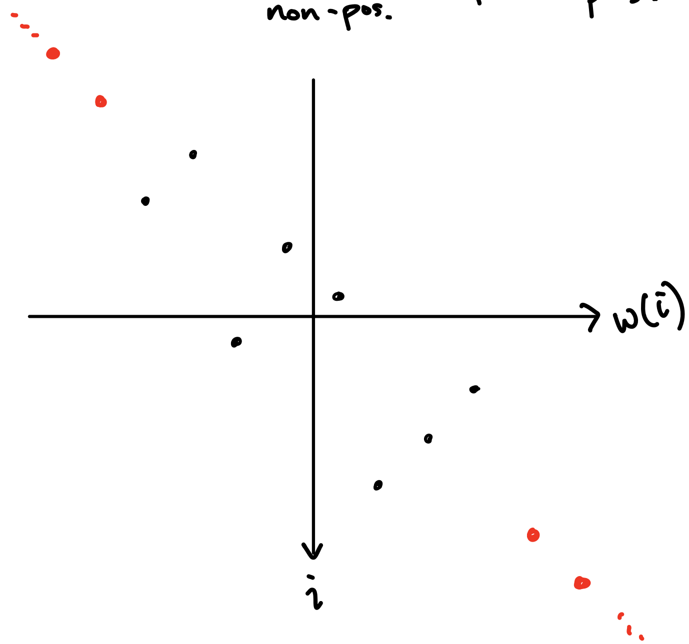
$$= \left\langle s_i \;:\; i\in\mathbb{Z} \;\middle|\; \begin{array}{l} s_i^2 = \text{identity} \\ s_i s_j = s_j s_i \iff |i-j|>1 \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \end{array} \right\rangle$$

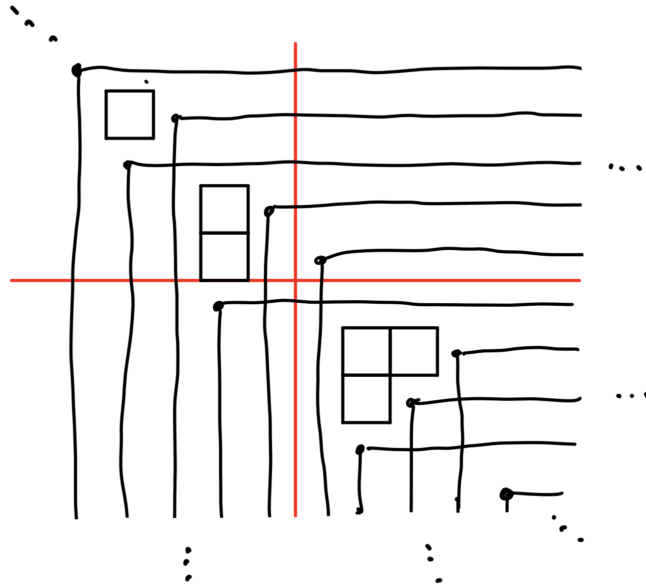
ex. $s_{-3}s_{-1}s_0s_2s_3s_2 \equiv \cdots -2\ \ -3\ \ 0\ \ 1 \mid -1\ \ 4\ \ 3\ \ 2 \cdots$

non-pos. pos.

plot:







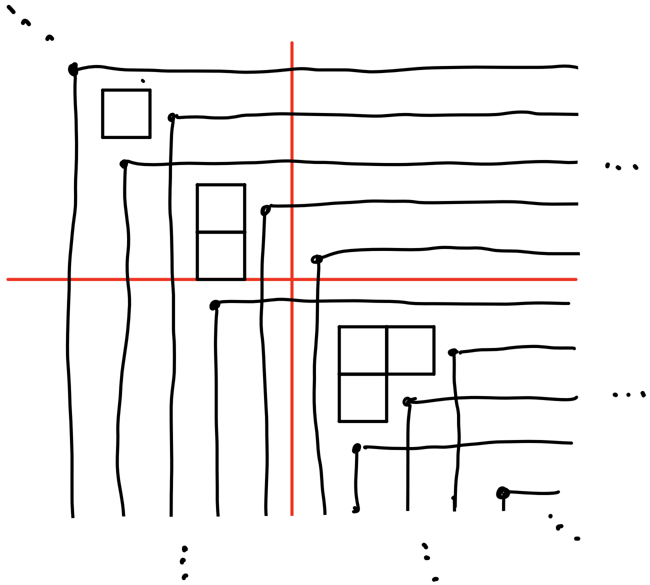
$$S_n = \{w\in S_{\mathbb{Z}} : \operatorname{supp}(w)\subseteq\{1,\dots,n\}\}$$

Backstable Schubert poly.

