Column-convex $\{0,1\}$-matrices, consecutive coordinate polytopes and flow polytopes

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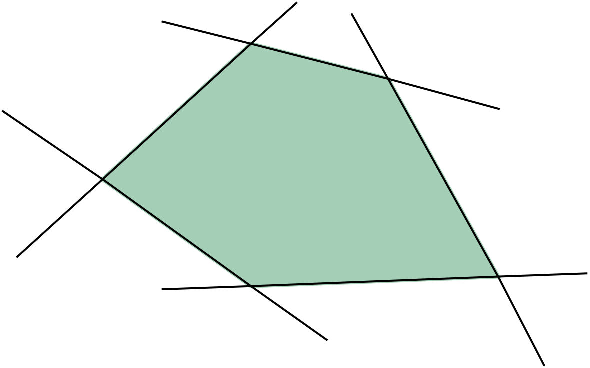
Joint work with Chris Hanusa, Alejandro Morales, Martha Yip

The usual suspects!

Polytopes



Polytopes

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$$Vol(\mathcal{B}_{M,\boldsymbol{b}}) = d!\, EuclideanVol(\mathcal{B}_{M,\boldsymbol{b}}).$$

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- Two lattice polytopes $\mathcal{P} \subset \mathbb{R}^m$ and $\mathcal{Q} \subset \mathbb{R}^n$ are **integrally equivalent** if there exists an affine transformation $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ whose restriction to $\mathcal{P}$ is a bijection $\varphi : \mathcal{P} \to \mathcal{Q}$ that preserves the lattice. This also implies that $Vol(\mathcal{P}) = Vol(\mathcal{Q})$.

A motivating question!

An exercise in Stanley's EC1

Find the volume of the polytope $\mathcal{C}_{d,2}$ in $\mathbb{R}^d$ defined by the inequalities $z_i \geq 0$ for all $i = 1, \ldots, d$, and

$$\begin{aligned} z_1 + z_2 &\leq 1 \\ z_2 + z_3 &\leq 1 \\ &\vdots \\ z_{d-1} + z_d &\leq 1. \end{aligned}$$

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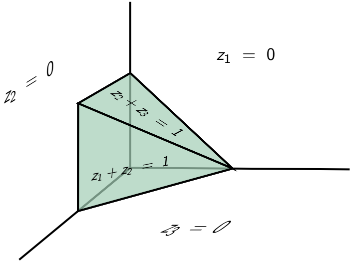

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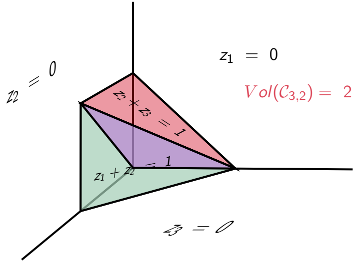

Proposition (Stanley)

We have that

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Euler numbers count up/down permutations

$$\begin{array}{lll} n=2 & 12 & E_2 = 1 \\ n=3 & 132, 231 & E_3 = 2 \\ n=4 & 1324, 1423, 2314, 2413, 3412 & E_4 = 5 \\ \vdots & & \vdots \end{array}$$

Stanley propose the more general question

Question (Stanley)

Find the volume of the polytope $\mathcal{C}_{d,k}$ in $\mathbb{R}^d$ defined by the inequalities $z_i \geq 0$ for all $i = 1, \ldots, d$, and

$$
\begin{aligned}
z_1 + z_2 + \cdots + z_k &\leq 1 \\
z_2 + z_3 + \cdots + z_{k+1} &\leq 1 \\
&\vdots \\
z_{d-k+1} + z_{d-k+2} \cdots + z_d &\leq 1.
\end{aligned}
$$

An even more general question

Question (Ayyer, Josuat-Vergès, Ramassamy 2019)

Find the volume of the polytope $\mathcal{B}_S$ in $\mathbb{R}^d$ defined by the inequalities $z_i \geq 0$ for all $i = 1, \ldots, d$, and

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The polytopes $\mathcal{B}_S$ are called **consecutive coordinate polytopes**.

Partial cyclic orders

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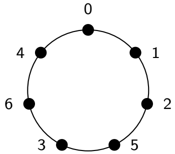
A partial cyclic order is called a **total cyclic order** if in addition it satisfies:

(d) for every $x, y, z \in X$, either $(x, y, z) \in \gamma$ or $(z, y, x) \in \gamma$ (comparability).

Total cyclic order

Total cyclic orders are easy to represent on a circle:

$\gamma =$



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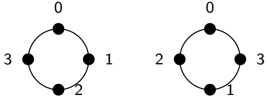

A beautiful answer to Stanley's question

Theorem (Ayyer, Josuat-Vergès, Ramassamy 2019)

For a collection S of intervals in $[d]$, the polytope $\mathcal{B}_S$ is a lattice polytope with

$$Vol(\mathcal{B}_S) = A_S,$$

where A_S is the number of total cyclic extensions to the partial cyclic order determined by S.

Can we generalize this further?

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For example, to cases where the sets in S are not necessarily intervals.

Column-convex $\{0, 1\}$-matrices

Column-convex and row-convex matrices

$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$
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Observations:

- $\mathcal{B}_S$ are $\mathcal{B}_{M,\mathbf{1}}$ polytopes with row-convex M.

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Observations:

- $\mathcal{B}_S$ are $\mathcal{B}_{M,\mathbf{1}}$ polytopes with row-convex M.
- We can remove redundant (nested) intervals in S without changing the polytope $\mathcal{B}_S$.
- The order of the inequalities does not matter in the definition of $\mathcal{B}_S$. So we could use lexicographic order in the nonredundant intervals giving also column-convexity.

Column-convex and row-convex matrices

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Moral:

Consecutive coordinate polytopes are polytopes from doubly-convex matrices.

From column-convex matrices to graphs

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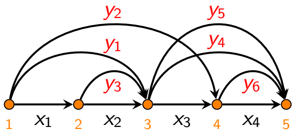
There is a bijection between column-convex matrices and spinal graphs.

$$M = \begin{array}{c} \\ \\ \\ \\ \end{array} \begin{matrix} {\color{red}y_1} & {\color{red}y_2} & {\color{red}y_3} & {\color{red}y_4} & {\color{red}y_5} & {\color{red}y_6} \end{matrix}$$

$$M = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{matrix} \quad \Longleftrightarrow \quad G =$$



Flow polytopes

Flows on a graph (network)

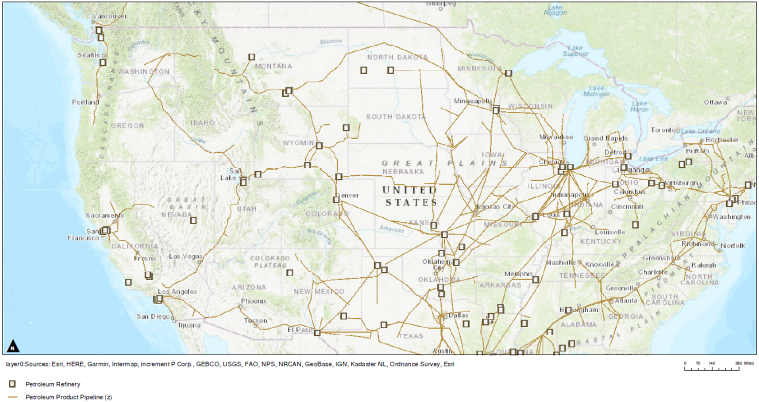
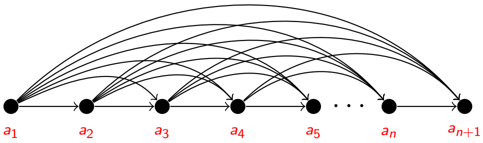


Figure: Image taken from the U.S. Energy Information Administration

Flows on a directed graph G



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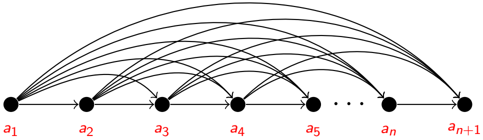

- We will think that the vector $\boldsymbol{a} = (a_1, a_2, \ldots, a_{n+1})$ is the **netflow** coming in/out of each vertex.

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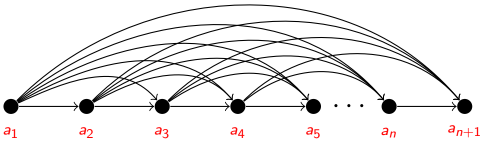
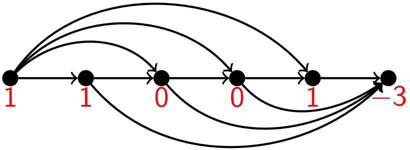