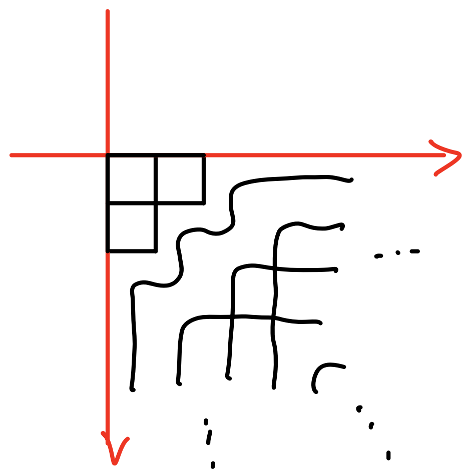
(Lam – Lee - Shimozono 2018)

in x_i, y_i for $i \in \mathbb{Z}$.

$$\overleftarrow{\mathfrak{S}}_w(x,y) = \sum_{B \in BPD_{\mathbb{Z}}(w)} wt(B)$$

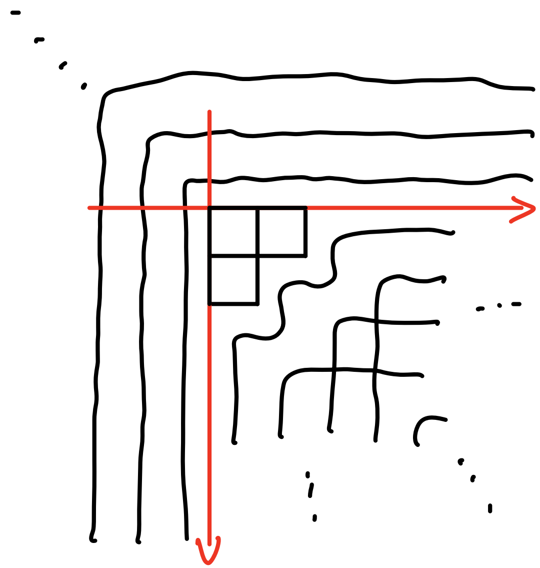


$BPD_{\mathbb{Z}}(w)$ gen. same way, but never stops.



before, no more moves

$\rightsquigarrow$



So $\overleftarrow{\mathfrak{S}}_w(x,y)$ is a formal power series in x,y.

$\boxed{F_w(x,y) \text{ defined from this, but need algebra.}}$

Algebra For $k \geq 1$, let $p_k(x,y) = \sum_{i \leq 0} (x_i^k - y_i^k)$

(k^{th} double power sum)

$\{p_j(x,y) : j \geq 1\}$ generates a $\mathbb{Q}[y]$-algebra $\Lambda(x,y)$.

ring of double symmetric functions.

Molev 2009

distinguished basis

$\{\overleftarrow{S}_\lambda(x,y)\}$

$\shortparallel$

$\overleftarrow{\mathfrak{S}}_{w_\lambda}(x,y)$

0-Grass.

$$w_\lambda(i) = i + \begin{cases} \lambda_{1-i} & i \leq 0 \\ -\lambda'_i & i > 0 \end{cases}$$

Def. • f is back symmetric if there is $k \in \mathbb{Z}$ such that $s_i(f) = f \ \forall\, i < k$.

• back stable double power series ring

$$\overleftarrow{R}(x,y) = \Lambda(x,y) \otimes_{\mathbb{Q}[y]} \mathbb{Q}[x,y].$$

$\boxed{\text{THM. } \overleftarrow{\mathfrak{S}}_w(x,y) \in \overleftarrow{R}(x,y).}$

• Let $y : \mathbb{Q}[x,y] \to \mathbb{Q}[y]$
$x_i \mapsto y_i$.