- We will think that the vector $\boldsymbol{a} = (a_1, a_2, \ldots, a_{n+1})$ is the **netflow** coming in/out of each vertex.
- The network is **balanced** in the sense that

$$\sum_{i=1}^{n+1} a_i = 0.$$

Flows on a directed graph G



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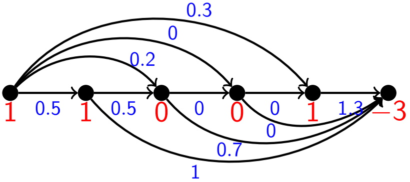

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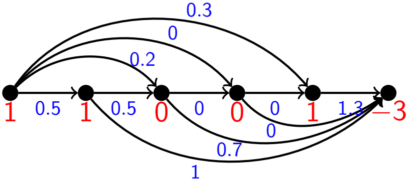
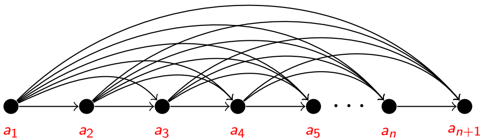


We have that $(0.5, 0.2, 0, 0.3, 0.5, 1, 0, 0.7, 0, 0, 1.3) \subseteq \mathbb{R}^{11}$ is an $(1, 1, 0, 0, 1, -3)$-flow on the graph G.

Flows on a directed graph G



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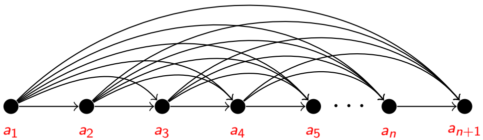

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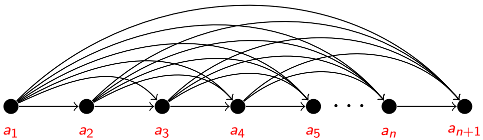
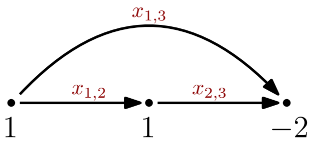


- We define $\mathcal{F}_G(\boldsymbol{a})$ to be the set of non-negative **$\boldsymbol{a}$-flows** on the edges of G.
- $\mathcal{F}_G(\boldsymbol{a})$ is a polytope.
- The **Kostant partition function** $K_G(\boldsymbol{a}) = \{\text{integer } \boldsymbol{a}\text{-flows on } G\}$.

An example of a simple graph



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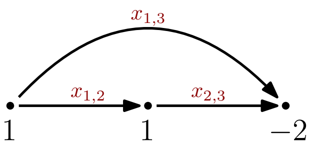

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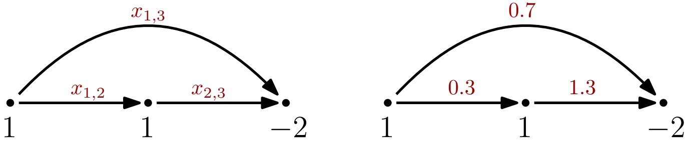

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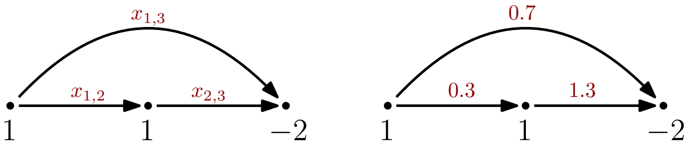

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The **nonnegative flow condition** gives

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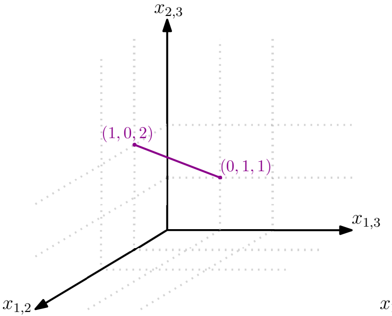
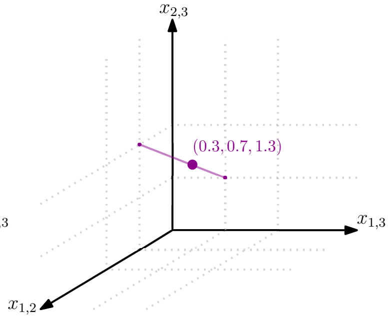
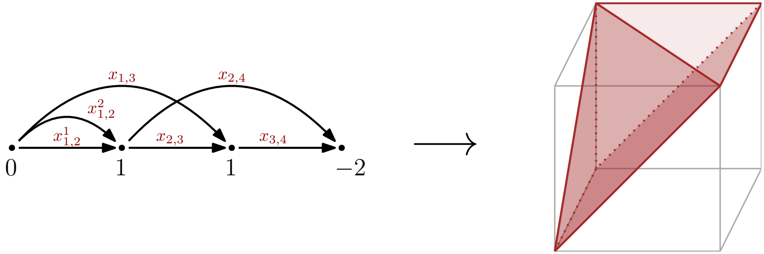




Another example



Open parenthesis (

Volumes of flow polytopes

For certain graphs G and vectors $\boldsymbol{a}$, the volumes of $\mathcal{F}_G(\boldsymbol{a})$ have nice combinatorial formulas.