• This induces $\eta_y : \overleftarrow{R}(x,y) \longrightarrow \Lambda(x,y) \otimes_{\mathbb{Q}[y]} \mathbb{Q}[y] \simeq \Lambda(x,y)$

(Think of $\overleftarrow{S}_\lambda(x,y) = \sum_{T \in SSYT_{\leq 0}(\lambda)} wt(T)$ ← double weight, defined soon.

- $\Lambda(x,y)$ ring of double symmetric functions
- $\overleftarrow{R}(x,y)$ backstable double power series ring

  have sym. part and polynomial part
- $\overleftarrow{\mathfrak{S}}_w(x,y) \in \overleftarrow{R}(x,y)$
- $\eta_y : \overleftarrow{R}(x,y) \longrightarrow \Lambda(x,y)$ substitutes $x_i \leftrightarrow y_i$ in the polynomial parts

Define $$\boxed{F_w(x,y) = \eta_y\left(\overleftarrow{\mathfrak{S}}_w(x,y)\right)}$$

double Stanley symmetric function

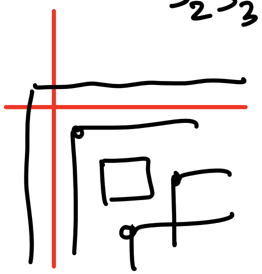
<u>Note</u>: when w 321-avoiding, $F_w(x,y)$ is Molev's (2009) skew double Schur function

Example Use transition equation for $\overleftarrow{\mathfrak{S}}_w(x,y)$ to compute:

1342

$$\overleftarrow{\mathfrak{S}}_{s_2s_3} = (x_3-y_2)\,\overleftarrow{\mathfrak{S}}_{s_2} + \overleftarrow{\mathfrak{S}}_{s_1s_2}$$

$\cdots\ 1|20\ \cdots$



$$= (x_3-y_2)\,\overleftarrow{\mathfrak{S}}_{s_2} + (x_2-y_1)\,\overleftarrow{\mathfrak{S}}_{s_1} + \overleftarrow{\mathfrak{S}}_{s_0s_1}$$

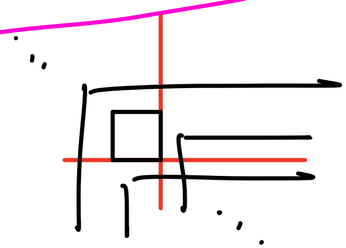
0-Grass $\Rightarrow\ = S_{(1,1)}$.

$$= (x_3-y_2)\,\overleftarrow{\mathfrak{S}}_{s_2} + (x_2-y_1)\,\overleftarrow{\mathfrak{S}}_{s_1} + (x_1-y_0)\,\overleftarrow{\mathfrak{S}}_{s_0} + \overleftarrow{\mathfrak{S}}_{s_{-1}s_0}$$

$$= (x_3-y_2)\big[\overleftarrow{\mathfrak{S}}_{s_0}+(x_1-y_1)+(x_2-y_2)\big] + (x_2-y_1)\big[\overleftarrow{\mathfrak{S}}_{s_0}+(x_1-y_1)\big] + (x_1-y_0)\,\overleftarrow{\mathfrak{S}}_{s_0} + \overleftarrow{S}_{(1,1)}.$$

$\overleftarrow{S}_{(1,1)} \in \Lambda(x\|y)$

$\overleftarrow{S}_{(1)} \in \Lambda(x\|y)$



so η_y ignores.

Apply η_y: $F_{s_2s_3} = \eta_y\big(\overleftarrow{\mathfrak{S}}_{s_2s_3}\big)$

$$= (y_3-y_2)\,\overleftarrow{S}_{(1)} + (y_2-y_1)\,\overleftarrow{S}_{(1)} + (y_1-y_0)\,\overleftarrow{S}_{(1)} + \overleftarrow{S}_{(1,1)}$$

telescope

$$= (y_3-y_0)\,\overleftarrow{S}_{(1)} + \overleftarrow{S}_{(1,1)}.$$

$$F_{s_2 s_3} = (y_3 - y_0)\, S_{(1)} + S_{(1,1)}.$$

$$\boxed{a^{31542}_{(1,1,1)} = (y_1-y_2)(y_1-y_4) \text{ for example}}$$

In general, $F_{s_{k-1}s_k} = S_{(1,1)} + (y_k - y_0)\, S_{(1)}$.