Volumes of flow polytopes

For certain graphs G and vectors $\boldsymbol{a}$, the volumes of $\mathcal{F}_G(\boldsymbol{a})$ have nice combinatorial formulas.

Theorem (Zeilberger 1999)

When G is the complete graph K_{n+1} and $\boldsymbol{a} = (1, 0, \ldots, -1)$, the (normalized) volume of $\mathcal{F}_G(\boldsymbol{a})$ (called the Chan–Robbins–Yuen "CRY" polytope) is given by

$$Vol\, \mathcal{F}_{K_{n+1}}(1, 0, \ldots, -1) = \prod_{i=1}^{n-2} C_i,$$

where $C_k := \frac{1}{k+1}\binom{2k}{k}$ is the k-th Catalan number.

Open problem

Find a combinatorial proof of Zeilberger's theorem.

A simpler graph, a simpler problem

The caracol graph

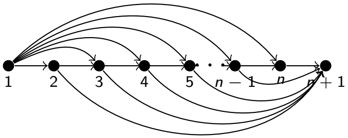


Figure: The caracol graph Car_{n+1}

Volumes of Caracol polytopes

Theorem (Stanley, Mészáros-Morales, Benedetti-G.D'L.-Hanusa-Harris-Khare-Morales-Yip)

For $n \geq 2$*, the volume of the flow polytope* $\mathcal{F}_{Car_{n+1}}(1, 0, \dots, -1)$ *is*

$$Vol\, \mathcal{F}_{Car_{n+1}}(1, 0, \dots, -1) = C_{n-2}.$$

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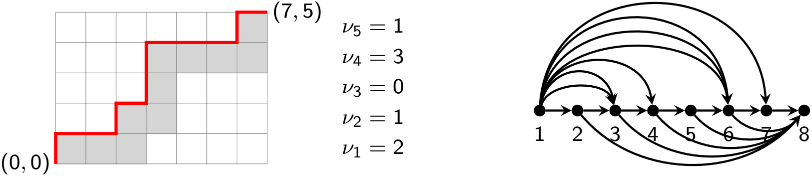
A more general Caracol!

The ν-Caracol graph



A lattice path $\nu = NE^2NE^1NE^0NE^3NE^1$ and its associated ν-caracol graph Car_ν.

Framed triangulations of the ν-Caracol polytope

Framed triangulations, the ν-Tamari lattice, and Young´s lattice

Theorem (von Bell-G.D'L.-Mayorga-Yip)

The length-framed triangulation of $\mathcal{F}_{\mathsf{Car}_\nu}(1, 0, \ldots, -1)$ *is a regular unimodular triangulation whose dual graph is the Hasse diagram of the* ν*-Tamari lattice* $\mathrm{Tam}(\nu)$ *defined by Préville-Ratelle and Viennot.*

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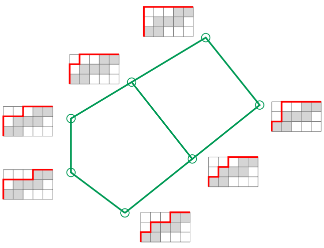
Corollary (Mészáros - Morales)

The normalized volume of the flow polytope $\mathcal{F}_{Car_\nu}$ is given by the number of ν-Dyck paths, that is, the ν-Catalan number $Cat(\nu)$.

The ν-Tamari lattice and the principal order ideal $I(\nu)$ in Young's lattice Y



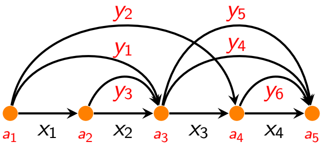
Close parenthesis)

A key integral equivalence

Theorem (González D'León-Hanusa-Morales-Yip)

When M is column-convex, the polytope $\mathcal{B}_{M,\mathbf{1}}$ is integrally equivalent to the flow polytope $\mathcal{F}_G$.