<u>THM.</u> (LLS18) For any $w \in S_{\mathbb{Z}}$, if $F_w(x,y) = \sum_\mu a^w_\mu(y)\, \overleftarrow{S}_\mu(x,y)$

then

$$a^w_\mu(y) \in \mathbb{Z}_{\geq 0}[\, y_i - y_j : \underbrace{i < j}\,]$$

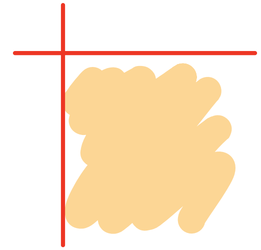
"Graham positivity"

$1 < 2 < 3 < \cdots < -3 < -2 < -1 < 0$

<u>Goal</u>: Explain $a^w_\mu(y) \in \mathbb{Z}_{\geq 0}[y_i - y_j : i < j]$ combinatorially.

Today: Explanation when w vexillary (2143-avoiding)
and has positive support. (fixes all non-pos. integers)

Fix a vexillary permutation v with pos. support.



THM. There is a (weight-preserving) bijection

(Weigandt 21, Kreiman 06)

$$\gamma : BPD_{\mathbb{Z}}(v) \longrightarrow SSYT^{f}_{\mathbb{Z}}(\lambda),$$

flagged SSYT
i.e. max entry in row $i \leq f_i$.

where $f = (f_1 \leq \dots \leq f_\ell)$ and $\lambda = (\lambda_1 \geq \dots \geq \lambda_\ell)$ are

determined by v.

(Note: for v with pos. support
one has $f > 0$.)

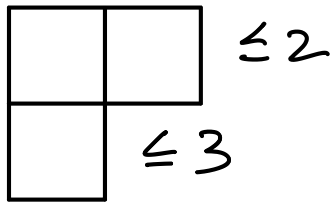
$$wt(T) = \prod_{(i,j)} (x_{T_{ij}} - y_{T_{ij}+j-i}).$$

Flagged tableaux and bijection to $BPD_{\mathbb{Z}}(v)$.

$v = 1432$

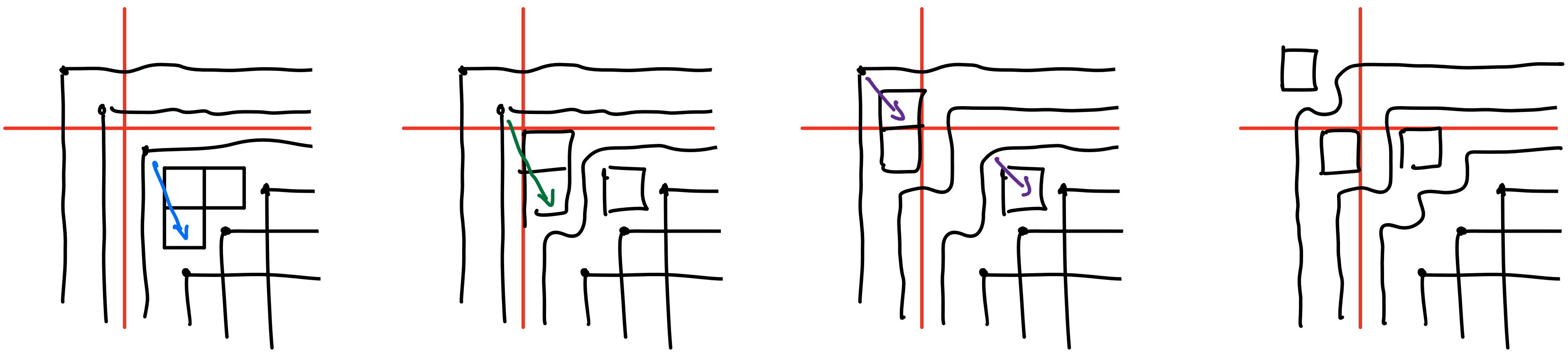
$\lambda = (2,1)$

$\phi = (2,3)$



$$\mathrm{SSYT}^f_{>0}(\lambda) = \left\{ \begin{array}{cc}1&1\\2&\end{array},\ \begin{array}{cc}1&1\\3&\end{array},\ \begin{array}{cc}1&2\\2&\end{array},\ \begin{array}{cc}1&2\\3&\end{array},\ \begin{array}{cc}2&2\\3&\end{array} \right\}$$

↳ $\mathrm{SSYT}^f_{\mathbb{Z}}(\lambda)$ allows for non-pos. as well

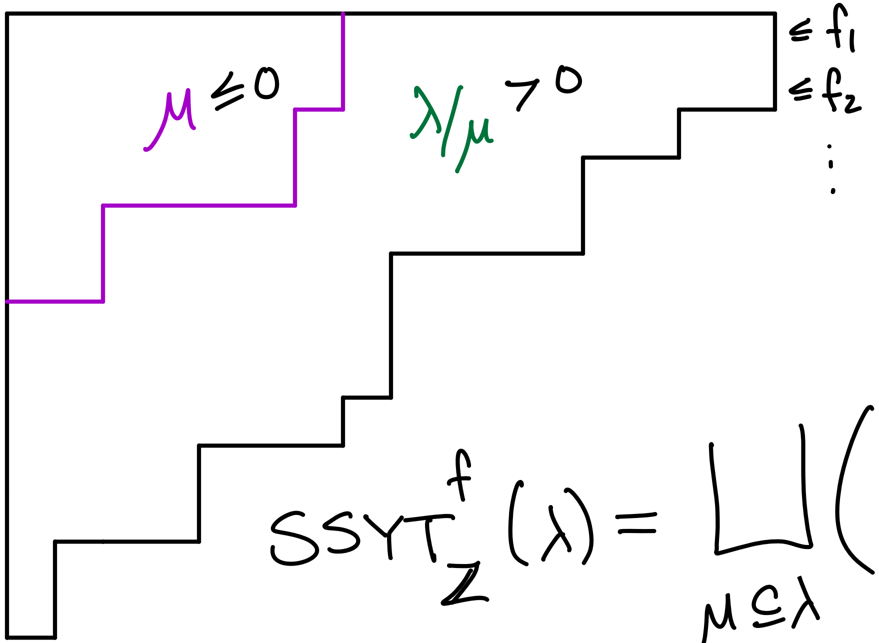


$\begin{array}{cc}2&2\\3&\end{array}$ $\quad$ $\begin{array}{cc}1&2\\2&\end{array}$ $\quad$ $\begin{array}{cc}0&2\\1&\end{array}$ $\quad$ $\begin{array}{cc}-1&1\\1&\end{array}$ $\quad \ldots$

So $$\overleftarrow{\mathfrak{S}}_v(x,y) = \sum_{B \in BPD_{\mathbb{Z}}(v)} wt(B) = \sum_{T \in \mathrm{SSYT}^f_{\mathbb{Z}}(\lambda)} wt(T).$$

We can partition these tableaux.

Given any $T \in SSYT^{f}_{\mathbb{Z}}(\lambda)$

$T =$ 

all these $f_i > 0$ and $f_1 \le f_2 \le \dots$

$$SSYT^{f}_{\mathbb{Z}}(\lambda) = \bigsqcup_{\mu \subseteq \lambda} \left(SSYT^{\le 0}_{\le 0}(\mu) \times SSYT^{f}_{>0}(\lambda/\mu) \right)$$

$\overleftarrow{s}_\mu$

$s^{f}_{\lambda}(x_1, x_2, \dots, x_N, y)$ polynomial

So $$\overleftarrow{\widetilde{G}}_v = \sum_{B} wt(B) = \sum_{T} wt(T) = \sum_{\mu \subseteq \lambda} s^{f}_{\lambda/\mu} \cdot \overleftarrow{s}_\mu .$$

need to show η_y of this is in $\mathbb{Z}[y_i - y_j \cdots]$

where $$s^{f}_{\lambda/\mu} = \sum_{T \in SSYT^{f}_{>0}(\lambda/\mu)} wt(T) .$$

Cor. For v vex. with pos. support, when

$$F_v(x\|y) = \sum_\mu a^v_\mu(y) \overleftarrow{s}_\mu(x\|y)$$

then we have that

$$a^v_\mu(y) = \eta_y\left(s^f_{\lambda/\mu}(x\|y)\right)$$

$$= \sum_{T \in SSYT^f_{>0}(\lambda/\mu)} \prod_{(i,j)\in\lambda/\mu} \left(y_{\underline{T(i,j)}} - y_{\underline{T(i,j)+j-i}} \right)$$

want $T(i,j) \prec T(i,j)+j-i$.