Idea behind the integral equivalence



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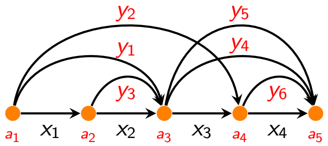
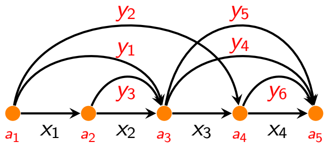


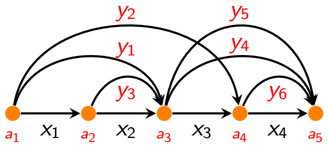
$$
\begin{aligned}
y_1 + y_2 + x_1 &= a_1 \\
y_3 + x_2 &= a_2 + x_1 \\
y_4 + y_5 + x_3 &= a_3 + x_2 + y_1 + y_3 \\
y_6 + x_4 &= a_4 + x_3 + y_2
\end{aligned}
$$

Idea behind the integral equivalence



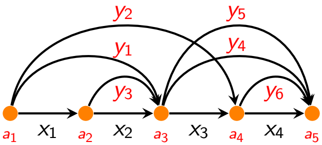
$$
\begin{aligned}
y_1 + y_2 + x_1 &= a_1 \\
y_1 + y_2 + y_3 + x_2 &= a_1 + a_2 \\
y_2 + y_4 + y_5 + x_3 &= a_1 + a_2 + a_3 \\
y_4 + y_5 + y_6 + x_4 &= a_1 + a_2 + a_3 + a_4
\end{aligned}
$$

Idea behind the integral equivalence



$$
\begin{aligned}
y_1 + y_2 &\le a_1 \\
y_1 + y_2 + y_3 &\le a_1 + a_2 \\
y_2 + y_4 + y_5 &\le a_1 + a_2 + a_3 \\
y_4 + y_5 + y_6 &\le a_1 + a_2 + a_3 + a_4
\end{aligned}
$$

Idea behind the integral equivalence



$$
\begin{aligned}
y_1 + y_2 &\leq 1 \\
y_1 + y_2 + y_3 &\leq 1 \\
y_2 + y_4 + y_5 &\leq 1 \\
y_4 + y_5 + y_6 &\leq 1
\end{aligned}
$$

when $\boldsymbol{a} = (1, 0, 0, 0, -1)$.

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Moral:

Consecutive coordinate polytopes "are" flow polytopes.

Lidskii formulas

Lidskii formulas

Baldoni and Vergne (2008) gave a beautiful set of formulas (which they call as the **Lidskii Formulas**) to compute volumes of flow polytopes.

Indegree Lidskii volume formula

Theorem (Baldoni-Vergne)

We have that

$$Vol\, F_G(a_1, \ldots, a_n, a_{n+1}) = \sum_{\mathsf{j} \leq \mathsf{i}} \binom{m-n}{\mathsf{j}} (-\boldsymbol{a})^{\mathsf{j}} K_G(\mathsf{i}-\mathsf{j}, 0),$$

where $\mathsf{i} = (\mathrm{indeg}_G(2)-1, \mathrm{indeg}_G(3)-1, \ldots, \mathrm{indeg}_G(n+1)-1)$.

Indegree version when $\boldsymbol{a} = (1, 0, \ldots, 0, -1)$

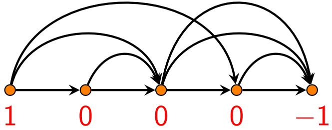
Theorem (Postnikov-Stanley, Baldoni and Vergne)

Let $u_i = \mathrm{indeg}_G(i) - 1$ *for* $i = 2, \ldots, n+1$. *Then*

$$Vol\,\mathcal{F}_G = K_G\left(0, u_2, \ldots, u_n, -\sum_{i=2}^{n} u_i\right).$$

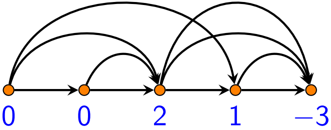
The spirit of this formula

Volume of



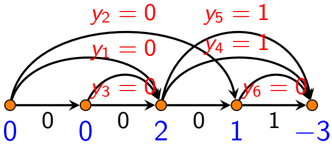
is equal to

Integer flows of

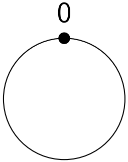


G-cyclic orders

Bijection between integral flows and upper (lower) G-cyclic orders



$\gamma \downarrow=$



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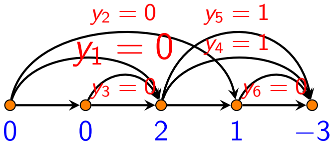
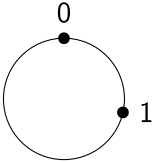


$\gamma \downarrow =$



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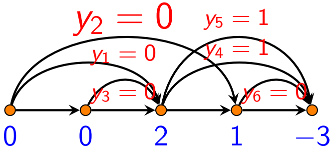
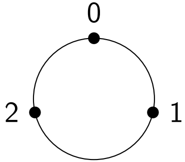


$\gamma \downarrow =$



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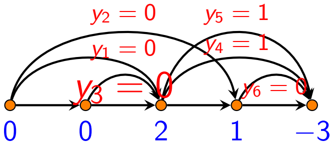

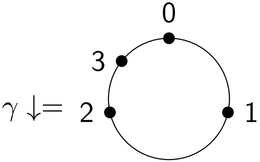

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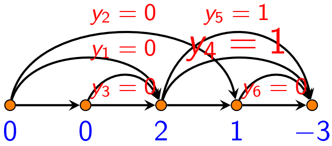
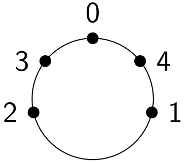


$\gamma\downarrow=$



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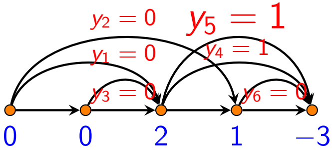

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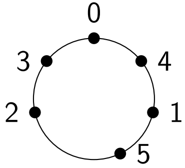

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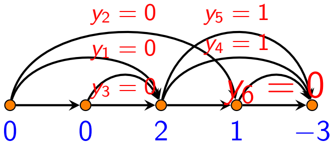

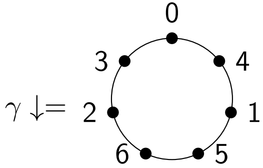

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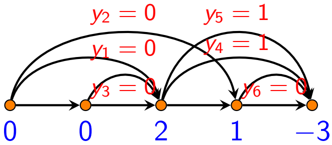
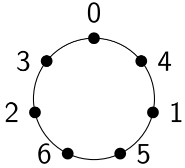
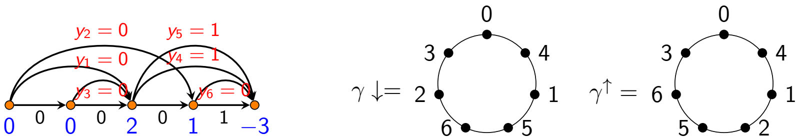


$\gamma \downarrow =$ 

Upper and lower G-cyclic orders



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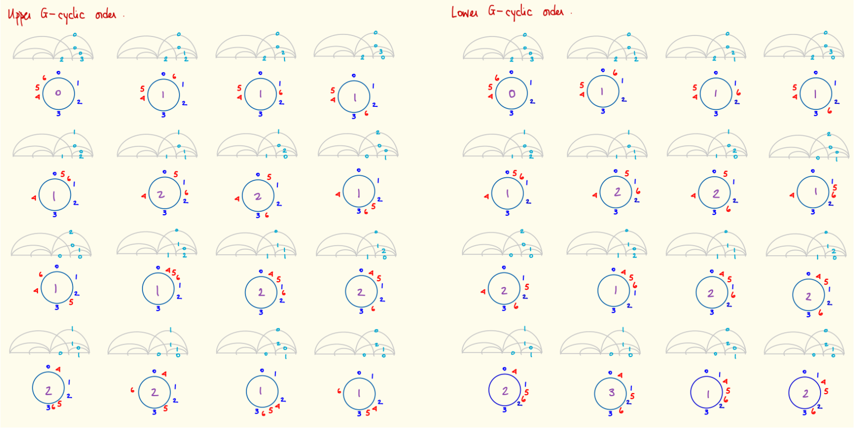

A more general answer

Theorem (González D'León-Hanusa-Morales-Yip)

For a spinal graph G, the normalized volume of the flow polytope $\mathcal{F}_G$ is the number of upper (or lower) G-cyclic orders. In other words,

$$Vol\,\mathcal{F}_G = A_G^{\downarrow} = A_G^{\uparrow}.$$

What else we gain in this translation?

Benefits of the flow polytope perspective

- We obtain a volume formula of the same flavor for a larger family of polytopes.

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- We obtain a volume formula of the same flavor for a larger family of polytopes.
- We gain all the combinatorics of flow polytopes (Lidskii formulas, a variety of combinatorial objects and triangulations)
- Because of the general Lidskii formula, that involves mixed volumes, and the Aleksandrov-Fenchel inequalities for mixed volumes one can prove certain log-concavity relations.

Back to the original polytopes $\mathcal{C}_{n,k}$.