$1 \prec 2 \prec \cdots \prec -2 \prec -1 \prec 0$

$T(i,j) > 0 \quad \forall i,j$ here

$$a_\mu^\nu(y) = \sum_{T \in SSYT^f_{>0}(\lambda/\mu)} \prod_{(i,j)\in\lambda/\mu} \left(y_{T(i,j)} - y_{T(i,j)+j-i} \right)$$

- λ/μ can be disconnected, but then
$$s^f_{\lambda/\mu} = \prod_\nu s^f_\nu .$$
- if ν lies above main diagonal then $j-i>0$. ✓

$\Rightarrow$ suffices to show that $\eta_y(s^f_\nu)$ is Graham positive whenever ν lies entirely beneath the main diagonal.

Cor. $a^{\vee}_{\mu} = \eta_y\left(s^{f}_{\lambda/\mu}\right) = 0$ whenever λ/μ contains a box in Durfree square of λ. (or $\mu \not\subseteq \lambda$)

Q: For which i is s^{f}_{λ} symmetric in x_i, x_{i+1} ?

$$\mathrm{SSYT}^{(2,3)}_{>0}(2,1) = \left\{ \begin{array}{cc} 1 & 1 \\ 2 & \end{array}, \begin{array}{cc} 1 & 1 \\ 3 & \end{array}, \begin{array}{cc} 1 & 2 \\ 2 & \end{array}, \begin{array}{cc} 1 & 2 \\ 3 & \end{array}, \begin{array}{cc} 2 & 2 \\ 3 & \end{array} \right\}$$

$$\mathrm{wt}(T) = (x_1 - y_1)(x_2 - y_3)(x_3 - y_2)$$

$s^{(2,3)}_{(2,1)}$ is symmetric in x_1, x_2.

$$\left(x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_2 x_3 + x_2^2 x_3 . \right)$$

Lemma: $s_\lambda^f(x,y)$ sym. in x_i, x_{i+1} whenever i not in f.

Proof: Bender-Knuth type argument.

Cor. $s_\lambda^f(x,y) = \tau \cdot s_\lambda^f(x,y)$ whenever τ permutes $\{f_i+1, \ldots, f_{i+1}\}$

τ acts on x-variable indices

THM. There is a perm. π and flag F such that

$$\eta_y\left[s_\lambda^f(x,y)\right] = \eta_y\left[\pi \cdot s_\lambda^F(x,y)\right].$$

Moreover, the right-hand side is in $\mathbb{Z}_{\geq 0}[y_i - y_j : i < j]$.

(Proof only needed that λ and f satisfy $f_i - f_{i+1} \leq \lambda_i - \lambda_{i+1} + 1$.)

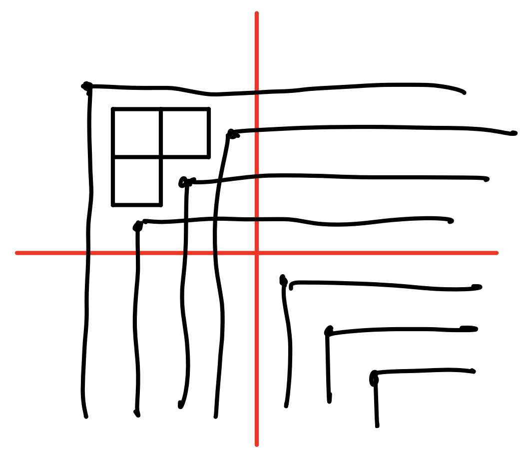
General vexillary case: Fix $v \in S_{\mathbb{Z}}$ vex.

Then (λ, f) satisfy $f_i - f_{i+1} \le \lambda_i - \lambda_{i+1} + 1$.

Outline: • Show that (λ, f) "compatible" $\Longrightarrow$ $\eta_y(S_\lambda^f) \in \mathbb{Z}_{\geq 0}[y_i - y_j \cdots]$ for any f.

(we have the $f > 0$ case above.)

- We proved it for $f < 0$.
- Can combine these to get for any f.



Future Directions

- What about non-vex.?

- what about K-theoretic setting?

  (our arguement for $f>0$ can be upgraded to set-valued tableaux.)