Distance graphs and k-Euler numbers

To the polytopes $\mathcal{C}_{d,k}$ we associate the **distance graphs** $G(k, n+1)$.

Distance graphs and k-Euler numbers

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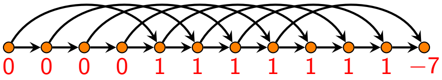
G(4,12)



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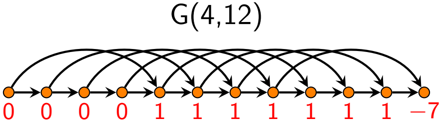
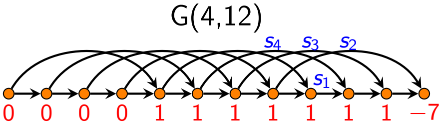


By analogy with the case $k = 2$, one can define the ***d*-th *k*-Euler number** E_d^k as the volume

$$E_d^k = Vol(\mathcal{F}_{G(k,d+k)}) = Vol(\mathcal{C}_{d,k}).$$

k-Entringer numbers

A further refinement are the **k-Entringer numbers** $E_{\boldsymbol{s}}$ indexed by weak compositions $\boldsymbol{s} = (s_1, \ldots, s_k)$.



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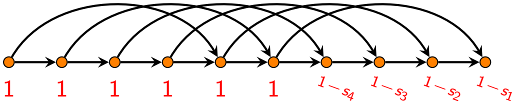

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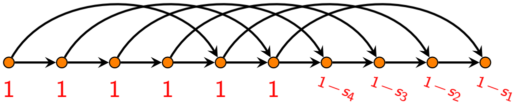


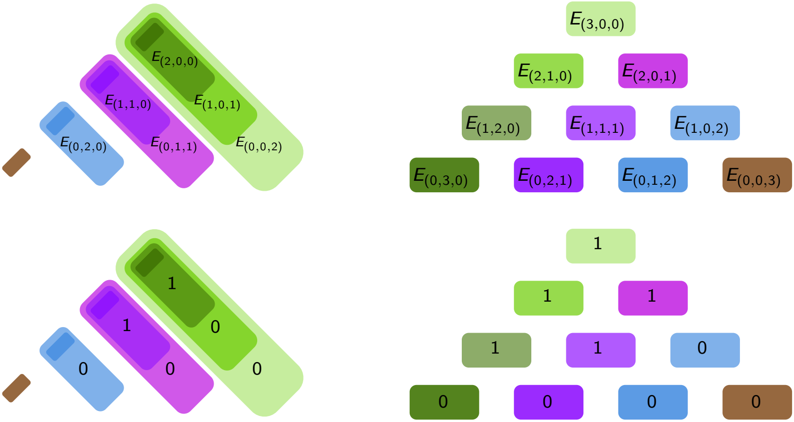
The recursivity of these graphs imply the **boustrophedon recursion**:

Theorem (c.f. Ayyer-Josuat-Vergès-Ramassamy 2019)

For $k \geq 2$, we have

$$E_{(s_1,\ldots,s_k)} = \begin{cases} 1 & \text{if } (s_1, \ldots, s_k) = (0, \ldots, 0), \\ \sum_{j=0}^{s_1 - 1} E_{(s_2+j, s_3, \ldots, s_k, s_1-j-1)}, & \text{if } s_1 > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Boustrophedon recursion



Nested boustrophedon recursions

$$
\begin{array}{ccccccc}
 & & & E_{(3,0,0)} & & & \\
 & & E_{(2,1,0)} & & E_{(2,0,1)} & & \\
 & E_{(1,2,0)} & & E_{(1,1,1)} & & E_{(1,0,2)} & \\
E_{(0,3,0)} & & E_{(0,2,1)} & & E_{(0,1,2)} & & E_{(0,0,3)}
\end{array}
\qquad
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & & 1 & & \\
 & 1 & & 1 & & 0 & \\
0 & & 0 & & 0 & & 0
\end{array}
$$

$$
\begin{array}{ccccccccc}
 & & & & E_{(4,0,0)} & & & & \\
 & & & E_{(3,1,0)} & & E_{(3,0,1)} & & & \\
 & & E_{(2,2,0)} & & E_{(2,1,1)} & & E_{(2,0,2)} & & \\
 & E_{(1,3,0)} & & E_{(1,2,1)} & & E_{(1,1,2)} & & E_{(1,0,3)} & \\
E_{(0,4,0)} & & E_{(0,3,1)} & & E_{(0,2,2)} & & E_{(0,1,3)} & & E_{(0,0,4)}
\end{array}
\qquad
\begin{array}{ccccccccc}
 & & & & 2 & & & & \\
 & & & 2 & & 2 & & & \\
 & & 2 & & 2 & & 1 & & \\
 & 1 & & 1 & & 1 & & 0 & \\
0 & & 0 & & 0 & & 0 & & 0
\end{array}
$$

$$
\begin{array}{ccccccccccc}
 & & & & & E_{(5,0,0)} & & & & & \\
 & & & & E_{(4,1,0)} & & E_{(4,0,1)} & & & & \\
 & & & E_{(3,2,0)} & & E_{(3,1,1)} & & E_{(3,0,2)} & & & \\
 & & E_{(2,3,0)} & & E_{(2,2,1)} & & E_{(2,1,2)} & & E_{(2,0,3)} & & \\
 & E_{(1,4,0)} & & E_{(1,3,1)} & & E_{(1,2,2)} & & E_{(1,1,3)} & & E_{(1,0,4)} & \\
E_{(0,5,0)} & & E_{(0,4,1)} & & E_{(0,3,2)} & & E_{(0,2,3)} & & E_{(0,1,4)} & & E_{(0,0,5)}
\end{array}
\qquad
\begin{array}{ccccccccccc}
 & & & & & 5 & & & & & \\
 & & & & 5 & & 5 & & & & \\
 & & & 5 & & 5 & & 3 & & & \\
 & & 4 & & 4 & & 3 & & 1 & & \\
 & 2 & & 2 & & 2 & & 1 & & 0 & \\
0 & & 0 & & 0 & & 0 & & 0 & & 0
\end{array}
$$

Log-concavity of k-Entringer numbers

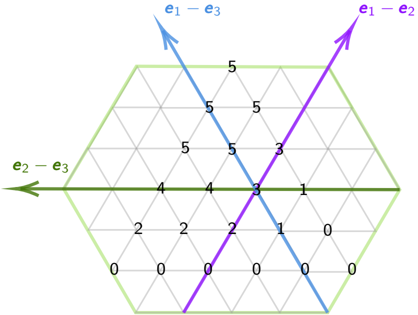
We obtain also this further log-concavity result about k-Entringer numbers

Theorem (González D'León-Hanusa-Morales-Yip)

Let $k \geq 2$ and $\mathbf{s} = (s_1, \ldots, s_{k-1}, s_k)$, then the numbers $E_{\mathbf{s}}$ are log-concave along root directions. That is,

$$E_{\mathbf{s}}^2 \geq E_{\mathbf{s}-\mathbf{e}_i+\mathbf{e}_j} \cdot E_{\mathbf{s}+\mathbf{e}_i-\mathbf{e}_j}.$$

Log-concavity along root directions



$$3^2 = E^2_{(2,1,2)} \geq E_{(1,2,2)} E_{(3,0,2)} = 2 \cdot 3$$

$$3^2 = E^2_{(2,1,2)} \geq E_{(3,1,1)} E_{(1,1,3)} = 5 \cdot 1$$

$$3^2 = E^2_{(2,1,2)} \geq E_{(2,2,1)} E_{(2,0,3)} = 4 \cdot 1$$

And a conjecture

The h^*-polynomial

A popular refinement of the volume of a lattice polytope that comes from Ehrhart theory is known as the h^*-**polynomial**.

The h^*-polynomial

A popular refinement of the volume of a lattice polytope that comes from Ehrhart theory is known as the h^***-polynomial**.

Theorem (Ayyer–Josuat-Vergès–Ramassamy 2019)

For a collection S of intervals in $[d]$ we have

$$h^*_{\mathcal{B}_S}(z) = \sum_{\gamma \in \mathcal{A}_S} z^{\mathrm{des}(\gamma)}.$$

The h^*-polynomial

A popular refinement of the volume of a lattice polytope that comes from Ehrhart theory is known as the **h^*-polynomial**.

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For a collection S of intervals in $[d]$ we have

$$h^*_{\mathcal{B}_S}(z) = \sum_{\gamma \in \mathcal{A}_S} z^{\mathrm{des}(\gamma)}.$$

Conjecture

Given a spinal graph G we have that

$$F_{\mathcal{A}_G^{\downarrow},\mathrm{des}}(z) \lhd h^*_{\mathcal{F}_G}(z) \lhd F_{\mathcal{A}_G^{\uparrow},\mathrm{des}}(z),$$

where $\lhd$ indicates dominance order in the vectors of coefficients.

Gracias!